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EIGENVALUE DISTRIBUTION OF INTEGRAL OPERATORS WITH WEAKLY SINGULAR KERNELS

HONG-LIANG ZHAO and DIAN-CHUAN JIN

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Abstract

Let $\Omega \subset R^N$ be a bounded domain. Let T_K denote the weakly singular integral operator of the form $T_K f(x) = \int_\Omega K(x, y) f(y) dy$ generated by the kernel $K(x, y) = l(x, y)/|x - y|^{N-\alpha}$. In case $N/2 < \alpha < N$, we prove that if $|l(x, y)| \leq g(y)$ and $g \in L_{N/\alpha, 1}(\Omega)$, then $(\lambda_n(T_K)) \in l_2$, which improves the corresponding results in [J. Funct. Anal. 37 (1980), 88-126], [Math. Ann. 268 (1984), 127-136].

1. Introduction

Let $\Omega \subset R^N$ be a domain, λ denote the Lebesgue measure on R^N and $K : \Omega^2 \to R$ be a measurable function given by

$$K(x, y) = \frac{l(x, y)}{|x - y|^{N-\alpha}}, \quad 0 < \alpha < N. \tag{1}$$

The so-called weakly singular integral operator T_K is the integral operator, with the weakly singular kernel $K(x, y)$, of the form

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$$T_K f(x) = \int_\Omega K(x, y) f(y) dy, \tag{2}$$

where f belongs to some suitable complex-valued function space. It is well known that a number of integral equations, derived from differential equations with boundary values, possess weakly singular kernels.

Let $\Omega \subset R^N$ be a bounded domain. In case, where $N/2 < \alpha < N$, König et al. in [6] proved that if $|l(x, y)| \leq g(y)$ and $g \in L_2(\Omega, \lambda)$, then the eigenvalues $\lambda_n(T_K)$ of T_K satisfy $(\lambda_n(T_K)) \in l_2$. Using the theory of entropy number, Carl and Kühn improved the result of [6]. They obtained in [1] that if $g \in L_q(\Omega)$ and $N/\alpha < q < +\infty$, then $(\lambda_n(T_K)) \in l_2$.

In this paper, we will prove, by means of the property of Lorentz function space and the theory of p-summing operators, that the condition $g \in L_{N/\alpha,1}(\Omega)$ alone is sufficient for $(\lambda_n(T_K)) \in l_2$. Background material can be found in [2-4].

2. A Property of Lorentz Function Space

Let p' denote the conjugate index defined by $1/p + 1/p' = 1$. Let (Ω, Σ, μ) be a measure space and $f : \Omega \to R$ be a measurable function. Putting $d_f(s) = \mu\{\omega \in \Omega \mid |f(\omega)| > s\}$, we recall from [5] that the decreasing rearrangement of f, $f^* : [0, \mu(\Omega)) \to [0, \infty)$ is given by $f^*(t) = \inf\{s > 0 \mid d_f(s) \leq t\}$. If $\mu(\Omega) < \infty$, then extend f^* to $[\mu(\Omega), \infty)$ by 0. For $0 < p < \infty$ and $0 < q \leq \infty$, the Lorentz function space $L_{p,q}(\Omega, \mu)$ consists of all measurable functions $f : \Omega \to R$ such that

$$\| f \|_{(p,q)} = \begin{cases} \left(\int_0^{+\infty} \frac{(f^*(t) t^{1/p})^q}{t} dt \right)^{1/q} < \infty, & q < \infty, \\ \sup\limits_{t \geq 0} t^{1/p} f^*(t) < \infty, & q = \infty. \end{cases} \tag{3}$$

Lemma 1. *Let* $f, g : \Omega \to R$ *be measurable functions. Then for any* $t \in [0, \mu(\Omega))$,

$$(f \cdot g)^*(2t) \leq f^*(t) \cdot g^*(t). \tag{4}$$

Proof. Since

$$\mu\{\omega \in \Omega \mid |f(\omega) \cdot g(\omega)| > f^*(t) \cdot g^*(t)\}$$
$$\leq \mu\{\omega \in \Omega \mid |f(\omega)| > f^*(t)\} + \mu\{\omega \in \Omega \mid |g(\omega)| > g^*(t)\} = 2t,$$

by the definition of rearrangement, we have $(f \cdot g)^*(2t) \leq f^*(t) \cdot g^*(t)$. The proof is complete.

Theorem 1. *Let* $0 < p_1,\ p_2 < \infty$ *and* $0 < q_1,\ q_2 \leq \infty$. *Let* $1/p = 1/p_1 + 1/p_2$ *and* $1/q = 1/q_1 + 1/q_2$. *Then*

$$L_{p_1, q_1}(\Omega) \circ L_{p_2, q_2}(\Omega) \subset L_{p, q}(\Omega). \tag{5}$$

Proof. Let $f \in L_{p_1, q_1}(\Omega)$ and $g \in L_{p_2, q_2}(\Omega)$. Then by Lemma 1 and Hölder inequality, we have

$$\begin{aligned} \| f \cdot g \|_{(p, q)} &= \left(\int_0^{+\infty} \frac{((f \cdot g)^*(t) t^{1/p})^q}{t} \, dt \right)^{1/q} \\ &\leq 2^{1/p} \left(\int_0^{+\infty} \frac{(f^*(t) \cdot g^*(t) t^{1/p})^q}{t} \, dt \right)^{1/q} \\ &\leq 2^{1/p} \| f \|_{(p_1, q_1)} \cdot \| g \|_{(p_2, q_2)}. \end{aligned} \tag{6}$$

The proof is complete.

Theorem 2. *Let* $0 < p_1,\ p_2 < \infty$ *and* $0 < q_1,\ q_2 \leq \infty$. *Let* $1/p = 1/p_1 + 1/p_2$ *and* $1/q = 1/q_1 + 1/q_2$. *If* $\mu(\Omega) < +\infty$ *and* $p_1/p_2 \leq q_1/q_2$, *then*

$$L_{p_1, q_1}(\Omega) \circ L_{p_2, q_2}(\Omega) = L_{p, q}(\Omega). \tag{7}$$

Proof. Since $p_1/p_2 \leq q_1/q_2$, it is easy to verify that $q_1/p_1 - q/p \geq 0$. Let $h(x) \in L_{p, q}(\Omega)$. Denote $f(x) = |h(x)|^{q/q_1}$ and $g(x) = \operatorname{sign}(h(x)) \cdot |h(x)|^{q/q_2}$. Then $f(x) \cdot g(x) = h(x)$. We shall prove that $f \in L_{p_1, q_1}(\Omega)$. Indeed, $d_f(s) = d_h(s^{q_1/q})$ and then $f^*(t) = \inf\{s > 0 \mid d_h(s^{q_1/q}) \leq t\} =$

$(h^*(t))^{q/q_1}$. Hence we have

$$\| f \|_{(p_1, q_1)} = \left(\int_0^{\mu(\Omega)} \frac{(f^*(t)t^{1/p_1})^{q_1}}{t} dt \right)^{1/q_1}$$

$$= \left(\int_0^{\mu(\Omega)} \frac{(h^*(t)t^{1/p})^{q}}{t} t^{q_1/p_1 - q/p} dt \right)^{1/q_1}$$

$$\leq \mu(\Omega)^{(q_1/p_1 - q/p)/q_1} \| h \|_{(p,q)}^{q/q_1} < +\infty. \tag{8}$$

Similarly, we can prove that $g \in L_{(p_2, q_2)}(\Omega)$. Thus, we have $L_{p,q}(\Omega) \subset L_{p_1, q_1}(\Omega) \circ L_{p_2, q_2}(\Omega)$. Our proof now follows from this and Theorem 1.

3. Eigenvalue Distribution of Weakly Singular Kernel

Using the above theorems and the theory of p-summing operator, we now discuss the eigenvalue distribution of (2). In the following, we always suppose that $\Omega \subset R^N$ is a bounded domain and $N/2 < \alpha < N$, and we write $p = N/\alpha$ and $e_x(y) = | x - y |^{\alpha - N}$ for the sake of simplicity.

Theorem 3. *If* $| l(x, y) | \leq g(y)$ *and* $g \in L_{N/\alpha, 1}(\Omega)$, *then* $(\lambda_n(T_K)) \in l_2$.

Proof. By [5, Lemma 1.c.7], for any $f \in L_{p,1}(\Omega)$, we have

$$\int_\Omega | e_x(y) f(y) | dy \leq \int_0^{\lambda(\Omega)} e_x^*(t) f^*(t) dt$$

$$\leq \sup_{t>0} e_x^*(t) t^{1/p'} \cdot \int_0^{\lambda(\Omega)} \frac{f^*(t)t^{1/p}}{t} dt$$

$$\leq \sigma_N^{1/p'} \| f \|_{(p,1)}, \tag{9}$$

where σ_N denotes the N-volume of the unit ball in R^N. Let $J(x, y) = K(x, y)/g(y)$. We show that the integral operator T_J, defined by J, is a continuous operator which maps $L_{p,1}(\Omega)$ into $L_\infty(\Omega)$. For any $f \in L_{p,1}(\Omega)$, from Hölder inequality and (9), we have

$$\| T_J f \|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} \left| \int_\Omega J(x, y) f(y) dy \right|$$

$$\leq \operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega | e_x(y) f(y) | dy$$

$$\leq \sigma_N^{1/p'} \| f \|_{(p,1)}. \tag{10}$$

Thus, $T_J : L_{p,1} \to L_\infty$ is a continuous operator with $\| T_J : L_{p,1} \to L_\infty \| \leq \sigma_N^{1/p'}$.

We now construct a factorization of g. Let $1/s = 1/p - 1/2$. From Theorem 2 and $g \in L_{p,1}(\Omega)$, there exists a factorization of the form

$$g(y) = g_1(y) \cdot g_2(y), \tag{11}$$

where $g_1 \in L_2(\Omega)$ and $g_2 \in L_{s,2}(\Omega)$. We define two multiplication operators: $M_{g_1} : L_\infty(\Omega) \to L_2(\Omega)$ by $M_{g_1} f = g_1 \cdot f$ and $M_{g_2} : L_2(\Omega) \to L_{p,1}(\Omega)$ by $M_{g_2} h = g_2 \cdot h$. It is not difficult to check that $\| M_{g_1} : L_\infty \to L_2 \| \leq \| g_1 \|_2$. By [5, Theorem 2.b.8], this implies that M_{g_1} is a 2-summing operator. From (6), we know that M_{g_2} is bounded and $\| M_{g_2} : L_2 \to L_{p,1} \| \leq 2^{1/p} \| g_2 \|_{(s,2)}$. Thus, by the ideal property of 2-summing operators, the multiplication operator $M_g = M_{g_2} \cdot M_{g_1} : L_\infty(\Omega) \to L_{p,1}(\Omega)$ defined by $M_g f = g \cdot f$ is also a 2-summing operators.

It is easy to verify that $T_K = T_J \cdot M_g$. Thus, by reason similar to that above, $T_K : L_\infty(\Omega) \to L_\infty(\Omega)$ is 2-summing. Consequently, by [5, Proposition 2.a.1], T_K has square-summable eigenvalues, and

$$\| (\lambda_n(T_K)) \|_2 \leq \pi_2(T_K) \leq \| T_J \| \cdot \| M_{g_2} \| \cdot \pi_2(M_{g_1})$$

$$\leq 2^{1/p} \sigma_N^{1/p'} K_G \| g_1 \|_2 \cdot \| g_2 \|_{(s,2)} = c \| g \|_{(N/a,1)}, \tag{12}$$

where $c = 2^{1/p} \sigma_N^{1/p'} K_G$ and K_G is the Grothendieck constant. The proof is complete.

Remark. König et al. in [6] proved that if $g \in L_2(\Omega)$, then $(\lambda_n(T_K)) \in l_2$, and Carl proved in [1] that if $g \in L_q(\Omega)$ and $N/\alpha < q < \infty$, then $(\lambda_n(T_K)) \in l_2$. Note that if $N/2 < \alpha < N$, then $1 < N/\alpha < 2$, and thus $L_2(\Omega) \subset L_{N/\alpha,1}(\Omega)$, $L_q(\Omega) \subset L_{N/\alpha,1}(\Omega)$. Hence Theorem 3 improves the results in [1, 6].

Theorem 4. *If* $|l(x, y)| \leq g(x)$ *and* $g \in L_{N/\alpha,1}(\Omega)$, *then* $(\lambda_n(T_K)) \in l_2$.

Proof. We first prove that $T_K : L_1(\Omega) \to L_1(\Omega)$ is bounded. For any $f \in L_1(\Omega)$,

$$\begin{aligned} \| T_K f \|_1 &= \int_\Omega \left| \int_\Omega K(x, y) f(y) dy \right| dx \\ &\leq \int_\Omega \int_\Omega | g(x) e_y(x) | \cdot | f(y) | dy dx \\ &= \int_\Omega \left(\int_\Omega | g(x) e_y(x) | dx \right) | f(y) | dy. \end{aligned} \tag{13}$$

From (13) and (9), we have

$$\| T_K f \|_1 \leq \int_\Omega \sigma_N^{1/p'} \| g \|_{(p,1)} | f(y) | dy \leq \sigma_N^{1/p'} \| g \|_{(p,1)} \cdot \| f \|_1. \tag{14}$$

Thus, $\| T_K : L_1(\Omega) \to L_1(\Omega) \| \leq \sigma_N^{1/p'} \| g \|_{(p,1)}$. Hence the conjugate operator of T_K, $T_K^* : L_\infty(\Omega) \to L_\infty(\Omega)$ is also bounded. Let K^* denote the kernel of T_K^*. Then $K^*(x, y) = K(y, x) = l(y, x)/| x - y |^{N-\alpha}$. Note that $\lambda_n(T_K) = \lambda_n(T_K^*)$, $n \in \mathbb{N}$ and $| l(y, x) | \leq g(y)$. Thus, in view of Theorem 1, we have

$$\| (\lambda_n(T_K)) \|_2 = \| (\lambda_n(T_K^*)) \|_2 \leq c \| g \|_{(N/\alpha,1)}, \tag{15}$$

where c is given as in Theorem 1. The proof is complete.

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Department of Mathematics
Qingdao Polytechnic University
Qingdao, Shandong 266033, P. R. China
e-mail: zhaohl@371.net

Department of Mathematics
Hebei Polytechnic University
Tangshan, Hebei 063000, P. R. China
e-mail: jdchuan@heut.edu.cn