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ON THE LATTICE OF PRINCIPAL *L*-TOPOLOGIES

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Abstract

We investigate the lattice structure of the set of principal L-topologies on a given set X. It is proved that the lattice of principal L-topologies is atomic and even not modular. It is complete and it has dual atoms if and only if membership lattice L has dual atoms and this lattice is semi-complemented.

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1. Introduction

The concept of fuzzy topology was introduced by Chang [2] in 1968, and later in a different way by Lowen [10] and Hutton [8]. Meanwhile, Goguen [7] introduced the concept of *L* fuzzy sets and consequently the Chang's definitions of a fuzzy topology have been extended to *L*-topology [15]. In this paper, we investigate the lattice structure of the set $\beta(X)$ of all principal *L*-topologies on a given set *X*. In [1], Birkhoff proved that the set $\sum(X)$ of all topologies on a fixed set *X*, forms a complete lattice with natural order of set inclusion. Vaidyanathswamy [12] showed that this lattice is not distributive in general. Steiner [11] proved that the lattice of topologies on a set with more than two elements is not even modular. Vaidyanathswamy [12] determined atoms in this lattice and proved that it is an atomic lattice. Frolich [4] determined dual atoms of this lattice and proved that it is also dually atomic. Van Rooji [13] and Steiner [11] independently proved that the lattice of topologies is complemented. In [9], Johnson studied the lattice structure of the set of all *L*-topologies on a given set *X*. It is quite natural to find sublattices in the lattice of *L*-topologies and study their properties. The collection $\beta(X)$ of all principal *L*-topologies on a given set *X* forms one of the sublattices of the lattice of *L*-topologies on *X*. Lattice of principal *L*-topologies is a complete sublattice of lattice of *L*-topologies. Also, $\beta(X)$ is not modular and complemented. The concept of principal topologies in the crisp case was studied by Steiner [11]. The lattice of principal topologies is a complete lattice whose least element is the indiscrete topology and the greatest element is the discrete topology. This lattice is both atomic and dually atomic. Its atoms coincide with those of $\sum(X)$. However, we prove that the lattice $\beta(X)$ has dual atoms if and only if the membership lattice *L* has dual atoms and it is not dually atomic in general. Also, it is proved that induced principal *L*-topologies have complements.

2. Preliminaries

Throughout this paper, *X* stands for a nonempty ordinary set and *L* for a fuzzy lattice with an order reversing involution. We denote the constant function in L^X taking the value $\alpha \in L$ by $\underline{\alpha}$. The fundamental definitions of L fuzzy set theory and L-topology are assumed to be familiar to the reader. A topological space is called *principal* if it is discrete or if it can be written as the meet of principal ultra topologies. Steiner [11] proved that this is equivalent to requiring that arbitrary intersection of open sets is open. Analogously, we define principal L-topology.

Definition 2.1. An L-topology is called *principal L-topology* if arbitrary intersection of open L subsets is an open L subset.

Definition 2.2 [15]**.** A fuzzy lattice is a *complete* and *completely distributive lattice* with an order reversing involution.

Definition 2.3 [15]**.** An element of a lattice L is called an *atom*, if it is the minimal element of $L\backslash\{0\}$.

Definition 2.4 [15]**.** An element of a lattice L is called a *dual atom*, if it is the maximal element of $L\backslash\{1\}$.

Definition 2.5 [3]**.** A lattice is said to be *bounded* if it possesses the smallest element 0 and the largest element 1.

Definition 2.6 [15]**.** A bounded lattice L is said to be *joint complemented* if for all x in L, there exists $y \in L$ such that $x \vee y = 1$.

Definition 2.7 [15]**.** A bounded lattice L is said to be *meet complemented* if for all $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$.

Definition 2.8 [15]**.** A bounded lattice is said to be *complemented* if it is both join complemented and meet complemented.

Definition 2.9. A bounded lattice L is said to be *semi-complemented* if it is either join complemented or meet complemented.

Definition 2.10 [6]**.** An element $p \in L$ is called *prime* if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leq p$, then $a \leq p$ or $b \leq p$. The set of all prime elements of L will be denoted by $\Pr(L)$.

Definition 2.11 [14]**.** The Scott topology on L is the topology S, generated by the sets of the form $\{t \in L : t \not\leq p\}$, where $p \in \Pr(L)$. Let (X, τ) be a topological space and L be a fuzzy lattice. Then $f : (X, \tau) \to L$ is said to be *Scott continuous* if $f : (X, \tau) \to (L, S)$ is continuous, i.e., if for every $p \in \Pr(L)$, $f^{-1}(\{t \in L : t \not\leq p\}) \in \tau$.

Remark 2.12. When $L = [0, 1]$, the Scott topology coincides with a topology of topologically generated spaces of Lowen [10]. The set $\omega_L(\tau) = \{f \in L^X : f : (X, \tau) \to L \text{ is Scott continuous}\}$ is an L-topology. An L-topology F on X is called an *induced L-topology*, if there exists a topology τ on X such that $F = \omega_L(\tau)$. If τ is a principal L-topology, then $\omega_L(\tau)$ is a principal L-topology and is denoted by $\omega_{PL}(\tau)$.

Note [5]**.** A lattice L is modular if it has no sublattice isomorphic to N_5, where N_5 is a standard non-modular lattice.

3. Lattice of Principal L-topologies

The family $\beta(X)$ of all principal L-topologies on a given set X forms a lattice under the natural order of set inclusion. The least upper bound of a collection of principal L-topologies belonging to $\beta(X)$ is the principal L-topology, which is generated by their union and the greatest lower bound is their intersection. The smallest principal L-topology is the indiscrete L-topology denoted by 0 and the largest principal L-topology is the discrete L-topology denoted by 1.

Theorem 3.1. *The lattice* $\beta(X)$ *is complete.*

Proof. Let S be a subset of $\beta(X)$, $G = \bigcap_{H_\alpha \in S} H_\alpha$ and $f_\alpha \in G$. Then $f_\alpha \in H_\alpha$ for each α. Since H_α is a principal L-topology, $\wedge f_\alpha \in F_\alpha$ for each α. Therefore, $\wedge f_\alpha \in \bigcap H_\alpha$ and so $\wedge f_\alpha \in G$. Thus, G is closed under arbitrary intersection. That is $G \in \beta(X)$ and G is the greatest lower bound of

S. Let K be the set of upper bounds of S. Then K is nonempty, since $1 \in K$. Using the above argument, K has the greatest lower bound, say H. Then this H is the least upper bound of S. Thus, every subset S of $\beta(X)$ has the greatest lower bound and least upper bound. Hence, $\beta(X)$ is complete. □

Theorem 3.2. $\beta(X)$ *is atomic*.

Proof. Atoms in $\beta(X)$ are of the form $\mathcal{F}(g) = \{\underline{0}, \underline{1}, g / g \text{ is an } L \text{ subset}\}$. Let P be any element of $\beta(X)$. Then $P = \bigvee_{g \in P} \mathcal{F}(g)$. Hence, $\beta(X)$ is atomic. □

Theorem 3.3. $\beta(X)$ *is not modular*.

For example, let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Suppose $F_1 = \{\underline{0}, \underline{1}\}$, $F_2 = \{\underline{0}, \underline{1}, \mu_{(a)}\}$, $F_3 = \{\underline{0}, \underline{1}, \mu_{(b)}\}$, $F_4 = \{\underline{0}, \underline{1}, \mu_{(a)}, \mu_{(a, b)}\}$, $F_5 = \{\underline{0}, \underline{1}, \mu_{(a)}, \mu_{(b)}, \mu_{(a, b)}\}$, where $\mu_{(a)}, \mu_{(b)}, \mu_{(a, b)}$ are the characteristic functions of open subsets $\{a\}, \{b\}, \{a, b\}$ of (X, τ). Then each element in the collection $S = [F_1, F_2, F_3, F_4, F_5]$ belongs to $\beta(X)$ and S is a sublattice of $\beta(X)$, isomorphic to N_5. Therefore, the $\beta(X)$ is not modular.

Theorem 3.4. *If L has no dual atoms, then the lattice of principal L-topologies $\beta(X)$ on a set X has no dual atoms.*

Proof. Let F be any principal L-topology on X other than 0 and 1. Being F is a principal L-topology it is closed for arbitrary intersection. Since $F \neq 1$, F cannot contain all constant L-subsets and all characteristic functions of open sets in τ. Let $S = L^X - F$. Then S is infinite since L has no dual atoms. Let $g \in S$ and G be the principal L-topology generated by $F \bigcup \{g\}$. Then $F \subset G \neq 1$. Hence, the proof of the theorem is completed. □

Theorem 3.5. *If L has dual atoms, then $\beta(X)$ has dual atoms.*

Proof. Let g be a dual atom in the lattice of principal topologies. Let A be a subset of X not in τ such that $S = \{B : B \in \tau\} \bigcup \{A\}$ generates the discrete topology on X. Let $\alpha \in L$ and assume that α is the dual atom of L. Then the L-topology generated by $\omega_{PL}(\tau) \cup f_A^{\alpha}$ is the dual atom in $\beta(X)$, where $\omega_{PL}(\tau)$ is the principal L-topology consists of all Scott continuous functions and

$$f_A^{\alpha}(x) = \begin{cases} \alpha, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$ □

By comparing Theorems 3.4 and 3.5, we get

Theorem 3.6. *Lattice of principal L-topologies* $\beta(X)$ *on a set X has dual atoms if and only if L has dual atoms*.

Theorem 3.7. *Lattice of principal L-topologies* $\beta(X)$, *on a set X is not dually atomic in general*.

Proof. This follows from Theorem 3.4. □

Theorem 3.8. *If F is any principal L-topology on X such that the topology corresponding to the characteristic functions in F is neither discrete nor indiscrete, then F has at least one join complement*.

Proof. Let τ be the principal topology corresponding to the characteristics functions in F. Since the lattice of principal topologies on X is complemented [11], we can find a principal topology τ' such that $\tau \wedge \tau' = 0$ and $\tau \vee \tau' = 1$ in the lattice of principal topologies. Then the principal L-topology generated by $F \cup \omega_{PL}(\tau') = 1$ in $\beta(X)$, where $\omega_{PL}(\tau') = \{f \in L^X / f : (X, \tau') \to L,$ is Scott continuous$\}$. Hence the proof. □

Theorem 3.9. *The lattice of principal L-topologies* $\beta(X)$ *on any set X, is semi-complemented*.

Proof. This follows from the above theorem. □

Theorem 3.10. *If F is any induced principal L-topology on X, then F has at least one complement in* $\beta(X)$.

Proof. Since F is induced, there exists a principal topology τ in the lattice of principal topologies such that $\omega_{PL}(\tau) = F$. Since lattice of principal topologies is complemented [11], there exists at least one topology τ' in the lattice of principal topologies such that $\tau \wedge \tau' = 0$ and $\tau \vee \tau' = 1$ in the lattice of principal topologies. Then $F \vee \omega_{PL}(\tau') = 1$ and $F \wedge \omega_{PL}(\tau') = 0$ in $\beta(X)$. □

4. Conclusion

For a given principal topology τ, the family F_τ of all principal L-topologies defined by families of Scott continuous functions from (X, τ) to L, form a lattice under the natural order of set inclusion. From this lattice, we can deduce the properties of $\beta(X)$.

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References

[1] G. Birkhoff, On the combination of topologies, Fund. Math. 26 (1936), 156-166.

[2] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 191-201.

[3] B. A. Davey and H. A. Priestley, Introduction to Lattices and Order, 2nd ed., Cambridge University Press, New York, 2002.

[4] O. Frolich, Das Halbordnungs system det Topologichen Raume auf einer Menge, Math. Ann. 156 (1964), 79-95.

[5] George Gratzer, Lattice Theory, University of Manitoba, 1971, p. 70.

[6] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove and D. S. Scott, A Compendium of Continuous Lattices, Springer-Verlag, Berlin, 1980.

[7] J. A. Goguen, *L*-fuzzy sets, J. Math. Anal. Appl. 18 (1967), 145-174.

[8] B. Hutton, Uniformities on fuzzy topological space, J. Math. Anal. Appl. 58(3) (1977), 559-571.

[9] T. P. Johnson, On lattice of *L*-topologies, Indian J. Math. 46(1) (2004), 21-26.

[10] R. Lowen, Fuzzy topological space and fuzzy compactness, J. Math. Anal. Appl. 56(3) (1976), 621-633.

[11] A. K. Steiner, The lattice of topologies, structure and complementation, Trans. Amer. Math. Soc. 122 (1966), 379-397.

[12] R. Vaidyanathaswamy, Set Topology, Chelsea Publ. Co., New York, 1960.

[13] A. C. M. Van Rooji, The lattice of all topologies is complemented, Canad. J. Math. 20 (1968), 805-807.

[14] M. Warner and R. G. McLean, On compact Hausdorff *L* fuzzy spaces, Fuzzy Sets and Systems 56 (1993), 103-110.

[15] Ying-Ming Liu and Mao-Kang Luo, Fuzzy Topology, World Scientific Publishing Company, River Edge, New Jersey, 1997.