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# MONOMIAL BASES FOR THE CENTRES OF THE GROUP ALGEBRA AND IWAHORI-HECKE ALGEBRA OF THE SYMMETRIC GROUP

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**Abstract**

A *monomial basis* for $Z(\mathbb{Z}S_n)$, the centre of the symmetric group algebra, or $Z(\mathcal{H}_n)$, the centre of the corresponding Iwahori-Hecke algebra, is a basis that consists solely of monomial symmetric polynomials in Jucys-Murphy elements. In a previous paper, we showed that there are only finitely many monomial bases for $Z(\mathbb{Z}S_3)$ and $Z(\mathcal{H}_3)$. In this paper, we prove that there exist infinitely many monomial bases for $Z(\mathbb{Z}S_4)$, which we characterize completely. Using this result, we are able to produce three new integral bases for $Z(\mathcal{H}_4)$, each of which is monomial. Based on extensive computer calculations, we conjecture that our list of monomial bases for $Z(\mathcal{H}_4)$ is exhaustive. In addition, we provide evidence to support the conjectures that the number of monomial bases for $Z(\mathbb{Z}S_5)$ is finite, and that no monomial bases exist for $Z(\mathcal{H}_5)$.



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## 1. Introduction

The appearance of symmetric polynomials in Jucys-Murphy elements in the context of the centre of the integral group ring of the symmetric group $S_n$ was implicit in work of Farahat and Higman [2], long before Jucys [8, 9, 10] and Murphy [12] had independently defined the elements that bear their names. Farahat and Higman showed that the centre of $\mathbb{Z}S_n$, which we denote here as $Z(\mathbb{Z}S_n)$, is generated over $\mathbb{Z}$ by sums of permutations of the same cycle type, sums that – once the definition of Jucys-Murphy elements is made – can be seen to be elementary symmetric polynomials in such elements. Monomial symmetric polynomials in the Jucys-Murphy elements have been of interest since Murphy [12] constructed a basis for $Z(\mathbb{Z}S_n)$ consisting solely of such elements. One natural question to ask then is whether the basis given by Murphy generalizes to $Z(\mathcal{H}_n)$, the centre of the corresponding Iwahori-Hecke algebra. Unfortunately, this generalization fails [1, p. 64]. But perhaps there are other bases for $Z(\mathbb{Z}S_n)$ consisting solely of monomial symmetric polynomials in the Jucys-Murphy elements that do generalize to $Z(\mathcal{H}_n)$. Such speculation led us to the investigation contained in this article. We refer to any basis for $Z(\mathbb{Z}S_n)$ or $Z(\mathcal{H}_n)$ that consists solely of monomial symmetric polynomials in the Jucys-Murphy elements as a *monomial basis*.

In this paper, we describe completely all monomial bases for $Z(\mathbb{Z}S_4)$. They fall into two related infinite families and eight exceptional cases (Theorem 4.5). To do this, we obtain explicit expressions for coefficients of the class sums in any monomial symmetric polynomial in the Jucys-Murphy elements. This largely involves finding and solving recursive relations among monomials, and takes up the bulk of this paper (Section 3). These computations have made use of the computer algebra package GAP [13] with CHEVIE [6], as well as Maple™ [11]. In Section 4, we use these closed forms to find sets of monomials that form bases for $Z(\mathbb{Z}S_4)$. This procedure requires solving a number of exponential Diophantine equations, sometimes using congruence arguments that would have been impractical without the use of a computer. Finally, in Section 6 we find only three monomial bases for $Z(\mathcal{H}_4)$, and conjecture that there are no more. On the basis of extensive computer calculations, we also report some results for $Z(\mathbb{Z}S_5)$ and $Z(\mathcal{H}_5)$, including our failure to find *any* monomial bases for $Z(\mathcal{H}_5)$, and our conjecture that there are no monomial bases for $Z(\mathcal{H}_n)$ when $n \geq 5$.

## 2. Definitions and Notation

Let $S_n$ be the symmetric group on $\{1, \ldots, n\}$, generated by the simple transpositions $\{(i\ i+1) | 1 \le i \le n-1\}$. Most calculations in this paper are done in $S_4$, and so we indicate these conjugacy classes using the set of the five generic class representatives $\mathcal{C} = \{1, (ab), (abc), (ab)(cd), (abcd)\}$, where $1 := (1)$. We will denote the conjugacy class sum by underlining the generic representative (except in the case of 1), so that the sum corresponding to the generic element (*abc*) is

$$\underline{(abc)} = (123) + (132) + (124) + (142) + (134) + (143) + (234) + (243).$$

The conjugacy class sums form a basis for the centre $Z(\mathbb{Z}S_n)$, and so any element of $Z(\mathbb{Z}S_n)$ can be written as an integral linear combination of the these sums. We define $\langle r, h \rangle \in \mathbb{Z}$ to be the coefficient of the class representative $r \in \mathcal{C}$ in the central element $h$, so that $h = \sum_{r \in \mathcal{C}} \langle r, h \rangle \underline{r}$. Note, the map $\mathbb{Z}S_n \times \mathbb{Z}S_n \to \mathbb{Z}$ defined by $(r, h) \mapsto \langle r, h \rangle$ satisfies the properties of an inner product, hence the notation.

**Definition 2.1.** The *Jucys-Murphy elements*, defined independently by Jucys [8, 9, 10] and Murphy [12], are $L_1 := 0$ and, for $2 \le i \le n$,

$$L_i := \sum_{1 \le k \le i-1} (k\ i).$$

**Definition 2.2.** A polynomial in the variables $\{X_1, \ldots, X_n\}$ is *symmetric* if it is fixed by the action of $S_n$ on the indices of the variables. Each partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $r \le n$ determines a *monomial symmetric polynomial*,

$$m_\lambda = m_\lambda(X_1, \ldots, X_n) := \sum_\sigma X_{\sigma(1)}^{\lambda_1} \cdots X_{\sigma(r)}^{\lambda_r},$$

where the sum is over permutations $\sigma \in S_n$ that give *distinct* monomials $X_{\sigma(1)}^{\lambda_1} \cdots X_{\sigma(r)}^{\lambda_r}$. Note that we also write $m_{\lambda_1, \lambda_2, \ldots, \lambda_r}$ for $m_\lambda$. For example, if $n = 3$ and $\lambda = (2, 1)$, then

$$m_\lambda = m_{2,1}(X_1, X_2, X_3) := \sum_{i \ne j} X_i^2 X_j.$$

In addition, we adopt the convention that $m_{0,0,\ldots,0} := 1$.

**Remark.** Jucys and Murphy both proved, via different methods, that the set of symmetric polynomials in $\{L_1, \ldots, L_n\}$ is precisely $Z(\mathbb{Z}S_n)$.

Although we postpone the definition and discussion of the Iwahori-Hecke algebra $\mathcal{H}_n$ until Section 5, we give the following definition.

**Definition 2.3.** A basis for $Z(\mathbb{Z}S_n)$, or $Z(\mathcal{H}_n)$ (the centre of the corresponding Iwahori-Hecke algebra), that consists solely of monomial symmetric polynomials in the Jucys-Murphy elements is called a *monomial basis*.

## 3. Closed Forms for Coefficients of Class Elements in Monomial Symmetric Polynomials in $Z(\mathbb{Z}S_4)$

We begin with some recursion relations that hold for monomial symmetric polynomials in three commuting variables, which are easy to verify.

**Lemma 3.1.** *The following relations hold for monomial symmetric functions in exactly three commuting variables*:

$$m_i = m_1 m_{i-1} - m_{1,1} m_{i-2} + m_{1,1,1} m_{i-3}, \quad i > 3,$$

$$m_{i,i} = m_{1,1} m_{i-1,i-1} - m_1 m_{1,1,1} m_{i-2,i-2} + m_{2,2,2} m_{i-3,i-3}, \quad i > 3,$$

$$m_{i,i,i} = m_{1,1,1} m_{i-1,i-1}, \quad i \geq 1.$$

While the above lemma is general, in what follows we present the monomial symmetric polynomials evaluated only at the Jucys-Murphy elements $L_1, \ldots, L_4$, recalling that $L_1 = 0$.

**Lemma 3.2.** *For monomial symmetric polynomials* $m_\lambda$ *evaluated at* $L_1, \ldots, L_4$, *we have the following recursive formulae*:

(1) *For* $i \geq 7$,

$$m_i = 14 m_{i-2} - 49 m_{i-4} + 36 m_{i-6}.$$

(2) *For* $i \geq 6$,

$$m_{i,i} = 8 m_{i-1,i-1} - 5 m_{i-2,i-2} - 50 m_{i-3,i-3} + 36 m_{i-4,i-4} + 72 m_{i-5,i-5}.$$

(3) *For* $i \geq 5$,

$$m_{i,i,i} = 40m_{i-2,i-2,i-2} - 144m_{i-4,i-4,i-4}.$$

**Proof.** These are all proved by induction, with base steps verified using the data in Tables 1, 2, and 3 in Section 7.

For (1), assume the claim is true for $i-1$, $i-2$ and $i-3$ (because we are attempting to prove the result for $i \geq 7$, this means the base step involves verifying that the recursion holds for $m_7$, $m_8$ and $m_9$). Using Lemma 3.1, we have

$$\begin{aligned} m_i &= m_1 m_{i-1} - m_{1,1} m_{i-2} + m_{1,1,1} m_{i-3} \\ &= m_1(14m_{i-3} - 49m_{i-5} + 36m_{i-7}) - m_{1,1}(14m_{i-4} - 49m_{i-6} + 36m_{i-8}) \\ &\quad + m_{1,1,1}(14m_{i-5} - 49m_{i-7} + 36m_{i-9}) \\ &= 14(m_1 m_{i-3} - m_{1,1} m_{i-4} + m_{1,1,1} m_{i-5}) - 49(m_1 m_{i-5} - m_{1,1} m_{i-6} + m_{1,1,1} m_{i-7}) \\ &\quad + 36(m_1 m_{i-7} - m_{1,1} m_{i-8} + m_{1,1,1} m_{i-9}) \\ &= 14m_{i-2} - 49m_{i-4} + 36m_{i-6}. \end{aligned}$$

The inductions for (2) and (3) are similar. □

Using standard techniques for solving linear recurrences, and the recurrence relations in Lemma 3.2, it is straightforward to derive the following closed forms for coefficients of class sums in the monomials.

**Lemma 3.3.** *For any* $i \geq 1$, *we have the following closed forms for the coefficients of the class sums in the monomial symmetric polynomials* $m_\lambda := m_\lambda(L_1, \ldots, L_4)$.

- $\langle -, m_i \rangle$

$$\langle 1, m_i \rangle = \frac{c_i}{24}(3^i + 10 \cdot 2^i + 23),$$

$$\langle (ab), m_i \rangle = \frac{c_{i+1}}{24}(3^i + 4 \cdot 2^i + 1),$$

$$\langle (abc), m_i \rangle = \frac{c_i}{24}(3^i + 2^i - 1),$$

$$\langle (ab)\,(cd),\, m_i \rangle = \frac{c_i}{24}(3^i - 2\cdot 2^i - 1),$$

$$\langle (abcd),\, m_i \rangle = \frac{c_{i+1}}{24}(3^i - 2\cdot 2^i + 1).$$

- $\langle -,\, m_{i,i} \rangle$

$$\langle 1,\, m_{i,i} \rangle = \frac{1}{12}(6^i + 3^i + 10\cdot 2^i + 9(-2)^i + 11(-1)^i),$$

$$\langle (ab),\, m_{i,i} \rangle = 0,$$

$$\langle (abc),\, m_{i,i} \rangle = \frac{1}{12}(6^i + 3^i + 2^i - (-1)^i),$$

$$\langle (ab)(cd),\, m_{i,i} \rangle = \frac{1}{12}(6^i + 3^i - 2\cdot 2^i - (-1)^i - 3(-2)^i),$$

$$\langle (abcd),\, m_{i,i} \rangle = 0.$$

- $\langle -,\, m_{i,i,i} \rangle$

$$\langle 1,\, m_{i,i,i} \rangle = \frac{c_i}{4}(6^{i-1} + 3\cdot 2^{i-1}),$$

$$\langle (ab),\, m_{i,i,i} \rangle = \frac{c_{i+1}}{4}(6^{i-1} - 2^{i-1}),$$

$$\langle (abc),\, m_{i,i,i} \rangle = \frac{c_i}{4}\cdot 6^{i-1},$$

$$\langle (ab)(cd),\, m_{i,i,i} \rangle = \frac{c_i}{4}(6^{i-1} - 2^{i-1}),$$

$$\langle (abcd),\, m_{i,i,i} \rangle = \frac{c_{i+1}}{4}(6^{i-1} + 2^{i-1}),$$

*where* $c_i = 1 + (-1)^i$.

To address general monomials, we need to consider coefficients in products of monomials.

**Lemma 3.4.** *The coefficients of the class sums from* $\mathbb{Z}S_4$ *in products of* *monomial symmetric polynomials in the Jucys-Murphy elements* $L_1, \ldots, L_4$ *are given by the following*:

$$\langle 1, m_\mu m_\lambda \rangle = \langle 1, m_\mu \rangle \langle 1, m_\lambda \rangle + 6\langle (ab), m_\mu \rangle \langle (ab), m_\lambda \rangle$$

$$+ 3\langle (ab)(cd), m_\mu \rangle \langle (ab)(cd), m_\lambda \rangle$$

$$+ 8\langle (abc), m_\mu \rangle \langle (abc), m_\lambda \rangle$$

$$+ 6\langle (abcd), m_\mu \rangle \langle (abcd), m_\lambda \rangle,$$

$$\langle (ab), m_\mu m_\lambda \rangle = \langle 1, m_\mu \rangle \langle (ab), m_\lambda \rangle$$

$$+ \langle (ab), m_\mu \rangle (\langle 1, m_\lambda \rangle + \langle (ab)(cd), m_\lambda \rangle + 4\langle (abc), m_\lambda \rangle)$$

$$+ \langle (abc), m_\mu \rangle (4\langle (ab), m_\lambda \rangle + 4\langle (abcd), m_\lambda \rangle)$$

$$+ \langle (ab)(cd), m_\mu \rangle (\langle (ab), m_\lambda \rangle + 2\langle (abcd), m_\lambda \rangle)$$

$$+ \langle (abcd), m_\mu \rangle (4\langle (abc), m_\lambda \rangle + 2\langle (ab)(cd), m_\lambda \rangle),$$

$$\langle (abc), m_\mu m_\lambda \rangle = \langle 1, m_\mu \rangle \langle (abc), m_\lambda \rangle$$

$$+ \langle (ab), m_\mu \rangle (3\langle (ab), m_\lambda \rangle + 3\langle (abcd), m_\lambda \rangle)$$

$$+ \langle (abc), m_\mu \rangle (\langle 1, m_\lambda \rangle + 3\langle (ab)(cd), m_\lambda \rangle + 4\langle (abc), m_\lambda \rangle)$$

$$+3\langle (ab)(cd), m_\mu \rangle \langle (abc), m_\lambda \rangle$$

$$+ \langle (abcd), m_\mu \rangle (3\langle s_1, m_\lambda \rangle + 3\langle (abcd), m_\lambda \rangle),$$

$$\langle (ab)(cd), m_\mu m_\lambda \rangle = \langle 1, m_\mu \rangle \langle (ab)(cd), m_\lambda \rangle$$

$$+ \langle (ab), m_\mu \rangle (2\langle (ab), m_\lambda \rangle + 4\langle (abcd), m_\lambda \rangle)$$

$$+ 8\langle (abc), m_\mu \rangle \langle (abc), m_\lambda \rangle$$

$$+ \langle (ab)(cd), m_\mu \rangle (\langle 1, m_\lambda \rangle + 2\langle (ab)(cd), m_\lambda \rangle)$$

$$+ \langle (abcd), m_\mu \rangle (4\langle s_1, m_\lambda \rangle + 2\langle (abcd), m_\lambda \rangle),$$

$$\langle (abcd),\ m_\mu m_\lambda\rangle = \langle 1,\ m_\mu\rangle\langle (abcd),\ m_\lambda\rangle$$

$$+ \langle (ab),\ m_\mu\rangle(4\langle (abc),\ m_\lambda\rangle + 2\langle (ab)(cd),\ m_\lambda\rangle)$$

$$+ \langle (abc),\ m_\mu\rangle(4\langle s_1,\ m_\lambda\rangle + 4\langle (abcd),\ m_\lambda\rangle)$$

$$+ \langle (ab)(cd),\ m_\mu\rangle(2\langle (ab),\ m_\lambda\rangle + \langle (abcd),\ m_\lambda\rangle)$$

$$+ \langle (abcd),\ m_\mu\rangle(\langle 1,\ m_\lambda\rangle + 4\langle (abc),\ m_\lambda\rangle + \langle (ab)(cd),\ m_\lambda\rangle).$$

**Proof.** The expansion for the coefficient of the identity follows by counting sizes of conjugacy classes. In other cases we must count elements in conjugacy classes that multiply to give the element in question. For instance, to find the coefficient of $(ab)$ in a product of monomials, we must find all pairs of elements in $S_4$ whose product is $(ab)$. A careful listing of such pairs in each case gives rise to the expansions in the statement. □

**Proposition 3.5.** *The coefficients of the class sums in monomials of the form* $m := m_{i,i,j}(L_1, \ldots, L_4)$ *for* $i \neq j$ *and* $i, j \geq 1$ *are given by the following closed forms*:

$$\langle 1,\ m\rangle = \frac{c_j}{24}(6^i + 2^i 3^j + 2^j 3^i + 9(-1)^i(2^i + 2^j) + 9 \cdot 2^i),$$

$$\langle (ab),\ m\rangle = \frac{c_{j+1}}{24}(6^i + 2^i 3^j + 2^j 3^i + 3(-1)^i(2^i + 2^j) - 3 \cdot 2^i),$$

$$\langle (abc),\ m\rangle = \frac{c_{j+1}}{24}(6^i + 2^i 3^j + 2^j 3^i),$$

$$\langle (ab)(cd),\ m\rangle = \frac{c_{j+1}}{24}(6^i + 2^i 3^j + 2^j 3^i - 3(-1)^i(2^i + 2^j) - 3 \cdot 2^i),$$

$$\langle (abcd),\ m\rangle = \frac{c_j}{24}(6^i + 2^i 3^j + 2^j 3^i - 3(-1)^i(2^i + 2^j) + 3 \cdot 2^i),$$

*where* $c_j = 1 + (-1)^j$.

**Proof.** These follow from the relation $m_{i,i,j} = m_{j,j,j} m_{i-j,i-j}$ if $i > j$ or $m_{i,i,j} = m_{i,i,i} m_{j-i}$ if $i < j$, together with the product expansions in Lemma 3.4

and the closed forms in Lemma 3.3. While the reduction is different depending on whether $i$ is greater than $j$ or not, the closed forms in terms of $i$ and $j$ have the same expression. □

**Proposition 3.6.** *The coefficients of the class sums in monomials of the form* $m := m_{i+j,i}(L_1, \ldots, L_4)$ *with* $i,\ j \geq 1$ *are given by the following closed forms*:

$$\langle 1, m\rangle = \frac{c_j}{24}(6^i 3^j + 6^i 2^j + 3^{i+j} + 3^i$$

$$+ (10 + 9(-1)^i)(2^{i+j} + 2^i) + 22(-1)^i),$$

$$\langle (ab), m\rangle = \frac{c_{j+1}}{24}(3^j 6^i + 2^j 6^i + 3^{i+j} + 3^i$$

$$+ (4 + 3(-1)^i)2^{i+j} + (4 - 3(-1)^i)2^i),$$

$$\langle (abc), m\rangle = \frac{c_j}{24}(3^j 6^i + 2^j 6^i + 3^{i+j} + 3^i + 2^{i+j} + 2^i - 2(-1)^i),$$

$$\langle (ab)(cd), m\rangle = \frac{c_j}{24}(3^j 6^i + 2^j 6^i + 3^{i+j} + 3^i(2 + 3(-1)^i)(2^i + 2^{i+j}) - 2(-1)^i),$$

$$\langle (abcd), m\rangle = \frac{c_{j+1}}{24}(3^j 6^i + 2^j 6^i + 3^i + 3^{i+j}$$

$$+ 3(-1)^i(2^i - 2^{i+j}) - 2(2^i + 2^{i+j})),$$

*where* $c_j = 1 + (-1)^j$.

**Proof.** These formulas all follow from the closed forms given in Lemma 3.3, together with the reduction relations give in Lemma 3.4 and the following relations among monomial symmetric polynomials in exactly three variables:

$$m_{i+j,i} = m_{i,i}m_j - m_{i,i,j} = \begin{cases} m_{i,i}m_j - m_{j,j,j}m_{i-j,i-j} & i > j \\ m_{i,i}m_j - m_{i,i,i}m_{j-i} & i < j \end{cases}$$

and

$$m_{2i,i} = m_{i,i}m_i - 3m_{i,i,i}.$$

For instance, for $i > j$ we have

$$\begin{aligned}\langle 1, m_{i+j,i}\rangle &= \langle 1, m_{i,i}m_j\rangle - \langle 1, m_{j,j,j}m_{i-j,i-j}\rangle\\
&= \langle 1, m_{i,i}\rangle\langle 1, m_j\rangle + 3\langle (ab)(cd), m_{i,i}\rangle\langle (ab)(cd), m_j\rangle\\
&\quad + 8\langle (abc), m_{i,i}\rangle\langle (abc), m_j\rangle - (\langle 1, m_{j,j,j}\rangle\langle 1, m_{i-j,i-j}\rangle\\
&\quad + 3\langle (ab)(cd), m_{j,j,j}\rangle\langle (ab)(cd), m_{i-j,i-j}\rangle\\
&\quad + 8\langle (abc), m_{j,j,j}\rangle\langle (abc), m_{i-j,i-j}\rangle)\\
&= \frac{1}{24}(1+(-1)^j)(6^i3^j + 6^i2^j + 3^{i+j} + 3^i\\
&\quad + (10+9(-1)^i)(2^{i+j}+2^i) + 22(-1)^i).\end{aligned}$$

Note that while the factorization of $m_{i,i,j}$ into two parts depends on whether $i > j$, $i < j$ or $i = j$, the resulting closed forms are equal and hence we obtain a single expression for all $i, j$. $\square$

**Proposition 3.7.** *The coefficients of the class sums in monomials of the form* $m := m_{k+i+j,k+i,k}(L_1, \ldots, L_4)$ *for* $i, j, k \geq 1$ *are given by*

$$\langle 1, m\rangle = \frac{c_{j+k}}{24}(6^k d_{ij} + 9(c_{k,i+k}2^{i+j+k} + c_{i,j}2^{i+k} + c_{j,j+k}2^k)),$$

$$\langle (ab), m\rangle = \frac{c_{j+k+1}}{24}(6^k d_{ij} + 3(c_{k,i+k}2^{i+j+k} + c_{i+1,k}2^{i+k} + c_{i+1,i+k}2^k)),$$

$$\langle (abc), m\rangle = \frac{c_{j+k}}{24}6^k d_{ij},$$

$$\begin{aligned}\langle (ab)(cd), m\rangle = \frac{c_{j+k}}{24}(6^k d_{ij} - 3(c_{j,k,i+j,i+k}2^{i+j+k} + c_{i,j,k,i+j+k}2^{i+k}\\
+ c_{i,i+j,i+k,i+j+k}2^k)),\end{aligned}$$

$$\begin{aligned}\langle (abcd), m\rangle = \frac{c_{j+k}}{24}(6^k d_{ij} + 3(c_{j,k+1,i+j,i+k+1}2^{i+j+k} + c_{i,j,k+1,i+j+k+1}2^{i+k}\\
+ c_{i,i+j,i+k+1,i+j+k+1}2^k)),\end{aligned}$$

*where* $c_x = 1 + (-1)^x$, $c_{x,y} = (-1)^x + (-1)^y$, $c_{x,y,z,w} = (-1)^x + (-1)^y + (-1)^z + (-1)^w$ *and* $d_{ij} = 3^j 6^i + 2^j 6^i + 3^{i+j} + 3^i + 2^{i+j} + 2^i$.

**Proof.** For monomial symmetric polynomials of three (non-zero) variables, we have $m_{i+j+k,i+k,k} = m_{k,k,k} m_{i+j,i}$. Combining this with the formulas for coefficients in products of monomials given in Lemma 3.4 gives the result. □

## 4. Monomial Bases for $Z(\mathbb{Z}S_4)$

An integral basis for $Z(\mathbb{Z}S_4)$ has a $5 \times 5$ transition matrix to the conjugacy class sum basis that is invertible over the integers. In other words, the determinant of the transition matrix is $\pm 1$. Because of Lemma 6.1 in [5], we can reorder the bases so that the transition matrix is block diagonal, with an "even" block and an "odd" block. The even $(3 \times 3)$ block consists of monomials whose partition is of an even integer, and can be written as a linear combination using only the sums 1, (*abc*) and (*ab*) (*cd*). We refer to these monomials as *even monomials*. The odd $(2 \times 2)$ block consists of monomials whose partition is of an odd integer, and can be written as a linear combination using only the sums (*ab*) or (*abcd*). We refer to these monomials as *odd monomials*. That is, the transition matrix of these reordered bases is of the form

$$\begin{array}{c|c} & \begin{matrix} 1 & (abc) & (ab)(cd) & (ab) & (abcd) \end{matrix} \\ \begin{matrix} 1 \\ (abc) \\ (ab)(cd) \\ (ab) \\ (abcd) \end{matrix} & \begin{pmatrix} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ & & & * & * \\ & & & * & * \end{pmatrix} \end{array}.$$

Some monomials have even coefficients on all class elements, and therefore they cannot be part of an integral basis (the corresponding column in the transition matrix would have a factor of 2). Using the closed forms for coefficients on class elements obtained above, we can determine which monomials have this property, and rule them out immediately as possible basis elements. We consider first the even monomials:

- $m_0$,
- $m_{i,i}$ for all $i$,
- $m_i$ and $m_{i,i,i}$ for $i$ even,
- $m_{i+j,i}$ and $m_{i,i,j}$ for $j$ even, and
- $m_{k+i+j,k+i,k}$ for $j+k$ even.

**Lemma 4.1.** *The even monomials that have at least one odd coefficient on a class element are* $m_0$, $m_2$, $m_{i,i}$ *and* $m_{i,i,2}$ *for* $i \geq 1$.

**Proof.** We break the set of possible even monomials into families as above, and claim that the following monomials have at least one odd coefficient on a class element:

(1) $m_0$,

(2) $m_{i,i}$ for all $i$,

(3) $m_2$ from $\{m_i \mid i \text{ even}\}$,

(4) $m_{2,2,2}$ from $\{m_{i,i,i} \mid i \text{ even}\}$,

(5) $m_{i,i,2}$ from $\{m_{i,i,j} \mid j \text{ even}, j \neq i\}$,

(6) none from $\{m_{i+j,i} \mid j \text{ even}\}$,

(7) none from $\{m_{k+i+j,k+i,k} \mid j+k \text{ even}\}$.

Since the proofs are similar, we present only the proofs of parts (3) and (5).

To prove part (3), we first observe that the coefficient $\langle 1, m_i \rangle$ of 1 in $m_i$ is even if and only if

$$3^i + 10 \cdot 2^i + 23 \equiv 0 \pmod 8$$

by Lemma 3.3. Since $i$ is even, $3^i \equiv 1 \pmod 8$, and $10 \cdot 2^i \equiv 0 \pmod 8$, this coefficient is always even.

Similarly, the coefficient on $(abc)$ is even if and only if $3^i + 2^i - 1 \equiv 0 \pmod 8$. This congruence holds if and only if $i \geq 4$, meaning that $m_2$ has odd coefficient on $(abc)$ but no others in this family do.

Finally, the coefficient on $(ab)(cd)$ is even if and only if $3^i - 2 \cdot 2^i - 1 \equiv 0 \pmod 8$. With $i$ even, we have $3^i \equiv 1 \pmod 8$ and $2^{i+1} \equiv 0 \pmod 8$. So this coefficient is even for all even $i$.

Thus, with the exception of $m_2$, all monomials in this family have even coefficients on all class elements, making $m_2$ the only candidate for inclusion in a basis.

For part (5), observe that for $j \geq 4$, the expressions

$$6^i + 2^i 3^j + 2^j 3^i + 9(-1)^i (2^i + 2^j) + 9 \cdot 2^i,$$

$$6^i + 2^i 3^j + 2^j 3^i \text{ and } 6^i + 2^i 3^j + 2^j 3^i - 3(-1)^i (2^i + 2^j) - 3 \cdot 2^i$$

are all divisible by 8, making the coefficients on class sums all even. For example,

$$6^i + 2^i 3^j + 2^j 3^i \equiv (-2)^i + 2^i (1) + 0 \equiv 2^i (1 + (-1)^i) \equiv 0 \pmod 8,$$

which accounts for the coefficient of $(abc)$.

This leaves the case $j = 2$, which means that $m_{i,i,2}$ is our only candidate for a basis element from this family. In this case, we have that $\langle (abc), m_{i,i,2} \rangle$ is odd since $6^i + 2^i 3^2 j + 2^2 3^i \equiv 4 \pmod 8$, while all other coefficients are even. □

We have the following table of candidate even monomials given in Lemma 4.1:

| | $0$ | $2$ | $(2, 2, 2)$ | $(2i, 2i)$ | $(2i+1, 2i+1)$ | $(i, i, 2)$ |
|---|---|---|---|---|---|---|
| $(1)$ | 1 | 6 | 6 | odd | even | even |
| $(abc)$ | 0 | 1 | 3 | even | odd | odd |
| $(ab)(cd)$ | 0 | 0 | 2 | even | odd | even |

The parities shown in the table are direct consequences of the closed forms obtained earlier.

Since these columns must be columns of a matrix whose determinant is $\pm 1$, we must have a monomial of form $m_{2i+1,2i+1}$ in a basis (see the coefficients of $(ab)(cd)$). Looking at the coefficients of the identity, we must also have either $m_0$ or one of the form $m_{2i,2i}$. Each of these three has the same parity on both $(abc)$ and $(ab)(cd)$, so we must include a monomial that has different parity on these two: either $m_2$, $m_{2,2,2}$ or $m_{i,i,2}$. Thus, we have six possible (families of) sets of monomials given below that could form a basis. For each of these cases, we examine the determinant of the $3 \times 3$ matrix to ascertain which have a determinant of $\pm 1$.

- $\{m_{2i+1,2i+1},\ m_0,\ m_2\}$

The determinant of this transition matrix is given by

$$\frac{1}{12}(6^{2i+1} + 3^{2i+1} - (2 + 3(-1)^{2i+1})2^{2i+1} - (-1)^{2i+1}).$$

This determinant is exactly 1 when $i = 0$ and is increasing with $i$.

Thus, $i = 0$ gives the only spanning subset here, namely $\{m_0,\ m_{1,1},\ m_2\}$.

- $\{m_{2i+1,2i+1},\ m_0,\ m_{2,2,2}\}$

The determinant here is

$$\frac{1}{12}(6^{2i+1} + 3^{2i+1} - (8 + 9(-1)^{2i+1})2^{2i+1} - (-1)^{2i+1}),$$

which is 1 when $i = 0$ and increasing with $i$. Thus, $\{m_0,\ m_{1,1},\ m_{2,2,2}\}$ is the only spanning set from this family.

- $\{m_{2i+1,2i+1},\ m_0,\ m_{j,j,2}\}$

The determinant simplifies to

$$-\frac{1}{48}(6^{2i+1} + 3^{2i+1} + 2^{2i+1} + 1)[(1 + (-1)^j)2^j + 4(-1)^j].$$

When $j$ is even, we have that $(1 + (-1)^j)2^j + 4(-1)^j > 4$. Then, since $6^{2i+1} + 3^{2i+1} + 2^{2i+1} + 1 \geq 12$, there are no values of $i$ and $j$ that give determinant $\pm 1$. When $j$ is odd, the determinant is 1 if and only if $i = 0$. Thus, the set $\{m_0,\ m_{1,1},\ m_{j,j,2} \mid j \text{ odd}\}$ gives all sets of this form that span the class elements here.

- $\{m_{2i+1,2i+1},\, m_{2j,2j},\, m_2\}$

The determinant simplifies to

$$\frac{1}{12}((2^{2j}-1)(6^{2i+1}+3^{2i+1}+2^{2i+1})+6\cdot 2^{2j}-6^{2j}-3^{2j}).$$

We claim that this determinant is never equal to –1, and is equal to 1 only when $i=0$ and $j=1$. To see this, we rewrite the determinant, replacing 2*j* with $w+3$, and set it equal to $\pm 1$. Multiplying by 12 produces the following two exponential Diophantine equations:

$$(2^{w+3}-1)(6^i+3^i+2^i)+6\cdot 2^{w+3}-6^{w+3}-3^{w+3}=\pm 12. \tag{4.1}$$

In the case when the right-hand side of (4.1) is –12, the equation has no solutions modulo 28. When the right-hand side of (4.1) is 12, we see that there are no solutions by reducing the equation modulo 1971. All calculations were done by computer. Thus, the equations in (4.1) have no solutions for any integers $i \geq 1$ and $w \geq 0$. This implies that the original determinant can only be $\pm 1$ when $j=1$. Substituting $j=1$ into the original determinant shows that *i* must be 0. Hence, the only spanning set here is $\{m_{1,1},\, m_{2,2},\, m_2\}$.

- $\{m_{2i+1,2i+1},\, m_{2j,2j},\, m_{2,2,2}\}$

The determinant simplifies to

$$\frac{1}{12}(17\cdot 2^{2j}-6^{2i+1}-6^{2j}-3^{2i+1}-3^{2j}-2^{2i+1}),$$

which is less than –1 when $j>1$ or $i>0$. The only solution is $i=0$ and $j=1$, giving the set of monomials $\{m_{1,1},\, m_{2,2},\, m_{2,2,2}\}$.

- $\{m_{2i+1,2i+1},\, m_{2j,2j},\, m_{k,k,2}\}$

The determinant simplifies to

$$\frac{1}{48}(2^{2j+3}3^k+2^{2j+1}6^k$$
$$-(1+(-1)^k)(2^{2i+k+1}+2^k6^{2i+1}+2^k3^{2j}+2^k3^{2i+1}+2^k6^{2j})$$
$$-4(-1)^k(6^{2i+1}+3^{2i+1}+2^{2i+1}+6^{2j}+3^{2j}+2^{2j})+(17-(-1)^k)2^{2j+k}).$$

As before, we require that this determinant be $\pm 1$. So, we need to solve the two exponential Diophantine equations:

$$\begin{aligned}
&2^{2j+3}3^k + 2^{2j+1}6^k \\
&-(1+(-1)^k)(2^{2i+k+1} + 2^k 6^{2i+1} + 2^k 3^{2j} + 2^k 3^{2i+1} + 2^k 6^{2j}) \\
&-4(-1)^k(6^{2i+1} + 3^{2i+1} + 2^{2i+1} + 6^{2j} + 3^{2j} + 2^{2j}) \\
&+(17-(-1)^k)2^{2j+k} = \pm 48. \qquad (4.2)
\end{aligned}$$

Reducing (4.2) modulo 45 shows that there are no solutions when the right-hand side is –48, while reduction modulo 1197 proves that there are no solutions to (4.2) when the right-hand side is 48. Hence, no spanning set arises in this situation.

We summarize the above computations on the even monomials in the following lemma.

**Lemma 4.2.** *The sets of even monomial symmetric polynomials in* $L_1, \ldots, L_4$ *that are bases for the subspace of* $Z(\mathbb{Z}S_4)$ *spanned by* $\{1, \underline{(abc)}, \underline{(ab)(cd)}\}$ *are*:

- $\{m_0, m_{1,1}, m_2\}$,
- $\{m_0, m_{1,1}, m_{2,2,2}\}$,
- $\{m_0, m_{1,1}, m_{i,i,2} \mid i \text{ odd}\}$,
- $\{m_{1,1}, m_{2,2}, m_2\}$,
- $\{m_{1,1}, m_{2,2}, m_{2,2,2}\}$.

We now turn our attention to the odd monomials.

**Lemma 4.3.** *The odd monomial symmetric polynomials in* $L_1, \ldots, L_4$ *that have at least one odd coefficient on a class element are* $m_i$, $m_{i,i,i}$ *and* $m_{i+j,i}$ *for* $i, j \geq 1$ *and odd*.

**Proof.** As in the proof of Lemma 4.1, we break the possibilities up into families, and show that the following have at least one odd coefficient on a class element:

(1) $\{m_i \mid i \text{ odd}\}$,

(2) $m_{1,1,1}$ from $\{m_{i,i,i} \mid i \text{ odd}\}$,

(3) $m_{i,i,1}$ from $\{m_{i,i,j} \mid j \text{ odd}, \; j \neq i\}$,

(4) $\{m_{i+j,i} \mid j \text{ odd}\}$,

(5) none from $\{m_{k+i+j,k+i,k} \mid j+k \text{ odd}\}$.

The proofs are similar to the proofs given of Lemma 4.1. For example, for part (1), observe that $3^i + 4 \cdot 2^i + 1 \equiv 4 \pmod 8$, which shows that $\langle (ab), m_i \rangle$ is odd. Also, $3^i - 2 \cdot 2^i + 1 \equiv 4 \pmod 8$ when $i \geq 3$, and $3^i - 2 \cdot 2^i + 1$ is divisible by 8 when $i = 1$, which shows that $\langle (abcd), m_i \rangle$ is even only when $i = 1$. $\square$

A consequence of Lemma 4.3 is that the parities of the candidate odd monomials are:

| | 1 | $(1, 1, 1)$ | $(i, i, 1)$ $i \geq 3$ | $i$ $i \geq 3$ | $(i+j, i)$ |
|---|---|---|---|---|---|
| $(ab)$ | 1 | 0 | even | odd | odd |
| $(abcd)$ | 0 | 1 | odd | odd | odd |

From this table we see that there are nine possibilities where the determinant of the $2 \times 2$ odd block is odd. We show below which of these possibilities actually yield a determinant equal to $\pm 1$. In certain cases, Maple was used in the computations.

- $\{m_1, m_{1,1,1}\}$

This is clearly a spanning set for the odd class elements.

- $\{m_i, m_{1,1,1}\}$ with $i \geq 3$

Since we require that the determinant is $\pm 1$, we derive the following equations:

$$3^i + 4 \cdot 2^i + 1 = \pm 12.$$

It is easy to see that there is no solution since $i \geq 3$.

- $\{m_{i+j,i},\ m_{1,1,1}\}$

Since $j$ is odd, the following equations are derived from requiring that the determinant be equal to $\pm 1$:

$$3^j 6^i + 2^j 6^i + 3^{i+j} + 3^i + (4 + 3(-1)^i)2^{i+j} + (4 - 3(-1)^i)2^i = \pm 12.$$

There are no solutions since the left-hand side is always larger than 12.

- $\{m_1,\ m_{i,i,1}\}$ with $i \geq 3$

The determinant here is

$$\frac{1}{12}(6^i + 2^i \cdot 3 + 2 \cdot 3^i + 3 \cdot (2^i + 2) + 3 \cdot 2^i),$$

which is clearly larger than 1. So, there are no solutions in this situation.

- $\{m_1,\ m_i\}$ with $i \geq 3$

Requiring that the determinant be $\pm 1$ produces the equations:

$$3^i - 2 \cdot 2^i + 1 = \pm 12.$$

It is easy to see that there is the single solution $i = 3$. Thus, the odd block can be $\{m_1,\ m_3\}$.

- $\{m_1,\ m_{i+j,i}\}$

Setting the determinant equal to $\pm 1$ gives the two equations

$$3^j 6^i + 2^j 6^i + 3^i + 3^{i+j} + 3(-1)^i(2^i - 2^{i+j}) - 2(2^i + 2^{i+j}) = \pm 12.$$

Reduction modulo 5 shows that the left-hand side is congruent to 0, 1 or 4, while the right-hand side is congruent to 2 or 3. Thus, there are no solutions here.

- $\{m_i,\ m_{j,j,1}\}$ with $i \geq 3$

Setting the determinant equal to $\pm 1$ leads to two exponential Diophantine equations: one with –48 on the right-hand side, and one with 48 on the right-hand side. The –48-equation has no solutions mod 819, while the 48-equation has no solutions mod 5.

- $\{m_{j+k,\,j},\ m_{i,i,1}\}$

As above we get two exponential Diophantine equations by equating the determinant to $\pm 1$. The –48-equation has no solutions mod 45, while the 48-equation has no solutions mod 85.

- $\{m_{i,\,i,1},\ m_{j,\,j,1}\}$ with $i,\ j \geq 3$

In this situation, we arrive at two exponential Diophantine equations: one with –144 on the right-hand side, and one with 144 on the right-hand side. Reduction modulo 91 shows that there are no solutions in either case since the left-hand side is congruent to 0 or 16 while the right-hand side is congruent to 38 or 53.

We summarize the above computations on the odd monomials in the following lemma.

**Lemma 4.4.** *The sets of odd monomial symmetric polynomials in* $L_1, \ldots, L_4$ *that are bases for the subspace of* $Z(\mathbb{Z}S_4)$ *spanned by* $\{\underline{(ab)}, \underline{(abcd)}\}$ *are* $\{m_1,\ m_{1,1,1}\}$ *and* $\{m_1,\ m_3\}$.

Lemma 4.2 and Lemma 4.4 determine all bases for $Z(\mathbb{Z}S_4)$ which consist solely of monomial symmetric polynomials in Jucys-Murphy elements. We get eight specific bases and two infinite families of bases. This result differs dramatically from $Z(\mathbb{Z}S_3)$, where there are only finitely such bases [5]. We state this main result in the following theorem.

**Theorem 4.5.** *The complete list of monomial bases for* $Z(\mathbb{Z}S_4)$ *is*

- $\{m_0,\ m_1,\ m_2,\ m_{1,1},\ m_{1,1,1}\}$,
- $\{m_0,\ m_1,\ m_{1,1},\ m_{1,1,1},\ m_{2,2,2}\}$,
- $\{m_1,\ m_2,\ m_{1,1},\ m_{1,1,1},\ m_{2,2}\}$,
- $\{m_1,\ m_{1,1},\ m_{1,1,1},\ m_{2,2},\ m_{2,2,2}\}$,
- $\{m_0,\ m_1,\ m_2,\ m_{1,1},\ m_3\}$ (*Murphy's basis* [12]),
- $\{m_0,\ m_1,\ m_{1,1},\ m_{2,2,2},\ m_3\}$,
- $\{m_1,\ m_2,\ m_{1,1},\ m_3,\ m_{2,2}\}$,

- $\{m_1, m_{1,1}, m_3, m_{2,2}, m_{2,2,2}\}$,
- $\{m_0, m_1, m_{1,1}, m_{1,1,1}, m_{i,i,2} \mid i \text{ odd}\}$,
- $\{m_0, m_1, m_{1,1}, m_3, m_{i,i,2} \mid i \text{ odd}\}$.

## 5. The Iwahori-Hecke Algebra

In this paper, we use a normalized version of the generators for the Hecke algebra.

**Definition 5.1.** The Iwahori-Hecke algebra $\mathcal{H}_n$ of $S_n$ is the associative $\mathbb{Z}[\xi]$-algebra generated by the set $\{\widetilde{T}_s \mid s \in S\}$ with identity $\widetilde{T}_1$ and subject to the relations

$$\begin{aligned} \widetilde{T}_{s_i}\widetilde{T}_{s_j} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i} && \text{if } |i-j| \geq 2, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_{i+1}}\widetilde{T}_{s_i} &= \widetilde{T}_{s_{i+1}}\widetilde{T}_{s_i}\widetilde{T}_{s_{i+1}} && \text{for } 1 \leq i \leq n-2, \\ \widetilde{T}_{s_i}^2 &= \widetilde{T}_1 + \xi\widetilde{T}_{s_i} && \text{for } 1 \leq i \leq n-1. \end{aligned}$$

**Remark.** The Iwahori-Hecke algebra $\mathcal{H}_n$ of type $A_{n-1}$ is a deformation of the symmetric group algebra $\mathbb{Z}S_n$. In particular, the specialization of $\mathcal{H}_n$ at $\xi = 0$ is isomorphic to $\mathbb{Z}S_n$.

The exact connection between this definition and the standard definition over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ is the following. Set $\widetilde{T}_s := q^{-1/2}T_s$ for $s \in S = \{(i\ i+1) | 1 \leq i \leq n-1\}$, and let $\xi = q^{1/2} - q^{-1/2}$. Then this normalized Hecke algebra, which is a subalgebra of the more standard Hecke algebra generated by $\{T_s \mid s \in S\}$ over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$, is an algebra over the ring $\mathbb{Z}[\xi]$. A principal reason for defining the algebra with normalized generators is that then many results concerning the centre have more natural statements and proofs [3, 4, 5]. The statements of this paper are all readily translated back to statements over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$.

Many of the approaches to $Z(\mathbb{Z}S_n)$ described above generalize neatly to the centre of the Iwahori-Hecke algebra $\mathcal{H}_n$, which we denote $Z(\mathcal{H}_n)$. To begin with, $Z(\mathcal{H}_n)$ has an integral basis of "class elements" $\{\Gamma_\lambda \mid w_\lambda \in S_n\}$, which specialize to a class sum [7, 3]. The Jucys-Murphy elements can directly be generalized, by setting $L_1 := 0$ and for $2 \le i \le n$,

$$L_i := \sum_{1 \le k \le i-1} \widetilde{T}_{(ki)}.$$

The main result of both Jucys and Murphy – that the centre is the set of symmetric polynomials in Jucys-Murphy elements – generalizes to a result known as the Dipper-James conjecture, shown for the semisimple case by Dipper and James in 1987 [1] and in generality by the first author and Graham in 2006 [4]. The result of Jucys giving the elementary symmetric polynomials in Jucys-Murphy elements as a sum of class sums has a direct analogue, and so the analogue of Farahat and Higman's generators for the centre also holds [4, Prop. 7.4, Cor. 7.6]. The fly in the ointment is that the basis for $Z(\mathbb{Z}S_n)$ given by Murphy does not generalise to a basis for $Z(\mathcal{H}_n)$, even for $Z(\mathcal{H}_3)$ [5]. While it is possible, as a result of the proof of the Dipper-James Conjecture, to construct an integral basis for $Z(\mathcal{H}_n)$ using linear combinations of monomial symmetric polynomials in Jucys-Murphy elements [5], it is still unclear in general whether there exists an integral basis for $Z(\mathcal{H}_n)$ using monomial symmetric polynomials alone.

## 6. Summary, Generalizations and Conjectures

Since, upon specialization at $\xi = 0$, any monomial basis for $Z(\mathcal{H}_n)$ gives a monomial basis for $Z(\mathbb{Z}S_n)$, we can identify monomial bases for $Z(\mathcal{H}_n)$ by restricting our attention to sets of monomials in $Z(\mathcal{H}_n)$ which correspond to monomial bases for $Z(\mathbb{Z}S_n)$. Using this strategy in [5], we showed that there are only four such bases for $Z(\mathbb{Z}S_3)$, only one of which "lifts to" an integral basis for $Z(\mathcal{H}_3)$. The motivation for the investigation in this paper was to produce a similar result for $n = 4$. Since Theorem 4.5 gives the complete list of monomial bases for $Z(\mathbb{Z}S_4)$, to find all monomial bases for $Z(\mathcal{H}_4)$, it is sufficient to check the sets of monomials in $\mathcal{H}_4$ corresponding to the bases for $Z(\mathbb{Z}S_4)$. However, the existence of the infinite families of monomial bases for $Z(\mathcal{H}_4)$ makes the identification of all monomial bases for $Z(\mathcal{H}_4)$ more difficult. Checking the infinite families for $i < 50$, and the remaining sporadic bases gives the following list of bases for $Z(\mathcal{H}_4)$ :

$$\{m_0, m_1, m_2, m_{1,1}, m_{1,1,1}\},$$

$$\{m_0, m_1, m_{1,1}, m_{1,1,1}, m_{2,1,1}\},$$

$$\{m_0, m_1, m_{1,1}, m_{1,1,1}, m_{2,2,2}\}.$$

In an attempt to extend the above findings to $n = 5$, we conducted a search using GAP up through partitions of 10, and found the following 12 monomial bases for $Z(\mathbb{Z}S_5)$:

$$\{m_0, m_1, m_2, m_{1,1}, m_3, m_{2,1}, m_4\},$$

$$\{m_0, m_1, m_2, m_{1,1}, m_3, m_{2,1}, m_{1,1,1,1}\},$$

$$\{m_0, m_1, m_2, m_{1,1}, m_3, m_{1,1,1}, m_4\},$$

$$\{m_0, m_1, m_2, m_{1,1}, m_3, m_{1,1,1}, m_{1,1,1,1}\},$$

$$\{m_0, m_1, m_2, m_{1,1}, m_{2,1}, m_4, m_5\},$$

$$\{m_0, m_1, m_2, m_{1,1}, m_{2,1}, m_{1,1,1,1}, m_5\},$$

$$\{m_0, m_1, m_2, m_{1,1}, m_{1,1,1}, m_4, m_{3,1,1}\},$$

$$\{m_0, m_1, m_2, m_{1,1}, m_{1,1,1}, m_{1,1,1,1}, m_{3,1,1}\},$$

$$\{m_0, m_1, m_{1,1}, m_3, m_{2,1}, m_{2,1,1}, m_{1,1,1,1}\},$$

$$\{m_0, m_1, m_{1,1}, m_3, m_{1,1,1}, m_{2,1,1}, m_{1,1,1,1}\},$$

$$\{m_0, m_1, m_{1,1}, m_{2,1}, m_{2,1,1}, m_{1,1,1,1}, m_5\},$$

$$\{m_0, m_1, m_{1,1}, m_{1,1,1}, m_{2,1,1}, m_{1,1,1,1}, m_{3,1,1}\}.$$

Note that despite checking up to partitions of 10, no partitions of greater than 5 appear in these bases. Together with the fact that none of these sets of monomials in $\mathcal{H}_5$ provides a basis for $Z(\mathcal{H}_5)$, this suggests the conjectures stated below. The following table summarizes what is currently known regarding monomial bases for $Z(\mathbb{Z}S_n)$ and $Z(\mathcal{H}_n)$ when $n = 3, 4, 5$.

| Algebra | Number of Monomial Bases | Reference |
|---|---|---|
| $Z(\mathbb{Z}S_3)$ | 4 | [5] |
| $Z(\mathcal{H}_3)$ | 1 | [5] |
| $Z(\mathbb{Z}S_4)$ | 8 + two infinite families | Theorem 4.5 |
| $Z(\mathcal{H}_4)$ | 3 known | end of Section 4 |
| $Z(\mathbb{Z}S_5)$ | 12 known | see above |
| $Z(\mathcal{H}_5)$ | None known | checked the 12 known bases for $Z(\mathbb{Z}S_5)$ |

We end by conjecturing the following:

**Conjecture 6.1.**

(1) There are only 12 monomial bases for $Z(\mathbb{Z}S_5)$.

(2) When $n \geq 5$, there are only finitely many monomial bases for $Z(\mathbb{Z}S_n)$.

(3) There are only 3 monomial bases for $Z(\mathcal{H}_4)$.

(4) When $n \geq 5$, there are no monomial bases for $Z(\mathcal{H}_n)$.

## 7. Tables of Data

**Table 1.** Coefficients for $m_i$ in $\mathbb{Z}S_4$, $1 \leq i \leq 9$, obtained using GAP

| | $m_0$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ | $m_8$ | $m_9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 6 | 0 | 22 | 0 | 116 | 0 | 762 | 0 |
| $(ab)$ | 0 | 1 | 0 | 5 | 0 | 31 | 0 | 225 | 0 | 1811 |
| $(abc)$ | 0 | 0 | 1 | 0 | 8 | 0 | 66 | 0 | 568 | 0 |
| $(ab)(cd)$ | 0 | 0 | 0 | 0 | 4 | 0 | 50 | 0 | 504 | 0 |
| $(abcd)$ | 0 | 0 | 0 | 1 | 0 | 15 | 0 | 161 | 0 | 1555 |

**Table 2.** Coefficients for $m_{i,i}$ in $\mathbb{Z}S_4$, $1 \le i \le 9$, obtained using GAP

| | $m_{1,1}$ | $m_{2,2}$ | $m_{3,3}$ | $m_{4,4}$ | $m_{5,5}$ | $m_{6,6}$ | $m_{7,7}$ | $m_{8,8}$ | $m_{9,9}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 11 | 20 | 141 | 670 | 4051 | 23520 | 140921 | 841490 |
| $(ab)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(abc)$ | 1 | 4 | 21 | 116 | 671 | 3954 | 23521 | 140536 | 841491 |
| $(ab)(cd)$ | 1 | 2 | 21 | 108 | 671 | 3922 | 23521 | 140408 | 841491 |
| $(abcd)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 3.** Coefficients for $m_{i,i,i}$ in $\mathbb{Z}S_4$, $1 \le i \le 8$, obtained using GAP

| | $m_{1,1,1}$ | $m_{2,2,2}$ | $m_{3,3,3}$ | $m_{4,4,4}$ | $m_{5,5,5}$ | $m_{6,6,6}$ | $m_{7,7,7}$ | $m_{8,8,8}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 6 | 0 | 120 | 0 | 3936 | 0 | 140160 |
| $(ab)$ | 0 | 0 | 16 | 0 | 640 | 0 | 23296 | 0 |
| $(abc)$ | 0 | 3 | 0 | 108 | 0 | 3888 | 0 | 139968 |
| $(ab)(cd)$ | 0 | 2 | 0 | 104 | 0 | 3872 | 0 | 139904 |
| $(abcd)$ | 1 | 0 | 20 | 0 | 656 | 0 | 23360 | 0 |

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