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VALIDITY OF OKA'S PRINCIPLE AND HOLOMORPHY OF DOMAINS WITH SMOOTH BOUNDARIES IN SEPARABLE HILBERT SPACES

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Abstract

The present Korean-Chinese-Japanese joint work leads the validity of Oka's principle to the holomorphy of a domain with smooth boundaries in separable Hilbert spaces.

1. Introduction

In one complex variable, a domain D in the complex plane $\mathbf{C}$ is unconditionally a domain of holomorphy, that is, the existence of the domain of a holomorphic function on D by a theorem of Weierstrass. H. Cartan [3] stated without proof that a Cousin-I domain in $\mathbf{C}^2$ is a domain of holomorphy and Behnke-Stein [2] proved it. Oka [56] proved that a domain of holomorphy in $\mathbf{C}^n$ is Cousin-I. So, the solvability of Cousin-I problem characterizes the holomorphy for a domain in the two dimensional complex space $\mathbf{C}^2$.

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Concerning Cousin-II problem, Thullen [67] stated that the punctured polydisc in $\mathbf{C}^2$ is not a domain of holomorphy but is a Cousin-II domain. Oka [57] proved that a Cousin-II problem in a domain of holomorphy in $\mathbf{C}^n$ is holomorphically solvable if and only if it is topologically solvable. J. P. Serre called this “Oka’s principle” and Thom praised it as the most beautiful principle in Analysis when he visited Kyushu University.

Let $\mathcal{O}$ and $\mathcal{O}^*$ be the sheaves of, respectively, additive and multiplicative groups of germs of holomorphic functions and of holomorphic functions, which take never the value zero. Any domain D with the vanishing cohomology $\mathrm{H}^1(D, \mathcal{O}) = 0$ or $\mathrm{H}^1(D, \mathcal{O}^*) = 0$ is Cousin-I or Cousin-II. Kajiwara [18] proved that a domain D in $\mathbf{C}^2$ is a domain of holomorphy if $\mathrm{H}^1(D, \mathcal{O}^*) = 0$. So, the punctured Thullen’s polydisc D is a Cousin-II domain for which $\mathrm{H}^1(D, \mathcal{O}^*) \neq 0$.

Concerning general Cousin problems, we are interested in deriving the holomorphy of a domain from the cohomology vanishing. Let L be a complex Lie group. We denote by $\mathcal{A}_L$ the sheaf of germes of holomorphic mappings into L. We denote by $\mathcal{E}_L^0$ or $\mathcal{E}_L^\infty$ the sheaf of germes of continuous or C^∞ mappings into L. When L is the additive group $\mathbf{C}$ or the multiplicative group $\mathbf{C}^* := \mathbf{C} - \{0\}$, the sheaf $\mathcal{A}_L$ coincides with the above $\mathcal{O}$ or $\mathcal{O}^*$. Kajiwara-Kazama [31] proved that a domain Ω in a two dimensional Stein manifold is also Stein if there exists a positive dimensional complex Lie group L with $\mathrm{H}^1(\Omega, \mathcal{A}_L) = 0$. Kajiwara [22] proved that a domain (D, φ) over the two dimensional complex projective space $\mathbf{P}_2$ is a domain of holomorphy over $\mathbf{P}_2$ if there exists an abelian complex Lie group L with $\mathrm{H}^1(S, \mathcal{A}_L) = 0$. For non abelian L, Kajiwara-Watanabe [42, 43] obtained similar results for domains in $\mathbf{P}_2$ and a product of two Riemann surfaces, which are not Stein.

In case that the dimension is larger than 2, by Cartan [4], the domain $D := \mathbf{C}^3 - \{(0, 0, 0)\}$ is not a domain of holomorphy but satisfies

$\mathrm{H}^1(D, \mathcal{O}) = 0$. So, the vanishing of the cohomology of degree 1 is not sufficient to let D be a domain of holomorphy. For an analytic coherent sheaf $\mathcal{F}$ over a Stein space D and for a positive integer p, the theorem of Oka-Cartan-Serre [58, 5, 64], gives the cohomology vanishing $\mathrm{H}^p(D, \mathcal{F}) = 0$. Moreover, by Serre [65], a domain D in $\mathbf{C}^n$ is a domain of holomorphy if $\mathrm{H}^p(D, \mathcal{O}) = 0$ for $p = 1, 2, \ldots, n-1$. In other words, vanishing of a suitable cohomology with degree from 1 to $n-1$ characterizes the Steinness of n dimensional domains. Kajiwara [25] proved that a domain D with real 1 codimensional continuous boundary in a finite dimensional Stein manifold S is Stein, if and only if there exists a positive dimensional complex Lie group L such that $\mathbf{H}^1(D \cap P, \mathcal{A}_L) = 0$, for any analytic polycylinder P in S.

In infinite dimensional case, Dineen [8] proved $\mathrm{H}^1(\Omega, \mathcal{O}) = 0$ for the structure sheaf $\mathcal{O}$ over a pseudoconvex domain Ω in a $\mathbf{C}$-linear locally convex space E equipped with the finite open topology τ_0. So, pseudoconvexity implies the cohomology vanishing similar to the finite dimensional case. Kajiwara-Shon [37] proved, however, for a pseudoconvex domain Ω in the $\mathbf{C}$-linear locally convex space E equipped with the topology τ_0, for an analytic subset A of Ω and for any positive integer $p \leq \operatorname{codim} A - 2$ the cohomology vanishing $\mathrm{H}^p(\Omega - A, \mathcal{O}) = 0$. The complement A of the open set $\Omega - A$ with respect to the pseudoconvex domain Ω has no interior point but in case $\operatorname{codim} = \infty$ the cohomology vanishing of all positive degree does not imply that the domain is a domain of holomorphy. Moreover, Ohgai [55] proved, for any positive integers p and q, for a pseudoconvex domain Ω in the $\mathbf{C}$-linear locally convex space E equipped with the topology τ_0, for a q-convex C^∞ function φ on Ω and for a negative number c, $\mathrm{H}^p(\{x \in \Omega;\ \varphi(x) > c\}, \mathcal{O}) = 0$. Kajiwara et al. [33] obtained similar results for a separable Hilbert space S instead of E with the topology τ_0. In this case, the complement $\{x \in \Omega;\ \varphi(x) \leq c\}$ of the open set $\{x \in \Omega;\ \varphi(x) > c\}$ with respect to the open set Ω may have interior points in $\{x \in \Omega;\ \varphi(x) < c\}$.

So, in infinite dimensional case, vanishing of cohomology of all positive degree of the structure sheaf $\mathcal{O}$ of the domain Ω does not assure the pseudoconvexity of the domain Ω.

Concerning more general Cousin problems, Grauert [14] established the validity of the above Oka's principle on the bijectivity and injectivity of the canonical mapping $j_a^0 : \mathrm{H}^1(S, \mathcal{A}_L) \to \mathrm{H}^1(S, \mathcal{E}_L^0)$, induced by the injection $\mathcal{A}_L \to \mathcal{E}_L^0$ for a Stein space S and a complex Lie group L, where $\mathcal{A}_L$ and $\mathcal{E}_L^0$ are, respectively, the sheaves over S of holomorphic and continuous mappings in L.

Concerning characterization of Steinness of domains by Oka's principle, Kajiwara-Nishihara [35] proved that a domain D in a two dimensional Stein manifold is also Stein if and only if there exists a positive dimensional complex Lie group L such that Oka's principle holds for the complex analytic fiber bundle with D as base space and with L as a structure group, by proving that the quasi-injectivity of the canonical mapping $j_a^0 : \mathrm{H}^1(\Omega, \mathcal{A}_L) \to \mathrm{H}^1(\Omega, \mathcal{E}_L^0)$ implies Steinness of D.

Thus, in two dimensional case, the existence of a positive dimensional complex Lie group for which Oka's principle holds in the above sense characterizes the holomorphy of a domain.

In higher dimensional case, Kajiwara-Takase [41] considered a product space of a Riemann surface and complex tori and Kajiwara [28] proved that a domain $D \neq \mathbf{P}_n$ of $\mathbf{P}_n$ is Stein if and only if there exists a positive dimensional complex Lie group L such that the canonical mapping $j_a^0 : \mathrm{H}^1(D \cap Z, \mathcal{A}_L) \to \mathrm{H}^1(D \cap Z, \mathcal{E}_L^0)$ is quasi-injective for any analytic polycylinder Z in $\mathbf{P}_n$, where quasi-injectivity means that the preimage of the neutral element of $\mathrm{H}^1(D \cap Z, \mathcal{E}_L^0)$ is also the neutral element of $\mathrm{H}^1(D \cap Z, \mathcal{A}_L)$. Kajiwara [29] proved that a domain Ω with real 1 codimensional continuous boundary in a Stein manifold S is also Stein, if and only if there exists a positive dimensional complex Lie group L such that $j_a^0 : \mathrm{H}^1(\Omega \cap P, \mathcal{A}_L) \to \mathrm{H}^1(\Omega \cap P, \mathcal{E}_L^0)$ is quasi-injective for any analytic polycylinder P in S. Thus, he characterized the holomorphy of Steinness of a domain Ω with real 1 codimensional smooth boundary in a Stein manifold S from validity of Oka's principle, i.e., quasi-injectivity of the canonical mapping $j_a^0 : \mathrm{H}^1(\Omega \cap P, \mathcal{A}_L) \to \mathrm{H}^1(\Omega \cap P, \mathcal{E}_L^0)$ for the cohomology of only the first degree with coefficient only one sheaf $\mathcal{A}_L$ but of the intersection of the domain Ω and any analytic polycylinder P in S.

Leiterer [45] replies immediately other 3 characterizations of Steinness of a domain Ω, the boundary of which is not necessarily real 1 codimensional continuous one, in an n dimensional Stein manifold S by the validity of Oka's principle. As the second one, a domain Ω in S with $\mathrm{H}^1(\Omega, \mathcal{O}) = 0$ is Stein if and only if, for the set $\sum_{2n}(\Omega)$ of holomorphic vector bundles over Ω which have characteristic fiber $\mathbf{C}^{2n}$ and which are subbundles of the product bundle $\Omega \times \mathbf{C}^{2n+1}$, an element E of $\sum_{2n}(\Omega)$, which is topologiacally stably trivial over Ω, there exists a holomorphic vector bundle F over Ω such that $E \oplus F$ is holomorphically trivial.

In the infinite dimensional case, Kajiwara [26] proved the validity of Oka's principle for analytic fiber bundles over pseudoconvex domains in $\mathbf{C}$-linear spaces with the topology τ_0. Kajiwara [27] proved $\mathrm{H}^1(\Omega, \mathcal{O}) = 0$ in a pseudoconvex domain Ω in a complex projective space E from the $\mathbf{C}$-linear spaces equipped with the topology τ_0 and recently Honda et al. [16] extended it to the case that the space E is the complex projective space induced from a complex Banach space with an unconditional Schauder basis.

Recent investigation of the school of Lempert is very remarkable: Lempert [47] solved $\bar{\partial}$-equations in pseudoconvex domains of spaces belonging to the category of Banach spaces, which are equipped with Schauder basis and satisfy his hypothesis (X), and gave cohomology vanishing theorems for pseudoconvex domains in those Banach spaces. And his ablest disciple Patyi [60, 61, 62] developed the theory of Grauert-Oka's principle. So, please visit his home page: http://www.math.uci.edu/ipatyi/.

The aim of the present paper is to extend the above Kajiwara's results [29] to separable Hilbert spaces, the dimensions of which are naturally infinite, as Theorem 3 in Section 3 and characterize the holomorphy of a domain Ω with real 1 codimensional smooth boundary even in a separable Hilbert space S by validity of Oka's principle, i.e., quasi-injectivity of the canonical mapping $j_a^0 : \mathrm{H}^1(\Omega \cap P, \mathcal{A}_L) \to \mathrm{H}^1(\Omega \cap P, \mathcal{E}_L^0)$ for the cohomology of only the first degree with coefficient only one sheaf $\mathcal{A}_L$ but of the intersection of the domain Ω and any analytic polycylinder P even in the infinite dimensional S. Theorem 3 is already proved by Li [52] for **C**-linear locally convex space E equipped with the topology τ_0 instead of the separable Hilbert space S equipped with the ℓ^2 norm, ℓ^2 satisfying the hypothesis (X), and by Kajiwara et al. [33] when the group L is the additive group **C** of complex numbers.

2. Limit of Cohomology Groups

In this section, we consider exclusively a finite p-dimensional Stein manifold S. A pair (D, ψ) of a Hausdorff space D and a local homomorphism $\psi : D \to S$ is called a *domain over* S. We induce canonically a complex structure on (D, ψ) so that the mapping ψ is locally biholomorphic. Two domains (D_1, ψ_1) and (D_2, ψ_2) are said to *satisfy* $(D_1, \psi_1) \prec (D_2, \psi_2)$ if there exists a locally biholomorphic mapping $\tau : D_1 \to D_2$ with $\psi_1 = \psi_2 \circ \tau$.

We consider a sequence $\{(D_n, \psi_n); n \geq 1\}$ of domains over S with $(D_n, \psi_n) \prec (D_{n+1}, \psi_{n+1})$ for any $n \geq 1$ and call it a *monotonically increasing sequence* over S. Then, for each pair of positive integers m and n with $n \leq m$, there exists canonically a holomorphic mapping τ_m^n of D_n in D_m with $\psi_n = \psi_m \circ \tau_m^n$ and $\tau_m^k = \tau_m^n \circ \tau_n^k$ for $k \leq n \leq m$. By Kajiwara [20], there exists the limit domain (D, ψ) over S. Let $\tau_n : D_n \to D$ be the canonical mapping.

Concerning the structure sheaf $\mathcal{O}$, i.e., the sheaf of germs of holomorphic functions, we induce, for any $m \geq n \geq 1$, by the local holomorphism $\tau_m^n : D_n \to D_m$, the canonical homomorphism

$$\pi_n^m : \mathrm{H}^1(D_m, \mathcal{O}) \to \mathrm{H}^1(D_n, \mathcal{O}) \tag{1}$$

satisfying $\pi_n^\ell = \pi_n^m \pi_m^\ell$ for any $\ell \geq m \geq n \geq 1$. Hence the sequence of cohomology **C**-modules $\{\mathrm{H}^1(D_n, \mathcal{O}); \pi_n^m\}$ is an inverse system of **C**-modules in the sense of Eilenberg-Steenrod [11]. Let $\lim_{n\to\infty} \mathrm{H}^1(D_n, \mathcal{O})$ be its limit in the sense of Eilenberg-Steenrod [11]. Kajiwara [20] proved the following theorem:

Theorem 1. *The canonical mapping*

$$\pi : \mathrm{H}^1(D, \mathcal{O}) \to \lim_{n\to\infty} \mathrm{H}^1(D_n, \mathcal{O}), \tag{2}$$

is injective for monotonically increasing sequences $\{(D_n, \psi_n); n \geq 1\}$ *of domains over a Stein manifold.*

As a corollary he obtained:

Corollary. *The limit of a monotonically increasing sequence of Cousin-I domains over a Stein manifold is Cousin-I.*

Let L be a finite dimensional complex Lie group, $\mathcal{A}_L$ and $\mathcal{E}_L^p$ $(0 \leq p \leq \infty)$ be, respectively, the sheaves of germs of holomorpic and C^p mappings in L. Now, we present the following theorem, which is proved by Li [52]. The canonical mapping π of (3) below is said to be *quasi-injective* if the preimage $\pi^{-1}(0)$ of the neutral element 0 is also neutral in $\mathrm{H}^1(D, \mathcal{A}_L)$.

Theorem 2. *The canonical mapping*

$$\pi : \mathrm{H}^1(D, \mathcal{A}_L) \to \lim_{n\to\infty} \mathrm{H}^1(D_n, \mathcal{A}_L), \tag{3}$$

is quasi-injective for a monotonically increasing sequence $\{(D_n, \psi_n); n \geq 1\}$ *of domains over a Stein manifold S, where quasi-injectivity means that any element* β *of* $\mathrm{H}^1(D, \mathcal{A}_L)$ *is neutral if the canonical image of* β *in* $\mathrm{H}^1(D_n, \mathcal{A}_L)$ *is neutral for any* $n \geq 1$.

Proof. For any $n \geq 1$, let $(\widetilde{D}_n, \widetilde{\psi}_n)$ be the envelope of holomorphy of the domain (D_n, ψ_n) over the Stein manifold S, (D, ψ) be the limit of the monotonically increasing sequence $\{(D_n, \psi_n); n \geq 1\}$, $\tau_n : D_n \to D$ be the canonical mapping with $\psi_n = \psi \circ \tau_n$, $\lambda_n : D_n \to \widetilde{D}_n$ be the canonical mapping with $\psi_n = \widetilde{\psi}_n \circ \lambda_n$, $(\widetilde{D}, \widetilde{\psi})$ be the limit of the monotonically increasing sequence of domains $\{(\widetilde{D}_n, \widetilde{\psi}_n); n \geq 1\}$ with canonical mappings $\widetilde{\tau}_m^n$ $(m \geq n)$ over the Stein manifold S and $\widetilde{\tau}_n : \widetilde{D}_n \to \widetilde{D}$ be the canonical mapping with $\widetilde{\psi}_n = \widetilde{\psi} \circ \widetilde{\tau}_n$ for any $n \geq 1$. Then, the limit domain $(\widetilde{D}, \widetilde{\psi})$ is the envelope of holomorphy of the domain (D, ψ) over the Stein manifold S. Let $\{Q_n; n \geq 1\}$ be a sequence of relatively compact open subsets of D such that the closure of each Q_n is contained in Q_{n+1} and $\widetilde{D} = \bigcup_{n=1}^{\infty} Q_n$. By the argument at page 42 of Kajiwara [20], we may assume that, for each $n \geq 1$, there exists a relatively compact subset $'Q_n$ of Q_n, P_n of D such that τ_n maps $'Q_n$ biholomorphically onto $'P_n$ and there holds

$$\tau_{n+1}^n('Q_n) \subset' Q_{n+1},\ \tilde{\tau}_{n+1}^n(P_n) \subset P_{n+1},\ \lambda_n('Q_n) \subset P_{n+1},\ \tilde{D} = \bigcup_{n=1}^{\infty} P_n \tag{4}$$

and moreover that each pair $(\tau_{n+1}^n(P_n), P_{n+1})$ is Runge.

Now we prove neutrality of the preimage β by the mapping π of the neutral element α of $\lim_{n\to\infty} \mathrm{H}^1(D_n, \mathcal{A}_L)$. There associate an open covering $\mathcal{U} = \{U_i; i \in I\}$ and an element $f = \{f_{ij}; i, j \in I\} \in \mathrm{Z}^1(\mathcal{U}, \mathcal{A}_L)$ such that f represents β. We consider the open covering $\tau_n^{-1}(\mathcal{U}) :=$ $\{\tau_n^{-1}(U_i); i \in I\}$ of the domain D_n for any $n \geq 1$. Since we assume that α is neutral, the cocycle $f \circ \tau_n := \{f_{ij} \circ \tau_n; i, j \in I\} \in \mathrm{Z}^1(\tau_n^{-1}(\mathcal{U}), \mathcal{A}_L)$ is a coboundary of a 0-cochain $h^n = \{h_i^n; i \in I\} \in \mathrm{C}^0(\tau_n^{-1}(\mathcal{U}), \mathcal{A}_L)$, there holds

$$f_{ij} \circ \tau_n = h_j^n (h_i^n)^{-1} \tag{5}$$

in $\tau_n^{-1}(U_i \cap U_j)$ for any $i, j \in I$.

For any $n \geq 1$, we put

$$k^n := (h_i^{n+1} \circ \tau_{n+1}^n)^{-1}(h_i^n) \tag{6}$$

in $\tau_n^{-1}(U_i)$ for each $i \in I$. Then $k_n : D_n \to L$ is a well-defined holomorphic mapping and, by Adachi et al. [1], there exists a holomorphic mapping $\widetilde{k}^n : \widetilde{D}_n \to L$ with $k_n = \widetilde{k}_n \circ \tau_n$ for any $n \geq 1$. There is no consistency between k^n and $\widetilde{k}_n$'s. Since each $(\tau_{n+1}^n(P_n), P_{n+1})$ is a Runge pair, in accordance with the fine arguments in Section 4 of Grauert [12], the mapping $\widetilde{k}_n$ is approximated by holomorphic mappings of P_{n+1} into L on any relatively compact subset of it. And, therefore $k_n = \widetilde{k}_n \circ \tau_n$ is approximated by holomorphic mappings k^n of Q_{n+1} into L on any relatively compact subset of it. We replace the mappings h_i^{n+1} by $h_i^{n+1}k^n$ and, for each $n \geq 1$ and i in I, we construct inductively a sequence $\{f_i^n;\, n \geq 1\}$ of holomorphic mappings $f_i^n : \tau_n^{-1}(U_i) \to L$ so that there holds $f_{ij} \circ \tau_n = f_j^n (f_i^n)^{-1}$ for any $n \geq 1$ and $i \in I$ and that each sequence $\{f_i^n;\, n \geq 1\}$ converges to a holomorphic mapping $f_i : U_i \to L$ uniformly on any compact subset of U_i for any in I. Then the cocycle $f = \{f_{ij};\, i, j \in I\} \in \mathrm{Z}^1(\mathcal{U}, \mathcal{A}_L)$ is the coboundary of the 0-cochain $g = \{g_i;\, i \in I\} \in \mathrm{C}^0(\mathcal{U}, \mathcal{A}_L)$.

Thus we have proved the neutrality of the preimage of the neutral element of $\lim_{n \to \infty} \mathrm{H}^1(D_n, \mathcal{A}_L)$ by the mapping π.

The canonical mapping $j_p^a : \mathrm{H}^1(D, \mathcal{A}_L) \to \mathrm{H}^1(D, \mathcal{E}_L^p)$ $(0 \leq p \leq \infty)$ is said to be *quasi-injective* if the preimage $(j_p^a)^{-1}(0)$ of the neutral element 0 in $\mathrm{H}^1(D, \mathcal{E}_L^p)$ is the neutral element 0 in $\mathrm{H}^1(D, \mathcal{A}_L)$.

Let L be a complex Lie group, $\mathcal{L}$ be its Lie algebra and $\exp : \mathcal{L} \to L$ be the exponential mapping. Let X be an element of $\mathcal{L}$. A domain D in a complex manifold S is said to be *X-regular* if, for any open covering $\mathcal{U} :=$

$\{U\alpha;\ \alpha \in A\}$ of D, any cocycle $\{f_{\alpha\beta} : \alpha, \beta \in A\} \in \mathrm{Z}^1(\mathcal{U}, \mathcal{O})$ and the element $X \in \mathcal{L}$, the cocycle $\{\exp(f_{\alpha\beta}X);\ \alpha, \beta \in A\} \in \mathrm{Z}^1(\mathcal{U}, \mathcal{A}_L)$ is a coboundary $\in \mathrm{B}^1(\mathcal{U}, \mathcal{A}_L)$.

3. Validity of Oka's Principle for Infinite Dimensional Domains with Smooth Boundaries Leads Holomorphy of Domains

Let E be a **C**-linear Hausdorff space and Λ be the set of finite dimensional **C**-linear subspaces of E. A complex valued function f on an open subset D of E is said to be *Gâtaux holomorphic* if, for any F in Λ, the restriction of f to $D \cap F$ is holomorphic on the open subset $D \cap F$ of the finite dimensional complex space F. A complex valued Gâtaux holomorphic function f on an open subset D of E is said to be *holomorphic* if f is continuous on D. An open subset D of E is said to be *pseudoconvex* if, for any F in Λ, the intersection $D \cap F$ is a pseudoconvex open set of the **C**-linear space F of finite dimension for any F in Λ. A topology on E is said to be *finite open* if the family of open sets consists of subsets O of E such that, for any F in Λ, the intersections $O \cap F$ are open sets.

Let E be a **C**-linear Hausdorff space and D be a domain of E. The boundary of D is said be *smooth*, if, for any boundary point x of D, there exist an open neighborhood U of x in E and a real valued function β of class C^1 on U such that $d\beta \neq 0$ at any $y \in U$ and that $D \cap U = \{y \in U;\ \beta(y) < 0\}$.

Quasi-injectivity of the canonical mapping (7) in Theorem 3 below means that the preimage of the neutral element by the canonical mapping (7) below is also the neutral element in the cohomology set.

Under these notations we have the following theorem, which is proved by Li [52] when the space S is the locally convex space equipped with the topology τ_0 and by Kajiwara et al. [33] when the Lie group L is the additive group of complex numbers.

Theorem 3. *Let S be a separable Hilbert space, Ω be a domain with smooth boundary in S. If there exists a complex Lie group L of dimension* *m finite and positive such that the canonical mapping*

$$\mathrm{j}_a^0 : \mathbf{H}^1(\Omega \cap P, \mathcal{A}_L) \to \mathbf{H}^1(\Omega \cap P, \mathcal{E}_L^0) \tag{7}$$

is quasi-injective for any open convex set P in S, then Ω is a domain of holomorphy in S.

Proof. Let $\mathcal{L}$ be the Lie algebra of L and $\exp : \mathcal{L} \to L$ be the exponential mapping. We denote by $\mathrm{Ad} : L \to \mathrm{GL}(m, \mathbf{C})$ and $\mathrm{ad} : \mathcal{L} \to g\ell(m, \mathbf{C})$ the adjoint representations, where $\mathrm{GL}(m, \mathbf{C})$, $g\ell(m, \mathbf{C})$ are, respectively, the Lie group of invertible $m \times m$ matrices and the Lie algebra of $m \times m$ matrices.

We take any non zero element X of $\mathcal{L}$.

By Noverraz [54], many definitions of pseudoconvexity are equivalent and by Gruman [15] a pseudoconvex domain in the Hilbert space S is a domain of holomorphy. So, it suffices to prove the pseudoconvexity of the cut $\Omega \cap S_f$ of the domain Ω by any finite dimensional affine subspace S_f of S. To begin with finite n pieces of orthonormal vectors of S, which span the affine subspace S_f, we put them in the opening of a complete orthonormal basis of S, we regard S as the space ℓ^2 of sequences $z = (z_1, z_2, \ldots, z_k, \ldots)$ of complex numbers z_k which satisfy $\| z \| := \sqrt{\sum_{k=1}^{\infty} | z_k |^2} < +\infty$, regard $\mathbf{C}^n$ as its subspace $\{z = (z_1, z_2, \ldots, z_n, \ldots) \in \ell^2; z_k = 0 \ (k > n)\}$ and regard S_f as an affine subspace of $\mathbf{C}^n$. Without loss of generality, we may assume that $S_f = \mathbf{C}^n$.

Let $\rho_n : S \cong \ell^2 \to \mathbf{C}^n$ be the projection defined by

$$\rho_n(z_1, z_2, \ldots, z_n, z_{n+1}, \ldots) = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n. \tag{8}$$

We put $\Omega_n := \Omega \cap \mathbf{C}^n$. Let Q_n be a convex domain in $\mathbf{C}^n$. Let $z^{(0,n)} = (z_1^{(0)}, z_2^{(0)}, \ldots, z_n^{(0)}) \in \mathbf{C}^n$ be an arbitrary boundary point of Ω_n in Q_n. We put $z^{(0)} := (z_1^{(0)}, z_2^{(0)}, \ldots, z_n^{(0)}, 0, 0, \ldots)$. Then, $z^{(0)}$ is also a boundary point of the domain Ω in $S \cong \ell^2$. There exist an open neighborhood U of $z^{(0)}$ in S and a real valued function β of class C^1 on U such that U is contained in $\rho_n^{-1}(Q_n)$, that $d\beta \neq 0$ at each $x \in U$ and that $\Omega \cap U = \{x \in U;\ \beta(x) < 0\}$. There exists a positive integer j such that either $\partial\beta/\partial x_j \neq 0$ at $z^{(0)}$ for the real part x_j of z_j or $\partial\beta/\partial y_j \neq 0$ at $z^{(0)}$ for the imaginary part y_j of z_j. In the latter case, we replace z_j by iz_j. We may assume that $j < n$ and we exchange z_j and z_1 and replace U by a closer neighborhood V of $z^{(0)}$. Thus, without loss of generality, we may assume that $S_f = \mathbf{C}^n$ and that there exists a real valued function $g(y_1, z_2, z_3, \ldots)$ of class C^1 on a neighborhood V, which is a subset of U and is the ball of radius $3r_0$ centered by $z^{(0)}$, of the boundary point $z^{(0)}$ of Ω so as there holds

$$\Omega \cap V = \{z = (z_1, z_2, z_3, \ldots) \in V;\ x_1 < g(y_1, z_2, z_3, \ldots)\}. \tag{9}$$

Let B be the ball of radius $2r_0$ centered by $z^{(0)}$. We put $B_n := \rho_n(B)$ and will show that the open set $\Omega_n \cap B_n$ is an open set of holomorphy in $\mathbf{C}^n$. For this purpose, we show that $\Omega_n \cap B_n$ is X-regular in the sense of Kajiwara [29].

For any nonnegative number t smaller than $1/2$, we denote by T_t and $T_{t,n}$ the translations

$$T_t(z_1, z_2, \ldots, z_n, z_{n+1}, \ldots) = (z_1 + r_0 t, z_2, \ldots, z_n, z_{n+1}, \ldots), \tag{10}$$

$$T_{t,n}(z_1, z_2, \ldots, z_n) = (z_1 + r_0 t, z_2, \ldots, z_n), \tag{11}$$

and put

$$P_t := \left\{(z_1, z_2, \ldots) \in S;\ \sum_{k=1}^{\infty} |z_k - z_k^{(0)}|^2 < 4(1-t)^2 r_0^2\right\}, \tag{12}$$

$$P_{t,n} := \left\{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n;\ \sum_{k=1}^{n} |z_k - z_k^{(0)}|^2 < 4(1-t)^2 r_0^2\right\}, \tag{13}$$

and

$$E_t := T_t^{-1}(\Omega \cap B \cap \rho_n^{-1}(Q_n) \cap P_t), \tag{14}$$

$$E_{t,n} := T_{t,n}^{-1}(\Omega_n \cap B_n \cap Q_n \cap P_{t,n}). \tag{15}$$

Then, for any positive number t smaller than 1/2, $E_{t,n}$ is a relatively compact open subset of $E_{0,n} = \Omega_n \cap B_n \cap Q_n = \Omega_n \cap B_n$.

Now, let $\mathcal{U}_n := \{U_{i,n};\ i \in I\}$ be any pseudoconvex covering of $E_{0,n}$ and $\{f_{ij,n};\ i, j \in I\}$ be any 1-cocycle of the covering $\mathcal{U}_n$ with coefficient in the structure sheaf $\mathcal{O}_n$ of S_f, which is regarded as $\mathbf{C}^n$. Since the n-dimensional subset $E_{t,n}$ is relatively compact in the infinite dimensional open set $\Omega \cap B$ in S, for any positive number t smaller than 1/2 we associate a positive number $\tau = \tau(t)$ such that, for the open convex set

$$W_t = \left\{ z = (z_1, z_2, z_3, \ldots) \in S;\ \sum_{k=n+1}^{\infty} |z_k|^2 < \tau^2 \right\} \tag{16}$$

in S, $\rho_n^{-1}(E_{t,n}) \cap W_t$ is contained in $\Omega \cap B$. We put $U_{t,i} := \rho_n^{-1}(U_{i,n}) \cap E_t \cap W_t$. Then, $\mathcal{U}_t := \{U_{t,i};\ i \in I\}$ is a pseudoconvex covering of $E_t \cap W_t$. For any $i, j \in I$, each function $f_{ij,n} \circ \rho_n$ is holomorphic in $U_{t,i} \cap U_{t,j} \cap E_t \cap W_t$.

Since T_t maps $E_t \cap W_t$ biholomorphically onto $\Omega \cap B \cap \rho^{-1}(Q_n) \cap P_t \cap W_t$, since there holds $\mathrm{H}^1(\Omega \cap B \cap \rho^{-1}(Q_n) \cap P_t \cap W_t, \mathcal{E}^{\infty}) = 0$ by the partition of unity and since the canonical homomorphism $\mathrm{H}^1(\mathcal{U}_t, \mathcal{O}) \to \mathrm{H}^1(E_t \cap W_t, \mathcal{O})$ is injective by Lemma L_0 of Scheja [63], the 1-cocycle $\{f_{ij,n} \circ \rho_n |_{U_{t,i} \cap U_{t,j} \cap E_t \cap W_t};\ i, j \in I\}$ of the covering $\mathcal{U}_t$ with value in $\mathcal{E}^{\infty}$ is a coboundary. Hence, there exists a 0-cochain $\{g_{t,i};\ i \in I\} \in \mathrm{C}^0(\mathcal{U}_t, \mathcal{E}^{\infty})$ such that $f_{ij,n} \circ \rho_n = g_{t,j} - g_{t,i}$ on $U_{t,i} \cap U_{t,j} \cap E_t \cap W_t$ for any $i, j \in I$. For any $i \in I$ and any positive number t smaller than 1/2, we put $U_{t,i,n}$

$:= U_{i,n} \cap E_{t,n}$. Then, $\mathcal{U}_{t,n} := \{U_{t,i,n};\, i \in I\}$ is a pseudoconvex covering of $E_t \cap W_t$. For any $i \in I$ and any positive number t smaller than 1/2, let $f_{t,i,n}$ be the restriction of $f_{t,i}$ to $U_{t,i,n}$. Then, the coboundary of the 0-cochain $\{g_{t,i,n};\, i \in I\} \in \mathrm{C}^0(\mathcal{U}_{t,n}, \mathcal{E}^\infty)$ is the 1-cocycle $\{f_{t,ij}|_{U_{t,i,n} \cap E_t} \in \mathrm{Z}^1(\mathcal{U}_{t,n}, \mathcal{O}_n)\}$, which is the restriction of the 1-cocycle $\{f_{ij,n};\, i, j \in I\} \in \mathrm{Z}^1(\mathcal{U}_n, \mathcal{O}_n)$.

Then, the 1-cocycle $\{\exp(f_{ij,n} \circ \rho_n|_{U_{t,i} \cap U_{t,j} \cap E_t \cap W_t} X);\, i, j \in I\}$ of the covering $\mathcal{U}_n$ with value in $\mathcal{A}_L$ is the coboundary of the 0-cochain $\{\exp(g_{t,i,n}X);\, i \in I\} \in \mathrm{C}^0(\mathcal{U}_{t,n}, \mathcal{E}_L^\infty)$. By the assumption of the present theorem, the 1-cocycle $\{\exp(f_{ij,n} \circ \rho_n|_{U_{t,i} \cap U_{t,j} \cap E_t \cap W_t} X);\, i, j \in I\} \in \mathrm{Z}^1(\mathcal{U}_{t,n}, \mathcal{A}_L)$ is a coboundary in the $\mathcal{A}_L$ category and there exists a 0-cochain $\{F_{t,i,n};\, i \in I\}$, which belongs to $\mathrm{C}^0(\mathcal{U}_{t,n}, \mathcal{A}_L)$ and the coboundary of which is the 1-cocycle $\{\exp(f_{ij,n} \circ \rho_n|_{U_{t,i} \cap U_{t,j} \cap E_t \cap W_t} X);\, i, j \in I\} \in \mathrm{Z}^1(\mathcal{U}_{t,n}, \mathcal{A}_L)$.

Now, we take a sequence $\{t(\nu);\, \nu = 1, 2, 3, \ldots\}$ of positive numbers $t(\nu)$ smaller than 1/2 such that $t(\nu) > t(\nu+1)$, $\tau(t(\nu)) > \tau(t(\nu+1))$ for $\nu = 1, 2, 3, \ldots,$ $t(\nu) \to 0$, $\tau(t(\nu)) \to 0$ as $\nu \to \infty$ and that the preceding open set $E_{t(\nu),n}$ is relatively compact in the rear open set $E_{t(\nu+1),n}$. Since the canonical homomorphism

$$\mathrm{H}^1(E_{0,n}, \mathcal{A}_L) \to \lim_{\nu \to \infty} \mathrm{H}^1(E_{t(\nu),n}, \mathcal{A}_L) \tag{17}$$

is quasi-injective by Theorem 2, the 1-cocycle $\{\exp(f_{ij,n}X);\, i, j \in I\}$ is the coboundary of a 0-cochain $\{F_{i,n};\, i \in I\} \in \mathrm{C}^0(\mathcal{U}_{t,n}, \mathcal{A}_L)$. Thus, we have proved that the open set $\Omega_n \cap B_n$ is X-regular. By Kajiwara [29], $\Omega_n \cap B_n$ is a domain of holomorphy. By Oka [59], the Cartan pseudoconvex domain Ω_n is a domain of holomorphy. By Noverraz [54], Ω is pseudoconvex in the infinite dimensional Hilbert space S. Lastly, by Gruman [15], Ω is a domain of holomorphy in the Hilbert space S.

As the above proof shows, we can replace (7) in Theorem 3 by

$$\mathrm{j}_a^\infty : \mathbf{H}^1(\Omega \cap P, \mathcal{A}_L) \to \mathbf{H}^1(\Omega \cap P, \mathcal{E}_L^\infty). \tag{18}$$

Concerning the converse, Patyi [60, 61, 62] proved the validity of Oka's principle for pseudoconvex domains in certain Banach spaces and infinite dimensional solvable complex Lie groups.

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