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NEW CHARACTERIZATIONS OF QUASI-DISCRETE (QUASI-CONTINUOUS) MODULES

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Abstract

Let R be a ring with identity. Then a right R-module M is called a Q-module if for any $A \leq_{cc} M$ and $B \leq_{cc} M$, $A + B = M$ yields $A \bigcap B \leq_{cc} M$. It is shown that a module M is quasi-discrete if and only if M is an amply supplemented Q-module. Dually, a right R-module M is called a P-module if for any $A \leq_c M$ and $B \leq_c M$, $A \bigcap B = 0$ yields $A \oplus B \leq_c M$. We prove that a module M is quasi-continuous if and only if M is a P-module.

1. Introduction

Quasi-discrete (quasi-continuous) modules play important roles in rings and

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categories of modules and they have been extensively studied by many authors (see [2-4, 6, 9, 11]). Let M be a module. Consider the following conditions:

(D_1) For every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \ll M$;

(D_2) If $A \leq M$ such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M;

(D_3) If A and B are direct summands of M such that $A + B = M$, then $A \cap B$ is a direct summand of M.

The module M is called *lifting* (see [2]) if it satisfies (D_1); M is called *discrete* if it satisfies (D_1) and (D_2); M is called *quasi-discrete* if it satisfies (D_1) and (D_3). According to [9], a module M is called a *T-module* if for $A \leq_{cc} M$ and $B \leq M$, $M/A \cong M/B$ implies $B \leq_{cc} M$. Keskin in [9] gave a new characterization of discrete modules by showing that a module M is discrete if and only if M is an amply supplemented T-module satisfying (D_{11}).

Note that the notion of T-modules is derived from the condition (D_2) by replacing "direct summand" by "coclosed submodule". Inspired by this, we introduce the concept of Q-modules on the basis of the condition (D_3) and use it to characterize quasi-discrete modules in Section 2. We call a right R-module M a *Q-module* if for any $A \leq_{cc} M$ and $B \leq_{cc} M$, $A + B = M$ yields $A \cap B \leq_{cc} M$. It is shown that a module M is quasi-discrete if and only if M is an amply supplemented Q-module.

Let M be a module. Dually, consider the following conditions:

(C_1) Every submodule of M is essential in a direct summand of M;

(C_2) If a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M;

(C_3) If A and B are direct summands of M such that $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

The module M is called *extending* [3] if it satisfies (C_1); M is called continuous if it satisfies (C_1) and (C_2); M is called *quasi-continuous* if it satisfies (C_1) and (C_3). Er in [4] gave the notion of SICC-modules and used it to characterize continuous modules. A module M is SICC if any submodule N which is isomorphic to a closed submodule of M is closed. It was shown in [5] that a module M is continuous if and only if M is an SICC-module satisfying (C_{11}). Inspired by the notion of SICC-modules, we introduce the concept of P-modules through the condition (C_3) and try to give a new characterization of quasi-continuous modules. We call a right R-module M a *P-module* if for any $A \leq_c M$ and $B \leq_c M$, $A \cap B = 0$ yields $A \oplus B \leq_c M$. It is shown that a module M is quasi-continuous if and only if M is a P-module in Section 3.

Throughout this paper, all rings are associative with unity and all modules will be unital right R-modules. Let M be a module and $S \leq M$. S is called *small* in M (denoted by $S \ll M$) if for any $T \leq M$, $S + T = M$ implies $T = M$. Dually, S is called *essential* in M (denoted by $S \leq_e M$) if for any $T \leq M$, $S \cap T = 0$ implies $T = 0$. For $N, L \leq M$, N is a supplement of L in M if N is a minimal element in the set of submodules $K \leq M$ with $K + L = M$. Equivalently, $N + L = M$ with $N \cap L \ll N$. Dually, N is a complement of L in M if it is maximal in the set of submodules $K \leq M$ with $K \cap L = 0$. M is called *supplemented* if every submodule of M has a supplement in M. M is *amply supplemented* if, for any submodules A, B of M with $M = A + B$ there exists a supplement P of A in M such that $P \leq B$. Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *coessential submodule* of A in M. A submodule A of M is called *coclosed* (denoted by $A \leq_{cc} M$) if A has no proper coessential submodule. Dually, a submodule A of M is called *closed* (denoted by $A \leq_c M$) if A has no proper essential extension in M. Also, we will call B an *coclosure* (or an *s-closure*) of A in M, if B is a coessential submodule of A and B is coclosed in M. All undefined concepts can be found in [2, 3, 14].

2. Q-modules

In this section, the notion of Q-modules is introduced and a new characterization of quasi-discrete modules is given. It is shown that a module M is quasi-discrete if and only if M is an amply supplemented Q-module. We start with the following.

Definition 2.1. A module M is called a *Q-module* if for any $A \leq_{cc} M$ and $B \leq_{cc} M$, $A + B = M$ implies $A \cap B \leq_{cc} M$.

Recall that a module M is said to be *weakly supplemented* if for each submodule A of M there is a submodule B of M such that $M = A + B$ and $A \bigcap B \ll M$.

Lemma 2.2 ([8, Lemma 1.1]). *Let M be a module and $N \leq M$. Consider the following conditions*:

(1) *N is a supplement submodule of M*;

(2) *N is coclosed in M*;

(3) *For all $X \leq N$, $X \ll M$ implies $X \ll N$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold*. *If M is a weakly supplemented module*, *then* (3) $\Rightarrow$ (1) *holds*.

Proposition 2.3. *Any direct summand of a weakly supplemented Q-module is a Q-module*.

Proof. Let N be a direct summand of a Q-module M and $A \leq_{cc} N$, $B \leq_{cc} N$ with $A + B = N$. Since N is coclosed in M, A and B are coclosed in M. Let $M = N \oplus N'$. Then $M = A + B \oplus N'$. It is easy to verify that $B \oplus N' \leq_{cc} M$ by Lemma 2.2, and hence $A \bigcap (B \oplus N') \leq_{cc} M$ by assumption. Since $A \bigcap (B \oplus N') = A \bigcap B$, $A \bigcap B \leq_{cc} M$. Thus $A \bigcap B \leq_{cc} N$, as desired. □

Theorem 2.4. *The following statements are equivalent for an amply supplemented module M.*

(1) *M is a Q-module*;

(2) *If $A \leq_{cc} M$, $B \leq_{cc} M$, $M = A + B$ holds and $A \bigcap B \ll M$, then $M = A \oplus B$.*

Proof. "(1) $\Rightarrow$ (2)" Let $A \leq_{cc} M$, $B \leq_{cc} M$ with $M = A + B$ and $A \bigcap B \ll M$. Since M is a Q-module, $A \bigcap B \leq_{cc} M$ and hence $A \bigcap B = 0$. Therefore, $M = A \oplus B$.

"(2) $\Rightarrow$ (1)" Let $A \leq_{cc} M$, $B \leq_{cc} M$ with $M = A + B$. Next we shall show that $A \bigcap B \leq_{cc} M$. Since M is an amply supplemented module, there is a supplement L of B such that $L \leq A$. Thus $M = L + B$ and $L \bigcap B \ll L$. Since L and B are coclosed in M, $M = L \oplus B$ by assumption, and so $A = A \bigcap M = (A \bigcap B) \oplus L$. Note that $A \leq_{cc} M$, $A \bigcap B \leq_{cc} A$, so $A \bigcap B \leq_{cc} M$, as required. □

Theorem 2.5. *A module M is quasi-discrete if and only if M is an amply supplemented Q-module.*

Proof. It follows by Theorem 2.4 and [12, Proposition 4.11]. □

Lemma 2.6 ([8, Lemma 2.5])**.** *Let* $M = M_1 \oplus M_2$. *Then* M_1 *is* M_2*-projective if and only if for any* $A \leq M$ *with* $A + M_2 = M$, *there exists some* $L \leq M$ *with* $L \leq A$ *and* $M_2 \oplus L = M$.

Proposition 2.7. *If* $M = A \oplus B$ *is an amply supplemented Q-module, then A and B are relatively projective.*

Proof. We shall show that *A* is *B*-projective, the fact that *B* is *A*-projective follows by symmetry. Let $X \leq M$ and $M = X + B.$ Since *M* is amply supplemented, there is a submodule *K* of *M* such that $M = K + B$ and $K \cap B \ll K \leq X$. Since *K* and *B* are coclosed in *M*, $K \cap B$ is coclosed in *M* by assumption. Therefore, $K \cap B = 0,$ and hence *A* is *B*-projective by Lemma 2.6. □

Corollary 2.8 ([12, Lemma 4.23])**.** *If* $M = A \oplus B$ *is a quasi-discrete module, then A and B are relatively projective.*

Corollary 2.9. *Let* $M = M_1 \oplus \cdots \oplus M_n$ *be a Q-module. Then M is lifting if and only if M is amply supplemented and* M_i *is lifting.*

Proof. It follows by Proposition 2.7 and [8, Corollary 2.9]. □

A quasi-discrete module is a *Q*-module, but the converse is not true. The following examples show that a *Q*-module need not be a quasi-discrete module.

Example 2.10. It is well known that a ring *R* is a right *V*-ring if and only if for any right *R*-module *M*, any submodule is coclosed in *M*. Thus any right *R*-module *M* over a right *V*-ring is a *Q*-module. However, any right *R*-module *M* over a right *V*-ring need not be a quasi-discrete module.

Example 2.11. Let *R* be a discrete valuation ring with field of fractions *K*. Assume K_R is not quasi-projective. By Keskin [8, Example 3.7], the *R*-module $M = K \oplus K$ is not amply supplemented. Hence it is not quasi-discrete. But every coclosed submodule is a direct summand of *M*. By Smith and Tercan [12, Lemma 3.6], *M* satisfies (D_2) and so *M* satisfies (D_3). Thus *M* is a *Q*-module.

3. *P*-modules

We define *P*-modules and use them to characterize quasi-continuous modules in this section. It is proved that a module *M* is quasi-continuous if and only if *M* is a *P*-module.

Definition 3.1. *A* module *M* is called a *P-module* if for any $A \leq_c M$ and $B \leq_c M$, $A \bigcap B = 0$ implies $A \oplus B \leq_c M$.

Proposition 3.2. *Any direct summand of a P-modules is a P-module*.

Proof. Let *N* be a direct summand of a *P*-module *M* and $A \leq_c N$, $B \leq_c N$ with $A \bigcap B = 0.$ Since *N* is closed in M, A and *B* are closed in *M*. Note that $A \bigcap B = 0$ and *M* is a *P*-module, $A \oplus B \leq_c M$ and hence $A \oplus B \leq_c N$, as required. □

Proposition 3.3. *Any P-module satisfies* (C_3).

Proof. Let *A* and *B* be direct summands of *M* with $A \bigcap B = 0.$ Then there is a complement *L* of *A* in *M* such that $B \leq L.$ Since *A* and *L* are closed in *M* with $A \bigcap L = 0,$ $A \oplus L$ is closed in *M* by assumption. Note that $A \oplus L \leq_e M$, so $A \oplus L = M$. Since *B* is a direct summand of *M*, it is a direct summand of *L*, and hence $A \oplus B$ is a direct summand of *M*. This completes the proof. □

Lemma 3.4 ([8, Lemma 7.5]). *Let* $M = M_1 \oplus M_2$. *Then* M_1 *is* M_2*-injective if and only if for any* $A \leq M$ *with* $A \bigcap M_1 = 0,$ *there exists some* $L \leq M$ *with* $A \leq L$ *and* $M_1 \oplus L = M$.

Proposition 3.5. *If M is a P-module and* $M = A \oplus B,$ *then A and B are relatively injective*.

Proof. We shall show that *A* is *B*-injective. (*B* is *A*-injective follows by symmetry.) Let $X \leq M$ and $X \bigcap A = 0.$ Then there is a closed submodule *L* of *M* such that $X \leq L,$ $A \bigcap L = 0$ and $A \oplus L \leq_e M$. Since $A \leq_c M$, $L \leq_c M$ and $A \bigcap L = 0,$ $A \oplus L \leq_c M$ by assumption. Note that $A \oplus L \leq_e M$, so $A \oplus L = M$. Thus *A* is *B*-injective by Lemma 3.4. □

It is known that a ring *R* is a *QF* ring if and only if the right *R*-module $R^{(\mathbb{N})}$ is injective. Now we have:

Corollary 3.6. *A ring R is a QF ring if and only if the right R-module* $R^{(\mathbb{N})}$ *is a P-module*.

Also, a well known result asserts that a ring R is semisimple Artinian if and only if every cyclic R-module is injective.

Corollary 3.7. *A ring R is semisimple Artinian if and only if every* 2-*generated R-module is a P-module*.

We know that a ring R is right self-injective if and only if $R \oplus R$ is quasi-continuous as a right R-module. Hence we have the following.

Corollary 3.8. *A ring R is right self-injective if and only if* $R \oplus R$ *is a P-module as a right R-module*.

Corollary 3.9 ([12, Proposition 2.10])**.** *If M is a quasi-continuous module and* $M = A \oplus B$, *then A and B are relatively injective*.

Proof. It is clear that a quasi-continuous module is a P-module. □

Corollary 3.10. *Let* $M = M_1 \oplus \cdots \oplus M_n$ *be a P-module*. *Then M is extending if and only if all* M_i *are extending*.

Proof. It follows by Proposition 3.5 and [3, Proposition 7.10]. □

Theorem 3.11. *A module M is quasi-continuous if and only if M is a P-module*.

Proof. "$\Rightarrow$" is clear.

"$\Leftarrow$" It suffices to proof that M satisfies the condition (C_1) by Proposition 3.3. Let N be any closed submodule of M. By Zorn Lemma, there is a closed submodule N' of M such that $N \cap N' = 0$ and $N \oplus N' \leq_e M$. Since M is a P-module, $N \oplus N' \leq_c M$, and hence $N \oplus N' = M$. That is to say that N is a direct summand of M. Thus M satisfies the condition (C_1). □

We end this section with the following. We know that the condition (C_2) implies the condition (C_3). It is natural to ask whether a SICC-module is a P-module. The following example shows that a SICC-module need not be a P-module.

Example 3.12. Let K be a field and $V = K \times K$. Consider the ring R of 2×2 matrix of the form (a_{ij}) with $a_{11}, a_{22} \in K$, $a_{12} \in V$, $a_{21} = 0$ and $a_{11} = a_{22}$. Now the only right ideals of R are 0, R_R, I_1, I_2, I_3, $I(x, y)$ for any nonzero x and y in

K, where I_1 is the set of (a_{ij}) with a_{11}, a_{22}, a_{21} all zero, and $a_{12} \in K \times 0$; I_2 is the set of (a_{ij}) with a_{11}, a_{22}, a_{21} all zero, and $a_{12} \in 0 \times K$; I_3 is the set of (a_{ij}) with a_{11}, a_{22}, a_{21} all zero, and $a_{12} \in V$; $I(x, y)$ is the set of (a_{ij}) with a_{11}, a_{22}, a_{21} all zero, and $a_{12} \in (x, y)K$. Now all the right ideals except I_3 are closed in R_R, and I_3 is not closed since it is essential in R_R. R_R is a SICC-module. However, it is easy to verify that R_R is not a P-module.

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