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THE a.s. WELL-POSEDNESS OF STOCHASTIC DELAY DIFFERENTIAL EQUATIONS WITH WHITE NOISES

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Abstract

A stochastic inequality between Itô integral and Lebesgue integral is obtained from the classic law of iterated logarithm, which provides a way to study stochastic delay differential equations with white noises directly in probability 1.

1. Introduction

We often call differential equations with time delay as *delay* (or *functional*) *differential equations*, which are widely used to model phenomena in physics, economics, biology, medicine, ecology and other sciences (e.g., [4, 7, 10]). However, in most cases, some kind of randomness can appear in the problem, so that the phenomena should be modeled by a stochastic form. The earliest work on SDEs was done to describe Wiener process in Einstein's famous paper [1], and at the same time by [9]. There is by now a rather comprehensive mathematical literature on the mathematical theory and on applications of stochastic differential equations driven by white noise (e.g., [5, 6, 8, 11]).

The well-posed problem stems from a definition given by Hadamard [3]. He

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believed that mathematical models of physical phenomena should have the properties that

(1) A solution exists

(2) The solution is unique

(3) The solution depends continuously on the initial data, in some reasonable topology.

For stochastic mathematical models, the probability sense must be considered in the definition of well-posedness. For example, we can define the almost sure well-posedness as the following:

Definition 1.1. If all the properties (1)-(3) hold in probability 1 for a stochastic mathematical model, then the model is *almost surely* (*a.s.*) *well-posed*.

The a.s. well-posedness is very important. It is a part of the definition of stochastic or random dynamical systems and most of the necessary tools are also needed in other parts of the study. Existence and uniqueness of global solutions have been established under global Lipschitz conditions (e.g., [5]) or under local Lipschitz and linear growth conditions (e.g., [6, 11]). [11] also considered the continuous dependence of solutions on initial values. The well-posedness problem for stochastic delay differential equations seems to be over. It is noteworthy that the continuous dependence on initial values is in mean square, not in probability 1. That is, (1) and (2) hold in probability 1, but (3) is true in mean square. Obviously, Definition 1.1 does not be satisfied. From the current literature, we find that the continuous dependence on initial values in probability 1 for Cauchy problem of stochastic delay differential equations with white noises has not been solved so far. It motivates us to think about the a.s. well-posedness problem of stochastic delay differential equations with white noises.

Suppose that $(\mathbf{\Omega}, \mathbf{F}, \{\mathbf{F}_t\}_{t\geq 0}, \mathbf{P})$ is a filtered complete probability space. We will always consider our problems in the probability space. Let $\mathbb{E}x$ be the mathematical expectation of a random variable x. From now on, $W(t) = W(t, \omega),\ t \geq 0$ is always a one dimensional Wiener process and fdW is an Itô's differential. By using the following properties of Itô integral,

$$\mathbb{E}\int_0^t g(s)dW(s) = 0, \tag{1.1}$$

$$\mathbb{E}\left(\int_0^t g(s)dW(s)\right)^2 = \mathbb{E}\int_0^t |g(s)|^2 ds, \tag{1.2}$$

where $g \in \mathcal{L}_{loc}^2$ and $\mathcal{L}_{loc}^2$ is as in Section 2, we can transform stochastic equations with white noises into the corresponding deterministic equations on some moments. By using some stochastic formulas or the relationship between different types of stochastic convergence again, we can obtain the existence of solutions of Itô stochastic differential equations. The way is well-known and it is used frequently by many authors (e.g., [2, 5, 6, 8, 11]). But, if $g \in L_{loc}^2$, where L_{loc}^2 is as in Section 2, then (1.1) and (1.2) may not be true for $t > 0$, which implies that the idea would not be suitable for stochastic differential equations with generalized Itô differential in L_{loc}^2. We also point out that the way is helpless to discuss the a.s. continuous dependence on initial data for the solutions of Itô stochastic differential equations though it leads to continuous dependence on initial data in mean square or in probability or in distribution. It motivates us to look for a new way to solve the a.s. well-posedness of Itô stochastic delay differential equations freely. In this paper, our way seems to be rather different from [5, 6, 11], but to be similar to deterministic delay differential equations [4, 7, 10]. That is, by using this way, we can use more ideas of deterministic analysis to solve Itô stochastic delay differential equations than those of stochastic analysis.

Here is the plan of this paper: in Section 2, we devote to building a bridge between Itô integral and Lebesgue integral from the classic law of iterated logarithm. In Section 3, by using the results of Section 2, we solve the existence, uniqueness, continuous dependence, extension and regularity of the solutions of stochastic delay differential equations with noises directly in probability 1.

2. A Stochastic Inequality

We will often use the following spaces in this paper:

$$L_{loc}^2 = \Big\{ f : f = \{f(t, \omega)\}_{t \geq 0},\ f(t, \omega) \text{ is a measurable and adapted process}$$
$$\text{satisfying } \int_\tau^t |f(s, \omega)|^2 ds < \infty \text{ a.s. for any } t \geq \tau \geq 0 \Big\},$$

$$\mathcal{L}^2_{loc} = \Big\{ f : f = \{f(t, \omega)\}_{t \geq 0},\ f(t, \omega) \text{ is a measurable and adapted process}$$

$$\text{satisfying } \mathbb{E}\int_\tau^t |f(s, \omega)|^2 ds < \infty \text{ for any } t \geq \tau \geq 0 \Big\}.$$

Obviously, $\mathcal{L}^2_{loc} \subset L^2_{loc}$. In this paper, we will use Itô integral in L^2_{loc}. The details on the generalized Itô integral can be found in Section 1.4 of [2].

We can assume without loss of generality that

$$\mathbf{\Omega} = \{\omega \in C(\mathbb{R}^+, \mathbb{R}) : \omega(t),\ t \geq 0 \text{ is a Wiener process with } \omega(0) = 0\}$$

that $\mathbf{P}$ is a Wiener measure and that $W(t, \omega) = \omega(t)$, $t \geq 0$, $\omega \in \mathbf{\Omega}$, where $\mathbb{R}^+ = [0, +\infty)$. In this setting, we define a family of mappings:

$$\theta_t : \mathbf{\Omega} \to \mathbf{\Omega}$$

for any fixed $t \geq 0$, where

$$\theta_t \omega(s) = \omega(s + t) - \omega(t),\ s \geq 0.$$

Obviously, $\theta_t \omega(s)$, $s \geq 0$ is also a Wiener process. Note that $\theta_t \omega$ is adapted and continuous in $t \in \mathbb{R}^+$.

From now on, we note that under normal circumstances, all conditions and conclusions hold almost surely and all random variables and stopping times are almost surely finite.

Suppose that $\sqrt{t \log|\log t|}$ takes the maximum value a_0 at $t = \sigma_0$ in the interval $\left(0, \frac{1}{e}\right]$, where log is the natural logarithm with the base e, $\sigma_0 \in \left(0, \frac{1}{e}\right)$ satisfies

$$1 + \log \sigma_0 \cdot \log|\log \sigma_0| = 0.$$

So, $a_0 = \sqrt{\sigma_0 \log|\log \sigma_0|}$. For the sake of brevity, we define the following two functions:

$$fan_1(t) = \begin{cases} 0, & t = 0, \\ \sqrt{t \log|\log t|}, & 0 < t < \sigma_0, \\ a_0\sqrt{\dfrac{t}{\sigma_0}}, & \sigma_0 \le t \le \dfrac{1}{\sigma_0}, \\ \sqrt{t \log\log t}, & t > \dfrac{1}{\sigma_0}, \end{cases} \qquad fan_2(t) = \begin{cases} 0, & t = 0, \\ \dfrac{fan_1(t)}{\sqrt{t}}, & t > 0. \end{cases}$$

It is verified that $fan_1(t)$ is continuous and increasing in $t \in \mathbb{R}^+$.

First, we provide a theorem, which will be very important in the paper.

Theorem 2.1. *There exists an adapted process* $C(\theta_\tau\omega) \ge \dfrac{\sqrt{2\sigma_0}}{a_0}$, $\tau \ge 0$, *which is sublinear and continuous in* τ *such that for any* $x \in L^2_{loc}$,

$$\left| \int_\tau^t x(s)\, dW(s) \right| \le C(\theta_\tau\omega)\, fan_1\left(\int_\tau^t |x(s)|^2 ds \right) fan_2(t - \tau) \tag{2.1}$$

for any $t \ge \tau \ge 0$.

Proof. By the classic law of iterated logarithm,

$$\mathbf{P}\left(\sup_{\substack{|a|>0 \\ t>0}} \frac{|aW(t)|}{fan_1(a^2 t) fan_2(t)} < +\infty \right) = 1.$$

It is verified that

$$c_0(\omega) = \sup_{\substack{|a|>0 \\ t>0}} \frac{|aW(t)|}{fan_1(a^2 t) fan_2(t)} \ge \frac{\sqrt{2\sigma_0}}{a_0}.$$

For $t \ge \tau \ge 0$,

$$\langle x \rangle_\tau^t = \int_\tau^t |x(s)|^2 ds,$$

where $x \in L^2_{loc}$. Denote by $\mathcal{L}^2_+$ a set of nonrandom step functions satisfying $\langle x \rangle_0^t > 0$ and there exist nonzero constants $t_1' > 0$, b such that $x(t) \equiv b$ for any $t \ge t_1'$.

For any $x \in \mathcal{L}_+^2$ and $t > 0$,

$$\mathbb{E}\left(\frac{\left|\int_0^t x(s)dW(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)}\right)^2 = \frac{1}{fan_2^2(\langle x\rangle_0^t)\,fan_2^2(t)} \le \frac{\sigma_0}{a_0^2}\frac{1}{fan_2^2(t)}. \tag{2.2}$$

By the definition of $\mathcal{L}_+^2$ for every $x \in \mathcal{L}_+^2$, there exists a constant $t_2' > 0$ such that for any $t \in (0, t_2']$,

$$\frac{\left|\int_0^t x(s)dW(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)} \le c_0(\omega). \tag{2.3}$$

By (2.2), (2.3) and by contradiction, we obtain that there exists a constant $t_0 > 0$ such that for any $t \in (0, t_0]$ and any $x \in \mathcal{L}_+^2$,

$$\frac{\left|\int_0^t x(s)dW(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)} \le c_0(\omega) + 1.$$

Let

$$c_1(\omega) = \sup_{\substack{x\in\mathcal{L}_+^2\\ t\in(0,\,t_0]}} \frac{\left|\int_0^t x(s)dW(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)}.$$

Since $\theta_{t_0}\omega(s),\ s \ge 0$ is also a Wiener process,

$$c_1(\theta_{t_0}\omega) = \sup_{\substack{x\in\mathcal{L}_+^2\\ t\in(0,\,t_0]}} \frac{\left|\int_0^t x(s)d\theta_{t_0}\omega(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)}.$$

Note that $fan_1(\langle x\rangle_0^t)\,fan_2(t)$ is continuous in $t \ge 0$ for any $x \in L_{loc}^2$. We obtain that for any nonrandom step function x and $t \in [t_0, 2t_0]$,

$$\left|\int_{t_0}^{t} x(s)\,dW(s) \le c_1(\theta_{t_0}\omega)\right| fan_1(\langle x\rangle_{t_0}^{t})\, fan_2(t-t_0). \tag{2.4}$$

By (2.4) and the increasing property of fan_1 for any $x \in \mathcal{L}_+^2$ and $t \in (t_0,\, 2t_0]$,

$$\frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\, fan_2(t)} \le c_1(\omega)\frac{fan_2(t_0)}{fan_2(t)} + c_1(\theta_{t_0}\omega)\frac{fan_2(t-t_0)}{fan_2(t)}.$$

So, for any fixed $t \in (t_0,\, 2t_0]$,

$$\mathbf{P}\left(\sup_{x\in\mathcal{L}_+^2} \frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\, fan_2(t)} < +\infty\right) = 1.$$

By induction and the continuity, we have that for any constant $\alpha > 0$,

$$\mathbf{P}\left(\sup_{\substack{x\in\mathcal{L}_+^2\\ t\in(0,\alpha]}} \frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\, fan_2(t)} < +\infty\right) = 1.$$

Set

$$c_n(\omega) = \sup_{\substack{x\in\mathcal{L}_+^2\\ t\in(0,nt_0]}} \frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\, fan_2(t)}.$$

For any given $x \in \mathcal{L}_+^2$, there exist constants $t_1' > 0$, $b \neq 0$, $n_0 \in \mathbb{N}$ such that for any $t \ge t_1'$, $x(t) \equiv b$ and

$$\frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\, fan_2(t)} \le \frac{c_{n_0}(\omega) fan_1(\langle x-b\rangle_0^{t_1'})\, fan_2(t_1')}{fan_1(\langle x\rangle_0^t) fan_2(t)} + \frac{fan_1(b^2 t)}{fan_1(\langle x\rangle_0^t)} c_0(\omega),$$

which implies that for every $x \in \mathcal{L}_+^2$, there exists a constant $t_3' > 0$ such that for any $t \geq t_3'$,

$$\frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)} \leq c_0(\omega) + 1. \tag{2.5}$$

By (2.2), (2.5) and by contradiction, we obtain that there exists a constant $T_0 > 0$ such that for any $t \geq T_0$ and any $x \in \mathcal{L}_+^2$,

$$\frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)} \leq c_0(\omega) + 2.$$

So,

$$\mathbf{P}\left(\sup_{\substack{x\in\mathcal{L}_+^2\\ t\geq T_0}} \frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)} < +\infty\right) = 1.$$

Together with the above results, we have

$$\mathbf{P}\left(\sup_{\substack{x\in\mathcal{L}_+^2\\ t>0}} \frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)} < +\infty\right) = 1.$$

Let

$$C(\omega) = \sup_{\substack{x\in\mathcal{L}_+^2\\ t>0}} \frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)}. \tag{2.6}$$

Set

$$L_+^2 = \left\{x \in L_{loc}^2 \,|\, x \text{ is a step function with } \int_0^t |x(s)|^2\,ds > 0\right\}.$$

By (2.6), for any given $x \in L_+^2$,

$$\frac{\left|\int_0^t x(s, \omega_1)\,dW(s, \omega)\right|}{fan_1(\langle x(\cdot, \omega_1)\rangle_0^t)\,fan_2(t)} \leq C(\omega)$$

uniformly on $\omega_1 \in \mathbf{\Omega}$ and $t > 0$, which implies

$$C(\omega) = \sup_{\substack{x \in L_+^2 \\ t>0}} \frac{\left|\int_0^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)}. \tag{2.7}$$

Since $\theta_\tau\omega(s),\ s \geq 0$ is also a Wiener process, by (2.6),

$$C(\theta_\tau\omega) = \sup_{\substack{t>0 \\ x \in \mathcal{L}_+^2}} \frac{\left|\int_0^t x(s)\,d\theta_\tau\omega(s)\right|}{fan_1(\langle x\rangle_0^t)\,fan_2(t)}. \tag{2.8}$$

By $\lim_{t\to+\infty} \frac{|W(t)|}{t} = 0$, $\lim_{\tau\to+\infty} \frac{|\theta_\tau\omega|}{\tau} = 0$. So, by (2.8), $\lim_{\tau\to+\infty} \frac{C(\theta_\tau\omega)}{\tau} = 0$, i.e., $C(\theta_\tau\omega)$ is sublinear in τ. By (2.8) and the definition of Itô integral, $C(\theta_\tau\omega)$ is continuous in τ. Since $\theta_\tau\omega$ is $\mathbf{F}_\tau$-measurable and $C(\theta_\tau\omega)$ only depends on $\theta_\tau\omega$, the process $C(\theta_\tau\omega),\ \tau \geq 0$ is adapted.

For any $x \in L_+^2$ with $\langle x\rangle_\tau^t > 0,\ t > \tau \geq 0,$ by (2.7), (2.8) and

$$\frac{\left|\int_\tau^t x(s)\,dW(s)\right|}{fan_1(\langle x\rangle_\tau^t)\,fan_2(t-\tau)} = \frac{\left|\int_0^{t-\tau} x(s+\tau)\,d\theta_\tau\omega(s)\right|}{fan_1\left(\int_0^{t-\tau} |x(s+\tau)|^2\,ds\right) fan_2(t-\tau)}$$

for $t > \tau \geq 0,$ we have (2.1). By limiting, we can prove that (2.1) holds for any $x \in L_{loc}^2$ and $t \geq \tau \geq 0$. So, the proof is completed. □

Remark 2.1. In Theorem 2.1, the process $C(\theta_\tau\omega)$ is independent of x and the length of the interval $[\tau, t]$.

Remark 2.2. Theorem 2.1 also holds for stopping times t, τ.

3. The a.s. Well-posedness Problem

The following lemma provides a stochastic inequality, which will play an important role in getting various estimates in stochastic differential equations with white noises.

Lemma 3.1. *Suppose that* $|g|^{\frac{1}{2}}$, $|y|^{\frac{1}{2}}$ *and* f *are in* L^2_{loc} *and satisfy*

$$dy = gyds + fydW, \quad s \geq \tau \geq 0. \tag{3.1}$$

Then

$$|y(t)| \leq |y(\tau)| e^{\int_\tau^t \left(g(s) - \frac{1}{2}|f(s)|^2\right) ds + C(\theta_\tau\omega)\, fan_1\left(\int_\tau^t |f(s)|^2 ds\right) fan_2(t-\tau)} \tag{3.2}$$

for $t \geq \tau$, *where* $C(\theta_\tau\omega)$ *is as in Theorem* 2.1.

Proof. We multiply two sides of (3.1) by

$$e^{-\int_\tau^s \left(g(\varsigma) - \frac{1}{2}|f(\varsigma)|^2\right) d\varsigma - \int_\tau^s f(\varsigma)\, dW(\varsigma)}.$$

We can obtain

$$y(t) = y(\tau) e^{\int_\tau^t \left(g(s) - \frac{1}{2}|f(s)|^2\right) ds + \int_\tau^t f(s)\, dW(s)}$$

for $t \geq \tau$. By Theorem 2.1, we get the inequality (3.2). □

Let $\|x\|_t$ be the supremum norm of x in $C[-\tau, t]$, where $\tau > 0$ is a constant. Then $C[-\tau, t]$ is a Banach space. Let b and $\sigma : \mathbb{R}^+ \times \mathbb{R} \times C[-\tau, 0] \times \mathbf{\Omega} \to \mathbb{R}$ be measurable. In this section, we always let $b(t, x, \varphi) = b(t, x, \varphi, \omega)$ and $\sigma(t, x, \varphi) = \sigma(t, x, \varphi, \omega)$. We always assume that $|b(\cdot, x, \varphi)|^{\frac{1}{2}}, \sigma(\cdot, x, \varphi) \in L^2_{loc}$ for every $(x, \varphi) \in \mathbb{R} \times C([-\tau, 0])$. Consider the following stochastic delay differential equations

$$dx = b(t, x, x_t)dt + \sigma(t, x, x_t)dW, \quad t > 0, \quad x_0 = \phi, \tag{3.3}$$

where $\phi \in C[-\tau, 0]$, $\tau > 0$; $x_t = x(t + s)$, $s \in [-\tau, 0]$.

We make the following assumptions on b and σ:

For any $\mathcal{B} = [T_1, T_2] \times [-M, M] \times B_M$, where $M > 0$, $T_2 > T_1 \geq 0$ and $B_M = \{\varphi \in C[-\tau, 0] : \|\varphi\|_0 \leq M\}$, there exists a positive process $L_{\mathcal{B}}(t)$, which is locally square integrable such that

$$|b(t, x_1, \varphi_1) - b(t, x_2, \varphi_2)| + |\sigma(t, x_1, \varphi_1) - \sigma(t, x_2, \varphi_2)|$$
$$\leq L_{\mathcal{B}}(t)(|x_1 - x_2| + \|\varphi_1 - \varphi_2\|_0), (t, x_1, \varphi_1), (t, x_2, \varphi_2) \in \mathcal{B}. \tag{3.4}$$

Furthermore, $L_{\mathcal{B}} \in L^2_{loc}$ if M is $\mathbf{F}_{T_1}$-measurable.

Theorem 3.1. *If* (3.4) *is satisfied, then for any* $\phi \in C[-\tau, 0]$ *with* $\mathbf{F}_0$-*measurable* $\|\phi\|_0$, *there exists a stopping time* $T > 0$ *such that* (3.3) *has a unique solution* $x \in C[-\tau, T]$ *satisfying* $x(t)$ *is adapted on* $[0, T]$ *and* $x_0 = \phi$.

Proof. If $\|\phi\|_0$ is $\mathbf{F}_0$-measurable and x exists, then by

$$x(t) = \phi(0) + \int_0^t b(s, x(s), x_s)ds + \int_0^t \sigma(s, x(s), x_s)dW(s),$$

we obtain that x is adapted and continuous on $[0, T]$.

Now, we prove the existence of solution of (3.3). Let $M = \|\phi\|_0 + 1$ and $\mathcal{B} = [0, 1] \times [-M, M] \times B_M$, where $B_M = \{\varphi \in C[-\tau, 0] : \|\varphi\|_0 \leq M\}$. Let $L(s) = L_{\mathcal{B}}(s)$, which is as in (3.4). Choose

$$T = \sup\left\{t \in (0, 1] : \sup_{\eta \in [0, t]} \left\{\int_0^\eta (2L(s)M + |b(s, 0, 0)|)ds\right.\right.$$
$$\left.\left. + C(\omega)\, fan_1\left(\int_0^\eta (2L(s)M + |\sigma(s, 0, 0)|)^2 ds\right) fan_2(\eta)\right\} < 1\right\},$$

where $C(\omega)$ is as in Theorem 2.1. By Theorem 3 of Preliminaries in [8], $T > 0$ is a stopping time.

Let $\|x\|$ be the supremum norm of x in $C[-\tau, T]$ and

$$\mathbb{C}[-\tau, T] = \{x \in C[-\tau, T] : \|x\| \leq M\}.$$

We define the mapping

$$F : \mathbb{C}[-\tau, T] \to \mathbb{C}[-\tau, T]$$

as the following:

$$(Fx)(t) = \begin{cases} \phi(0) + \int_0^t b(s, x(s), x_s)ds + \int_0^t \sigma(s, x(s), x_s)dW(s), & t > 0, \\ \phi(t), & t \in [-\tau, 0]. \end{cases}$$

For any $x \in \mathbb{C}[0, T]$, by Theorem 2.1, we have

$$|(Fx)(t)| \leq \|\phi\|_0 + \int_0^t (2L(s)M + |b(s, 0, 0)|)ds$$

$$+ C(\omega)\, fan_1\left(\int_0^t (2L(s)M + |\sigma(s, 0, 0)|)^2 ds\right) fan_2(t)$$

$$\leq \|\phi\|_0 + 1.$$

So, $F : \mathbb{C}[-\tau, T] \to \mathbb{C}[-\tau, T]$.

For any $x, y \in \mathbb{C}[-\tau, T]$, by (3.4), we obtain

$$|b(s, y, y_s) - b(s, x, x_s)| + |\sigma(s, y, y_s) - b(s, x, x_s)| \leq 2L(s)\|y - x\|.$$

By Theorem 2.1,

$$|(Fy)(t) - (Fx)(t)| \leq \int_0^t |b(s, y(s), y_s) - b(s, x(s), x_s)|ds$$

$$+ C(\omega)\, fan_1\left(\int_0^t |\sigma(s, y(s), y_s) - b(s, x(s), x_s)|^2 ds\right)$$

$$\leq \|y - x\| \int_0^t L(s)ds$$

$$+ C(\omega)\, fan_1\left(\|y - x\|^2 \int_0^t |L(s)|^2 ds\right) fan_2(t).$$

So,

$$\|Fy - Fx\| \to 0 \text{ as } \|y - x\| \to 0,$$

which implies that F is a continuous mapping from $\mathbb{C}[-\tau, T]$ to itself.

For any constant $\varepsilon > 0$, let

$$\delta = \sup\Bigg\{\eta \in (0, T] : \sup_{t\in[0,T]} \sup_{\kappa\in[0,\eta]} \Bigg(\int_t^{t+\kappa} (2L(s)M + |b(s, 0, 0)|)ds$$

$$+ C(\theta_t\omega)\, fan_1\Bigg(\int_t^{t+\kappa} (2L(s)M + |\sigma(s, 0, 0)|)^2 ds\Bigg) fan_2(\kappa)\Bigg) < \varepsilon\Bigg\}.$$

It is checked that $\delta > 0$ and is a stopping time. For any $x \in \mathbb{C}[-\tau, T]$, by Theorem 2.1, we have

$$|(Fx)(t_2) - (Fx)(t_1)| < \varepsilon,$$

where $t_1, t_2 \in [0, T]$ satisfying $|t_2 - t_1| < \delta$, which implies that F is an equicontinuous mapping from $\mathbb{C}[-\tau, T]$ to itself.

From all the above, we obtain that F is completely continuous on $\mathbb{C}[-\tau, T]$. So, by the stochastic Schauder theorem, F has at least a fixed point $x \in \mathbb{C}[-\tau, T]$ for almost every $\omega \in \Omega$, which is a solution of (3.3) satisfying $x \in C[-\tau, T]$ and $x_0 = \phi$.

We prove the solution of (3.3) is unique. Suppose that $x, y \in \mathbb{C}[-\tau, T]$ are two solutions of (3.3) satisfying $x_0 = y_0 = \phi$. Obviously, $y - x$ satisfies

$$d(y - x) = (b(s, y, y_s) - b(s, x, x_s))ds + (\sigma(s, y, y_s) - \sigma(s, x, x_s))dW. \tag{3.5}$$

Let $t_1 = \inf\{t : y(t) \neq x(t)\}$. Obviously, t_1 is a stopping time. With no generality, let $t_1 \in (0, T)$. By Theorem 2.1, (3.4) and (3.5), we have $|y(t_1) - x(t_1)| = 0$. Let $t_2 = \sup\{t \in (0, T) : y(t) - x(t)$ is monotonous on $[t_1, t]\}$. Obviously, $t_2 > 0$ is a stopping time. With no generality, let $y(t) - x(t)$ be strictly increasing $[t_1, t_2]$. Since $y - x$ is adapted on $[t_1, t_2]$, by Lemma 3.1, (3.4) and (3.5), there exists a positive process $L_1(t)$, which is locally square integrable such that

$$|y(t) - x(t)|$$

$$\leq |y(t_1) - x(t_1)|\, e^{2\int_{t_1}^{t_2} (L_1(s) + |L_1(s)|^2) ds + C(\theta_{t_1}\omega)\, fan_1\left(4\int_{t_1}^{t_2} |L_1(s)|^2 ds\right) fan_2(t_2 - t_1)}$$

for any $t \in [t_1, t_2]$. So, $y(t) = x(t)$, $t \in [t_1, t_2]$, which in contradiction with the definition of t_2. So, $y(t) = x(t)$, $t \in [-\tau, T]$. We make the conclusion. □

Remark 3.1. After some simple modification for the proof of Theorem 3.1, we can make the following result:

Let x be a solution of (3.3) satisfying $|x(t)| \leq M$ for any $t \in [-\tau, T]$, where M is $\mathbf{F}_T$-measurable. If (3.4) is satisfied, then there exists a stopping time $S > 0$ such that x can be prolonged from $[-\tau, T]$ to $[-\tau, T + S]$.

Theorem 3.2. *If* b, σ *satisfy* (3.4), *then for any* $\phi \in C[-\tau, 0]$ *with* $\mathbf{F}_0$-*measurable* $\|\phi\|_0$, *there exists a predictable time* $\xi > 0$ (*finite or infinite*) *such that* (3.3) *has a unique solution* $x \in C[-\tau, \xi)$ *satisfying* $x(t)$ *is adapted on* $[0, \xi)$ *and*

$$\mathbf{P}\left(\xi < \infty \text{ and } \lim_{t \to \xi^-} |x(t)| = \infty\right) = \mathbf{P}(\xi < \infty).$$

Proof. By Remark 3.1, we can prolong $x(t)$ as possible. So, there exists a predictable time ξ (finite or infinite) such that $[0, \xi)$ is the maximal existing interval of the solution of (3.3). Let

$$\Omega_0 = \{\xi < \infty\},$$

$$\Omega_1 = \{\omega \in \Omega_0 : \lim_{t \to \xi^-} \|x\|_t = \infty),$$

$$\Omega_2 = \{\omega \in \Omega_0 : \lim_{t \to \xi^-} |x(t)| = \infty\}.$$

Obviously, $\Omega_2 \subset \Omega_1 \subset \Omega_0$.

First, we prove that $\mathbf{P}(\Omega_0/\Omega_1) = 0$. By contradiction, suppose $\mathbf{P}(\Omega_0/\Omega_1) > 0$. So, there exists a stopping time T_0 satisfying $T_0 = \xi$ in Ω_0/Ω_1 and $\mathbf{F}_{T_0}$-measurable $M > 0$ such that $|x(t)| \leq M$ for $t \in [0, T_0)$. By Remark 3.1, there exists a stopping time $T_1 > 0$ such that we can prolong x from $[0, T_0)$ to $[0, T_0 + T_1)$, which is in contradiction with the definition of $[0, \xi)$ in Ω_0/Ω_1. So, $\mathbf{P}(\Omega_0) = \mathbf{P}(\Omega_1)$.

Second, from the continuity of $x(t)$, we know that in Ω_1, for enough large $n \in \mathbb{N}$, $\tau_n = \sup\{t \geq 0 : |x(t)| \leq n\}$ exists and it satisfies that $\tau_n \leq \tau_{n+1} < \xi$ and $|x(t)| > n$ for all $t \in (\tau_n, \xi)$. By the continuity of $x(t)$ in $[0, \xi)$ and the increasing property of $\{\tau_n\}$, we have $\tau_n \to \xi^-$ as $n \to \infty$ for almost every $\omega \in \Omega_1$. So, $\lim_{t \to \xi^-} |x(t)| = \infty$ for almost every $\omega \in \Omega_1$, which implies that $\mathbf{P}(\Omega_1) = \mathbf{P}(\Omega_2)$. So, $\mathbf{P}(\Omega_2) = \mathbf{P}(\Omega_0)$. □

Theorem 3.3. *If* (3.4) *is satisfied by* b *and* σ, *then the solution mapping* $x_0 \mapsto x$ *is continuous*, *where* $x_0 \in C[-\tau, 0]$ *with* $\mathbf{F}_0$*-measurable* $\| x_0 \|_0$.

Proof. Assume the opposite, i.e., if there exists a set $\Omega_0 \subset \mathbf{\Omega}$ with $\mathbf{P}(\Omega_0) > 0$ such that for almost every fixed $\omega \in \Omega_0$, there exists a sequence $\phi^k \in C[-\tau, 0]$ with $\|\phi^k - x_0\|_0 < \delta \in (0, \varepsilon_0)$ and $\|x^k - x\|_T \geq \varepsilon_0$, where ε_0 is a constant, x^k is the solution of (3.3) with initial value ϕ^k, $[0, T]$ is the existing interval. Without loss of generality, let Ω_0 be the maximal set satisfying the condition.

For some sequence of increasing stopping times $t^k \in (0, T]$, x^k satisfies

$$|x^k(t) - x(t)| < \varepsilon_0, \quad -\tau \leq t < t^k, \qquad \begin{cases} |x^k(t^k) - x(t^k)| = \varepsilon_0, & \omega \in \Omega_0, \\ |x^k(t^k) - x(t^k)| < \varepsilon_0, & \omega \in \mathbf{\Omega} \backslash \Omega_0. \end{cases}$$

The set of $x^k : [-\tau, t^k] \to \mathbb{R}$ is uniformly bounded and equicontinuous for almost every fixed $\omega \in \mathbf{\Omega}$. Therefore, without loss of generality, after passing to the subsequence, we can assume that $t^k \to \bar{t} > 0$ and $x^k \to \bar{x}$ as $k \to \infty$ uniformly on each interval $[0, \tilde{t}] \subset [0, \bar{t}]$. In addition, $\bar{x}$ is uniformly continuous on $[0, \bar{t})$ and therefore can be continuously prolonged on $[0, \bar{t}]$ with $|\bar{x}(\bar{t}) - x(\bar{t})| = \lim_{k \to \infty} |\bar{x}(t^k) - x(t^k)| = \varepsilon_0$ in Ω_0.

Let $\mathcal{B} = [0, T+1] \times [-M, M] \times B_M$, where $M = \sup \| x \|_T + \varepsilon_0$, $B_M = \{\varphi \in C[-\tau, 0] : \| \varphi \|_0 \leq M\}$. Let $L(s) = L_{\mathcal{B}}(s)$, which is as in (3.4). By Theorem 2.1 and (3.4),

$$\left| \int_0^t b(s, x^k(s), x_s^k)\, ds - \int_0^t b(s, \bar{x}(s), \bar{x}_s)\, ds \right|$$

$$\leq \int_0^t L(s)\left(|x^k(s) - \bar{x}(s)| + |x_s^k - \bar{x}_s|\right) ds, \tag{3.6}$$

$$\left| \int_0^t \sigma(s, x^k(s), x_s^k)\, dW(s) - \int_0^t \sigma(s, \bar{x}(s), \bar{x}_s)\, dW(s) \right|$$

$$\leq C(\omega)\, fan_1\left(\int_0^t |L(s)|^2 \left(|x^k(s) - \bar{x}(s)| + |x_s^k - \bar{x}_s|\right)^2 ds \right) fan_2(t), \tag{3.7}$$

where $t \in [0, \tilde{t}]$. The terms at the right-hand side of (3.6) and (3.7) tend to zero as $k \to \infty$. So, $\bar{x}$ is the solution of the following equation:

$$d\bar{x} = b(t, \bar{x}, \bar{x}_t)\, dt + \sigma(t, \bar{x}, \bar{x}_t)\, dW, \quad t > 0, \quad \bar{x}_0 = x_0 = \phi. \tag{3.8}$$

Since the solution of (3.3) is unique, $\bar{x}(t) = x(t)$, $t \in [0, \tilde{t}]$. Since $\tilde{t} \in [0, \bar{t}]$ is arbitrary, we get $\bar{x}(t) = x(t)$, $t \in [0, \bar{t}]$, which is impossible in Ω_0. So, we complete the proof. □

Theorem 3.4. *Suppose that* b, σ *satisfy* (3.4). *Then for any* $\phi \in C[-\tau, 0]$ *with* $\mathbf{F}_0$*-measurable* $\|\phi\|_0$, *there exists a global solution* $x \in C[-\tau, \infty)$ *of* (3.3) *if one of the following conditions holds*:

(i) *There exists a continuous adapted process* $k_1 : [0, \infty) \to (0, \infty)$ *such that*

$$|x(t)| \leq k_1(t), \quad t \geq 0.$$

(ii) *There exists a positive process* $k_2 \in L_{loc}^2$ *such that*

$$|b(t, x_1, \varphi_1) - b(t, x_2, \varphi_2)| + |\sigma(t, x_1, \varphi_1) - \sigma(t, x_2, \varphi_2)|$$

$$\leq k_2(t)\left(|x_2 - x_1| + \|\varphi_2 - \varphi_1\|\right)$$

for $t \geq 0$, $x_1, x_2 \in \mathbb{R}$ *and* $\varphi_1, \varphi_2 \in C[-\tau, 0]$.

Proof. By Theorem 3.1 and Remark 3.1, the conclusion can be made easily. □

Theorem 3.5. *Suppose that* $b, \sigma \in C([0, T] \times \mathbb{R} \times C[-\tau, 0], \mathbb{R})$ *and* (3.3) *has a solution* $x \in C[-\tau, T]$. *Then* $x(t), \ t \geq 0$ *has a version with continuous sample paths*, *whose increment* $x(t+h) - x(t)$ *is an infinitesimal of the order not less than* $|h|^{\frac{1}{2}} \log|\log|h||$ *when* $h \to 0$, *where* $t, t+h \in [0, T]$.

Proof. Let $M = \max\{|x(t)|, -\tau \leq t \leq T\}$ and

$$N = \max\{|b(t, x, x_t)| + |\sigma(t, x, x_t)|, \|x\|_T \leq M, 0 \leq t \leq T\}.$$

By (3.3),

$$x(t+h) - x(t) = \int_t^{t+h} b(s, x(s), x_s)\, ds + \int_t^{t+h} \sigma(s, x(s), x_s)\, dW(s)$$

for any $t, t+h \in [0, T]$. By Theorem 2.1,

$$|x(t+h) - x(t)| \leq N|h| + \sup_{t \in [0, T]} C(\theta_t \omega) \cdot fan_1(N^2|h|)\, fan_2(|h|).$$

By $\lim\limits_{h \to 0+} \dfrac{fan_1(N^2 h)}{\sqrt{h \log|\log h|}} = N$, we make the conclusion. □

Remark 3.2. After some light modification, our results can be generalized to $\mathbb{R}^n$ $(n \geq 2)$ and stochastic delay differential equations such as

$$dx = b(t, x, x_t)\, dt + \sum_{i=1}^{n} \sigma_i(t, x, x_t)\, dW_i, \quad t > 0, \quad x_0 = \phi,$$

where $\phi \in C[-\tau, 0]$, $\tau > 0$; b and $\sigma_i : \mathbb{R}^+ \times \mathbb{R} \times C[-\tau, 0] \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ are measurable and $|b(\cdot, x, \varphi)|^{\frac{1}{2}}, \sigma_i(\cdot, x, \varphi) \in L^2_{loc}$ for every $(x, \varphi) \in \mathbb{R} \times C[-\tau, 0]$; $(W_1, W_2, \ldots, W_n)^T$ is an n-dimensional Wiener process.

References

[1] A. Einstein, Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen, Annalen der Physik 17 (1905), 549-560.

[2] G. Gong, Theory on Stochastic Differential Equations, 2nd ed., Peking University Press, 1995.

[3] J. Hadamard, Sur les problèmes aux dérivés partielles et leur signification physique, Princeton University Bulletin 13 (1902), 49-52.

[4] V. Kolmanovskii and A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations, Springer-Verlag, New York, 1999.

[5] X. Mao, Stochastic Differential Equations and Applications, Horwood, Chichester, 1997.

[6] S. E. A. Mohammed, Stochastic Functional Differential Equations, Pitman, Boston, 1984.

[7] W. Ruess, Existence of solutions to partial functional differential equations with delay, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Lecture Notes Pure Appl. Math. 178, A. G. Kartsatos, ed., Marcel-Dekker, New York, 1996, pp. 259-288.

[8] P. Protter, Stochastic Integration and Differential Equations, Springer-Verlag, Berlin, Heidelberg, New York, 1990.

[9] M. Smoluchowski, Zur kinetischen Theorie der Brownschen Molekularbewegung und der Suspensionen, Annalen der Physik 21 (1906), 756-780.

[10] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.

[11] D. Xu, Z. Yang and Y. Huang, Existence-uniqueness and continuation theorems for stochastic functional differential equations, J. Differential Equations 245 (2008), 1681-1703.