Advances in Differential Equations and Control Processes
Volume 8, Number 1, 2011, Pages 23-32
Published Online: November 2011
Available online at http://pphmj.com/journals/adecp.htm
Published by Pushpa Publishing House, Allahabad, INDIA

INTEGRAL COMPARISON THEOREM FOR HALF-LINEAR THIRD-ORDER DIFFERENTIAL EQUATIONS

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Abstract

Integral comparison theorem for half-linear differential equations of the third-order is shown by means of Riccati change of variable and the technique of successive approximations. In the end, a conjecture concerning the generalized Euler differential equation is provided.

1. Introduction

Birkhoff in [1] pioneered the study of separation and comparison theorems for equations of third-order. However, papers directly connected with comparison and oscillation theorems appeared later, e.g., [7]. It is largely due to Hanan [5], whose contribution was enormous, especially in connection to the theory of conjugate points. Reader can

2010 Mathematics Subject Classification: 34C10.
Keywords and phrases: comparison theorem, half-linear, third-order differential equations, oscillatory solution.
Received July 5, 2011

find these and other results on separation and comparison theorems for linear differential equations of the third-order summarized in [6 Chapters 4.2. and 4.3. respectively]. There are several tests for oscillation and disconjugacy in the linear case. Many of them are using the Euler equation in connection with known comparison theorems. As it will be clear later, some analogous methods from the theory of the linear third-order differential equations

$$y''' + p_1(t)\, y' + p_0(t)\, y = 0, \tag{1}$$

will be presented. It is most general in the sense that equation $x''' + s_2(t)\, x'' + s_1(t)\, x' + s_0(t)\, x = 0$ can be always transformed into (1) by

$$y = x\, \mathrm{e}^{-\frac{1}{3}\int_{t_0}^{t} s_2(u)\, \mathrm{d}u}$$

(under the assumption that s_1' and s_0'' are continuous). One can convince oneself that $p_1 = s_1 - \frac{s_2^2}{3} - s_2'$ and $p_0 = s_0 - \frac{s_2\, s_1}{3} + \frac{2\, s_2^3}{27} - \frac{s_2''}{3}$ (see [4, Section 1]). The Riccati equation corresponding to (1) is

$$w'' + 3\, w\, w' + w^3 + p_1(t)\, w + p_0(t) = 0, \tag{2}$$

where standard substitution $w = \frac{y'}{y}$ was made. In [5] is shown the following: if equation (1) has $p_1 \equiv 0$ and p_0 of constant sign, then it is oscillatory if

$$\lim_{t\to\infty} \inf t^3 |p_0| > \frac{2}{3\sqrt{3}}$$

and nonoscillatory if

$$\lim_{t\to\infty} \sup t^3 |p_0| < \frac{2}{3\sqrt{3}}.$$

Note that these conditions are sharp and cannot be weakened. The following result from [3] generalizes previous assertion. If $p_1 \equiv 0$ and p_0 has a constant sign, then reduced equation (1) is disconjugate on $[a, \infty)$

if

$$\left|\int_a^t (t-s)\,s^3\,p_0(s)\,\mathrm{d}s\right| \le \frac{(t-a)^2}{3\sqrt{3}}, \quad t > a.$$

The following ideas are largely helpful. The Euler differential equation

$$y''' + \frac{\tilde{\gamma}}{t^3}\,y = 0, \tag{3}$$

where $\tilde{\gamma} \in \mathbb{R}$, possesses a solution in the form t^λ, exponents of which are determined from the corresponding algebraic (indical) equation. Therefore there are critical constants, which determine equation (3) to be conditionally oscillatory equation. Being more specific, equation (3) is oscillatory iff $\tilde{\gamma} \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$.

2. Integral Comparison Theorem

We are concerned here with the third-order half-linear differential equation

$$\left(\phi_{\alpha_1}[y']\right)'' + q(t)\,\phi_{\alpha_1}[y] = 0, \tag{E}$$

where

$$\phi_{\alpha_1}[x] := |x|^{\alpha_1 - 1}\,x = |x|^{\alpha_1}\,\mathrm{sgn}\,x,\ \ \alpha_1 > 0,$$

known as "signed power function". Furthermore, we suppose that $q \in C(I),\ I = [a,b)\,,\ a < b \le \infty$ is nonnegative on I and $q \not\equiv 0$ on any subinterval of I. Analysis is based on conversion of (E) into an integral equation (8) with the help of Riccati substitution and Euler-type differential equation. This approach is inspired by paper [3]. The second-order generalized Riccati equation corresponding to (E) is

$$z'' + (2\alpha_1 + 1)\,|z|^{\frac{1}{\alpha_1} - 1} z\,z' + \alpha_1^2\,|z|^{\frac{2}{\alpha_1}}\,z + q(t) = 0, \tag{5}$$

where transformation $z = \phi_{\alpha_1}\left[\frac{y'}{y}\right]$ is used. Now, we take into account the generalized Euler differential equation

$$(\phi_{\alpha_1}[y'])'' + \frac{\gamma}{t^{\alpha_1+2}}\,\phi_{\alpha_1}[y] = 0. \tag{6}$$

It is true that for specific γ there always exist, solution t^{λ}, where $\lambda \in (1,2)$. Moreover, if $z(t) = \phi_{\alpha_1}\left[\frac{\lambda}{t}\right]$, then for arbitrary α_1 (with corresponding λ), it is true that $\lambda^{\alpha_1} \in (1,2)$ and therefore we choose $w = 2 - t^{\alpha_1}\, z$ to ensure that $w \in (0,1)$. It changes (5) into

$$\begin{aligned} t^2\, w'' + (2\,\alpha_1 + 1)\, t\, |2-w|^{\frac{1}{\alpha_1}-1}(2-w)\, w' - 2\,\alpha_1\, t\, w' = \alpha_1^2\, |2-w|^{\frac{2}{\alpha_1}} \\ (2-w) + (\alpha_1^2 + \alpha_1)(2-w) - \alpha_1\,(2\,\alpha_1+1)\,|2-w|^{\frac{1}{\alpha_1}+1} + t^{\alpha_1+2}\, q(t). \end{aligned} \tag{7}$$

Let $\omega = \frac{\alpha_1(2\,\alpha_1+1)}{\alpha_1+1}$, $\sigma = \omega + \alpha_1(2\alpha_1+1)$, $C = -\sigma\, 2^{\frac{1}{\alpha_1}+1} + \alpha_1^2\, 2^{\frac{2}{\alpha_1}+1}$ $+2\alpha_1(\alpha_1+1)$ and $D = \omega\, 2^{\frac{1}{\alpha_1}+1}$ be a constant. Now, we use integration by parts twice to obtain the integral equation

$$\begin{aligned} t^2\, w = a^2\, w(a) + g(a)(t-a) + \frac{C}{2}(t-a)^2 + \frac{D}{2}\left(t^2 - a^2\right) \\ + \int_a^t [(t-s)\, H(w) + G(w,s)]\,\mathrm{d}s + Q(t), \end{aligned} \tag{8}$$

where

$$g(a) = a^2\, w'(a) - 2\,(\alpha_1+1)\, a\, w(a) - \omega\, a\, |2 - w(a)|^{\frac{1}{\alpha_1}+1}, \tag{9}$$

$$\begin{aligned} H(w) = -\sigma\, |2-w|^{\frac{1}{\alpha_1}+1} + \alpha_1^2\, |2-w|^{\frac{2}{\alpha_1}}(2-w) - (\alpha_1+1)(\alpha_1+2)\, w \\ + 2\alpha_1(\alpha_1+1) - C, \end{aligned} \tag{10}$$

$$G(w,s) = (4 + 2\,\alpha_1)\, s\, w + \omega\, s\, |2-w|^{\frac{1}{\alpha_1}+1} - Ds, \tag{11}$$

$$Q(t) = \int_a^t (t-s)\, s^{\alpha_1+2}\, q(s)\,\mathrm{d}s. \tag{12}$$

Hence, $w(t)$ solves (7) iff w solves (8). Further, we use the technique of successive approximations to obtain a solution. Define the sequence

$w_n = w_n(t), \quad n = 0, 1, 2, \dots$ by

$$t^2 w_0(t) = a^2 w(a) + g(a)(t-a) + \frac{C}{2}(t-a)^2 + \frac{D}{2}\left(t^2 - a^2\right) + Q(t),$$
$$t^2 w_k(t) = t^2 w_0(t) + \int_a^t (t-s)\, H(w_{k-1})\,\mathrm{d}s + \int_a^t G(w_{k-1}, s)\,\mathrm{d}s, \quad k \geq 1. \tag{13}$$

We need to know some properties of the integrands in (13) and one should know that this "decomposition" is unique in the sense that in order to preserve properties (in the next lemma) of G and H, one must "choose" it in this form.

Lemma 2.1. *Assume that $\alpha_1 \geq 1$. Function $G(w,s)$ is non-negative in $[0,1] \times [a,b)$ and function $H(w)$ is non-negative and increasing in $[0,1]$. Further, $G(w,s)$ is increasing function of variable w.*

Proof. First we show the monotonicity of H on $[0,1]$. One can show that $H'(0) \geq 0$ and $H'(1) = 0$ for $\alpha_1 \in [1, \infty)$ (actually $H'(0) \leq 0.3$ holds for such α_1). But $H''(w)$ has exactly one root in the interval $[0,1]$, namely $w^* = 2 - \left(\frac{2\alpha_1+1}{2\alpha_1}\right)^{\alpha_1}$, what implies $H'(w) \geq 0, \ w \in [0,1]$ for all α_1. Now we look at the positivity of H. Obviously, we have $H(0) = 0$. Now, $H(1) = \frac{1}{2}$ for $\alpha_1 = 1$ and $\lim\limits_{\alpha_1 \to \infty} H(1) = 6 \ln 2 - 3 - 2 \ln 2^2 > 0$. Moreover, $H'(1)$ has no root for $\alpha_1 \geq 1$ ($H(1)$ is decreasing in α_1) and thus $0 < H(1) \leq \frac{1}{2}$ for all $\alpha_1 \geq 1$. But from the fact that H is increasing on $[0,1]$ the result is obvious.

Monotonicity of $G(w,s)$ follows directly from the inequality

$$\frac{\partial G(w,s)}{\partial w} = s\left(2(2+\alpha_1) - (2\alpha_1+1)(2-w)^{\frac{1}{\alpha_1}}\right) \geq 0$$

for $(w,s) \in [0,1] \times [a,b), \ \alpha_1 \geq 1$. Further, we have $G(0,s) = 0$ and $G(1,s) \geq \frac{3}{2}s$ for all $s \in [a,b)$. The result thus follows from the monotonicity of G in w. This completes the proof. $\square$

Remark 2.1. Notice that for $\alpha_1 < 1$, $H(w)$ is no more increasing on $[0,1]$. Moreover, $G(w,s)$ is not increasing for all $w \in [0,1]$ and is not positive for all $s \in [a,t]$. For $\alpha_1 = 1$, we have, in coincidence with [3], $H(w) = \frac{3}{2}\, w^2 - w^3$ and $G(w,s) = \frac{3}{2}\, s\, w^2$. Therefore, from this time forth, we assume that $\alpha_1 \geq 1$.

Lemma 2.2. *Let $w(t)$ be a solution of (8) s.t. $0 < w < 1$ on $[a,b),\ a < b \leq \infty$ and suppose that*

$$g(a)(t-a) + \frac{C}{2}(t-a)^2 + \frac{D}{2}\left(t^2 - a^2\right) + Q(t) \geq 0, \quad t \geq a.$$

Then the sequence $\{w_k\}_{k=0}^{\infty}$ converges uniformly to w on each compact subset of $[a,b)$.

Proof. From the assumptions of lemma, it is clear that $t^2\, w_0(t) > 0,\ t \geq a$. Thus,

$$t^2(w_1(t) - w_0(t)) = \int_a^t (t-s)\, H(w_0)\, \mathrm{d}s + \int_a^t G(w_0, s)\, \mathrm{d}s > 0$$

holds. As well, we have for $k > 1$,

$$\begin{aligned} t^2(w_k(t) - w_{k-1}(t)) \quad &= \quad \int_a^t (t-s)\, [H(w_{k-1}) - H(w_{k-2})]\, \mathrm{d}s \\ &\quad\ + \int_a^t [G(w_{k-1}, s) - G(w_{k-2}, s)]\, \mathrm{d}s, \end{aligned}$$

which is positive for $t > a$, by induction, provided $w_k(t) < w(t),\ t > a$. But $t^2\,(w(t) - w_0(t)) > 0$ and $t^2\,(w(t) - w_k(t)) > 0$, again by induction and assumptions of lemma. It follows from the Monotone Convergence Theorem that

$$\lim_{k\to\infty} w_k(t) = w^*(t) \leq w(t), \quad \text{for each} \quad t,$$

and w^* solves (8) with the same initial conditions as w. Thus $w^* = w$ and by the Dini's theorem, the convergence is uniform on compact subset of $[a,b)$. $\square$

Now consider related equation to (E)

$$\left(\phi_{\alpha_1}[y']\right)'' + q_1(t)\,\phi_{\alpha_1}[y] = 0, \tag{E_1}$$

$q_1 \in C(I)$. The generalized Riccati equation corresponding to (E_1) is

$$z_1'' + (2\alpha_1 + 1)\,|z_1|^{\frac{1}{\alpha_1}-1} z_1\, z_1' + \alpha_1^2\,|z_1|^{\frac{2}{\alpha_1}} z_1 + q_1(t) = 0, \tag{15}$$

and the corresponding integral equation after transformation $v = 2 - t^{\alpha_1} z_1$ is

$$\begin{aligned} t^2\, v &= a^2\, v(a) + g_1(a)(t-a) + \frac{C}{2}(t-a)^2 + \frac{D}{2}\left(t^2 - a^2\right) \\ &+ \int_a^t \left[(t-s)\, H_1(w) + G_1(w,s)\right] \mathrm{d}s + Q_1(t), \end{aligned} \tag{16}$$

where

$$\begin{aligned} g_1(a) &= a^2\, v'(a) - 2\,(\alpha_1 + 1)\, a\, v(a) - \omega\, a\, |2 - v(a)|^{\frac{1}{\alpha_1}+1} && (17) \\ H_1(v) &= -\sigma\, |2 - v|^{\frac{1}{\alpha_1}+1} + \alpha_1^2\, |2 - v|^{\frac{2}{\alpha_1}}\, (2 - v) && \\ &\quad -(\alpha_1 + 1)(\alpha_1 + 2)\, v + 2\alpha_1(\alpha_1 + 1) - C, && (18) \\ G_1(v,s) &= (4 + 2\,\alpha_1)\, s\, v + \omega\, s\, |2 - v|^{\frac{1}{\alpha_1}+1} - Ds, && (19) \\ Q_1(t) &= \int_a^t (t-s)\, s^{\alpha_1+2}\, q_1(s)\, \mathrm{d}s. && (20) \end{aligned}$$

At first, notice that H_1 coincides with H, as well G_1 with G. It is clear that if v solves (16) on $[a,b)$, then $z_1(t) = \dfrac{2 - v(t)}{t_1^{\alpha}}$ solves (15) and moreover $y(t) = \mathrm{e}^{\int_a^t \phi_{\alpha_1}^{-1}[z_1(s)]\,\mathrm{d}s}$ is a positive solution of (E_1). Let w be a solution of (8) with $0 < w < 1$ on I and the sequence $\{v_k\}_{k=0}^{\infty}$ be defined for $a \le t < b$ by

$$\begin{aligned} t^2\, v_0(t) &= a^2\, w(a) + g(a)(t-a) + \frac{C}{2}(t-a)^2 + \frac{D}{2}\left(t^2 - a^2\right) + Q_1(t), \\ t^2\, v_k(t) &= t^2\, v_0(t) + \int_a^t (t-s)\, H_1(v_{k-1})\, \mathrm{d}s + \int_a^t G_1(v_{k-1}, s)\, \mathrm{d}s, \quad k \ge 1. \end{aligned} \tag{21}$$

Notice, that the first element of defined sequence contains terms of w_0, which is the first element of the sequence (13) defined above.

Proposition 2.3. *Let* $w(t)$ *be solution of* (8) *such that* $0 < w(t) < 1$ *on* I. *Assume*

$$g(a)(t-a) + \frac{C}{2}(t-a)^2 + \frac{D}{2}\left(t^2 - a^2\right) + Q_1(t) \geq 0, \quad t \geq a.$$

Further assume

$$Q_1(t) \leq Q(t), \quad t \geq a. \tag{22}$$

Then the sequence $\{v_k\}_{k=0}^{\infty}$ *converges, uniformly on compact substets of* I*, to a solution* v *of* (16) *with* $0 < v \leq w$ *on* I.

Proof. Again it is clear that $t^2 v_0(t) > 0,\ t \geq a$. It can be shown similarly as in lemma 2.2 that $v_k(t) \geq v_{k-1}(t),\ k = 1, 2, \dots$. Further, if we define w_k as in (13), then $t^2(w_0(t) - v_0(t)) \geq 0$ by condition (22). For $k > 1$ we obtain

$$\begin{aligned} t^2(w_k(t) - v_k(t)) \quad &= \quad \int_a^t (t-s)(H(w_k) - H(v_k))\,\mathrm{d}s \\ &\quad + \int_a^t (G(w_k, s) - G(v_k, s))\,\mathrm{d}s \\ &\quad + Q(t) - Q_1(t) \geq 0 \end{aligned}$$

by assumptions and induction. Thus,

$$\lim_{k \to \infty} v_k(t) = v(t) \leq w(t),$$

uniformly on compact subsets $[a, b)$ and $v(t)$ is a solution of (16).

From the remark 2.1 is clear that the technique used here cannot be used for $\alpha_1 < 1$. Now we present a conjecture, which is true if so is the following conjecture. Equation (6) is nonoscillatory iff $\gamma \in [\gamma_-, \gamma_+]$.

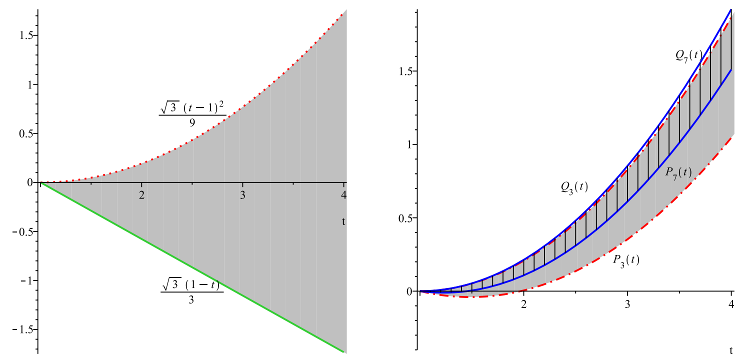

Figure 1.

Unfortunately, it cannot be proved as in the case of the second-order half-linear equations, where the implicit (exact) type of solutions are used, see [2]. This seems to be a challenge. We emphasize here that the assertion plays an important role in the oscillation theory and one should target the approval of it.

Conjecture 2.4. *Let constants* λ_+, γ_+ *be defined as*

$$\lambda_+ = \frac{(\alpha_1+1)(2\,\alpha_1+1)+\sqrt{(\alpha_1+1)(5\,\alpha_1+1)}}{2\,\alpha_1\,(\alpha+2)},$$

and

$$\gamma_+ = -\alpha_1\lambda_+^{\alpha_1}(\lambda_+-1)(\alpha_1\lambda_+-\alpha_1-1),$$

respectively. Then

$$\frac{\gamma_+}{2}(t-a)^2 \geq Q_1(t) \geq \Big(2\,a\,(\alpha_1+1)(2-\lambda_+^{\alpha_1})+\omega\,a\,\lambda_+^{\alpha_1+1}\Big)$$

$$(t-a)-\frac{C}{2}(t-a)^2-\frac{D}{2}\left(t^2-a^2\right)$$

implies disconjugacy of (E_1) *on* $[a,\infty)$.

Conjecture 2.4 says that sufficient condition for equation to be disconjugate on $[a,\infty)$ is falling $q(t)$ into the area between two parabolic curves, see picture 2. We notice the interesting fact that with the increasing α_1 a parabolic curves of the region achieve the bound.

Remark 2.2. Limiting cases:

$\alpha_1 \to \infty$

$$\mathcal{R}_h = \left\{ q_1 : (\tilde{c}_1\, t + a\, \tilde{c}_2)(t-a) \le Q_1(t) \le \frac{\left(\sqrt{5}-2\right) \mathrm{e}^{\frac{\sqrt{5}-1}{2}}}{2}\,(t-a)^2 \right\}$$

where $\tilde{c}_1 = \ln 2\,(1-\ln 2)$, $\tilde{c}_2 = \ln 2\,(\ln 2 - 5) + 6 + \left(\sqrt{5}-4\right)\, \mathrm{e}^{\frac{\sqrt{5}-1}{2}}$

$\alpha_1 \to 1$

$$\mathcal{R}_l = \left\{ q_1 : -\frac{a\,(t-a)}{\sqrt{3}} \le Q_1(t) \le \frac{(t-a)^2}{3\sqrt{3}} \right\}$$

The question of $\alpha_1 < 1$ is left as an open question and it seems to be a big challenge.

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