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WHEN IS A SUBSET OF A COMPACT SPACE COMPACT?

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Abstract

In the present work we give a necessary and sufficient condition for the compactness of a not closed subset A of a compact space X.

It is well known that a closed subset of a compact space is compact (see [1, 2] for an elementary introduction to topology). What about subsets which are not closed? It is well known that there exist compact subsets of a compact space which are not closed. In the present note, we give a necessary and sufficient condition for the compactness of a not closed subset A of a compact topological space X. We begin by giving some examples. Consider the interval $S = [0, 1]$ with the finite complement topology (see [3, p. 49]), that is, the topology generated by declaring open the set [0, 1], the empty set $\varnothing$ and all the sets with finite complements. Each subset A of [0, 1] is compact. Indeed, if $\mathcal{C}$ is a cover of A, each open set $G \in \mathcal{C}$ contains all the points of A but a finite number of points $\{x_i\}_{i=1, ..., n}$. Therefore, the family of sets $\{G, G_1, ..., G_n\}$, where $G_i \in \mathcal{C}$ and $x_i \in G_i$ is a finite cover of A.

Another example is the following. Let us consider the set $X = [-1, 1]$ with the

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overlapping interval topology ([3, p. 77]), that is, the topology generated by declaring open the sets of the form $[-1, b)$ for $b > 0$ and $(a, 1]$ for $a < 0$. This means that also the sets of the form (a, b) are open. The space X is compact, since for any open cover, the open sets containing 1 and -1 cover X.

Now, let us consider the subset $A = [-1/2, 1/2]$ and the relative topology of A (a set $Z \subset A$ is open if $Z = G \cap A$, where G is an open set in the topology of X). The subset A is not closed since it is neither of the form $[-1, a]$ nor of the form $[b, 1]$ but it is compact. In order to see this, notice that any covering of A must contain the points $-1/2$ and $1/2$. Moreover, the neighborhoods of $-1/2$ are the sets of the form $[-1, b)$ and (a, b), with $a < -1/2, b > 0$, while the neighborhoods of $1/2$ are the sets of the form $(c, 1]$ and (c, d), with $c < 0, d > 1/2$. Therefore, any cover of A contains one of the following sub-covers:

$$\{[-1, b) \cap A, (c, 1] \cap A\},$$

$$\{[-1, b) \cap A, (c, d] \cap A\},$$

$$\{(a, b) \cap A, (c, 1] \cap A\},$$

$$\{(a, b) \cap A, (c, d) \cap A\}.$$

Notice that the topological spaces considered above are not Hausdorff (or T_2) and this is a general consequence of the fact that a compact subset of a compact Hausdorff space is closed [1, 2]. Moreover, it is worth remarking that X is T_0 but it is not T_1 while S is T_1 and thus T_0.

Theorem 1. *Let (T, τ) be a compact space, $A \subset T$ be a not closed subset of T and τ' be the relative topology of A. Then A is compact if and only if for any open cover $\{G_l \mid G_l \in \tau'\}_{l \in L}$ of A and for any $x \in \overline{A} - A$, there exist a neighborhood V_x of x and a finite subset L_x of L such that $V_x \cap A \subseteq \bigcup_{l \in L_x} G_l$.*

Proof. *Sufficiency*. First, we suppose that the hypothesis of the theorem is true and prove that A is compact. Let $\mathcal{C} := \{G_l \mid G_l \in \tau'\}_{l \in L}$ be an open cover of A. For any $x \in \overline{A} - A$, there exist V_x and $L_x \subset L$, such that L_x is finite and $V_x \cap A \subseteq$

$\bigcup_{l\in L_x} G_l$. Now, we consider the family of sets $\mathcal{C}_1 = \mathcal{C} \bigcup_{x\in\overline{A}-A} \{\bigcup_{l\in L_x} G_l\}$ which we denote by $\mathcal{C}_1 = \{G_m^{(1)}\}_{m\in M}$. The family $\mathcal{C}_1$ is a cover of A such that for any $x \in \overline{A} - A$, there exist a neighborhood V_x of x and a set $G_{m_x}^{(1)} \in \{G_m^{(1)}\}_{m\in M}$ for which $V_x \bigcap A \subset G_{m_x}^{(1)}$. Now, we start from the family $\mathcal{C}_1$ and for each $x \in \overline{A} - A$, we replace the sub-family of sets $\{G_{m_x}^{(1)}\}_{x\in\overline{A}-A} \subset \mathcal{C}_1$ by the family of sets $\{(H^{(1)}(x) \bigcup V_x) \bigcap \overline{A}\}_{x\in\overline{A}-A}$, where $H^{(1)}(x) \in \tau$ is the open set such that $H^{(1)}(x) \bigcap A = G_{m_x}^{(1)}$. We get a family of sets $\widetilde{\mathcal{C}}_1$ which we denote by $\widetilde{\mathcal{C}}_1 = \{\widetilde{G}_s^{(1)}\}_{s\in S}$. Moreover, $\widetilde{\mathcal{C}}_1$ is a cover of $\overline{A}$. Since $\overline{A}$ is closed, it is compact [1, 2]. Therefore, there exists a finite sub-cover $\{\widetilde{G}_j^{(1)}\}_{j=1,\dots,m} \subset \widetilde{\mathcal{C}}_1$ of $\overline{A}$. By setting $G'_j := \widetilde{G}_j^{(1)} \bigcap A$, we get a cover of A. Indeed,

$$\bigcup_{i=1}^m G'_i = \bigcup_{i=1}^m (\widetilde{G}_i^{(1)} \bigcap A) = (\bigcup_{i=1}^m \widetilde{G}_i^{(1)}) \bigcap A = \overline{A} \bigcap A = A.$$

Moreover, $\{G'_j\}_{j=1,\dots,m}$ is a sub-cover of $\mathcal{C}_1$, i.e. $\{G'_j\}_{j=1,\dots,m} \subset \{G_m^{(1)}\}_{m\in M}$. In order to prove this, we recall that $\widetilde{\mathcal{C}}_1$ was obtained from $\mathcal{C}_1$ by replacing the family of sets $\{G_{m_x}^{(1)}\}_{x\in\overline{A}-A}$ by the family of sets $\{(H^{(1)}(x) \bigcup V_x) \bigcap \overline{A}\}_{x\in\overline{A}-A}$. Therefore, $\widetilde{G}_j^{(1)} \in \mathcal{C}_1$ or there exists a point $x_j \in \overline{A} - A$ such that $\widetilde{G}_j^{(1)} = (H^{(1)}(x_j) \bigcup V_{x_j}) \bigcap \overline{A}$. This means that

$$G'_j = [(H^{(1)}(x_j) \bigcup V_{x_j}) \bigcap \overline{A}] \bigcap A = (H^{(1)}(x_j) \bigcup V_{x_j}) \bigcap A$$

$$= (H^{(1)}(x_j) \bigcap A) \bigcup (V_{x_j} \bigcap A) = G_{m_{x_j}}^{(1)} \bigcup (V_{x_j} \bigcap A) = G_{m_{x_j}}^{(1)},$$

where the fact that $(V_{x_j} \bigcap A) \subseteq G_{m_{x_j}}^{(1)}$ was used. Therefore, $\{G'_j\}_{j=1,\dots,m}$ is a cover of A and $\{G'_j\}_{j=1,\dots,m} \subset \mathcal{C}_1$. We recall that $\mathcal{C}_1 = \mathcal{C} \bigcup \{\bigcup_{l\in L_x} G_l\}_{x\in\overline{A}-A}$. This means that the sets in $\mathcal{C}_1$ are either contained in $\mathcal{C}$ or are finite unions of sets contained in $\mathcal{C}$. Since, as we proved above, $\{G'_i\}_{i=1,\dots,m} \subset \mathcal{C}_1$, the same is true for the sets G'_i, that is, either $G'_i \in \mathcal{C}$ or $G'_i = \bigcup_{l \in L_x} G_l$ for some $x \in \overline{A} - A$. Therefore, if we replace in $\{G'_i\}_{i=1,\ldots,n}$ each set of the form $G'_i = \bigcup_{l \in L_x} G_l$ by the family of sets $\{G_l\}_{l \in L_x}$ we get a finite cover of A which is a sub-cover of $\mathcal{C}$.

Necessity. Let $\mathcal{C} = \{G_l \mid G_l \in \tau'\}$ be a cover of A. Since A is compact, there exists a finite sub-cover $\mathcal{C}_1 = \{G_{l_i}\}_{i=1,\ldots,n} \subset \mathcal{C}$. Therefore, for any $x \in \overline{A} - A$ and for any neighborhood V_x of x,

$$V_x \cap A \subset A = \bigcup_{i=1}^{n} G_{l_i}. \qquad \square$$

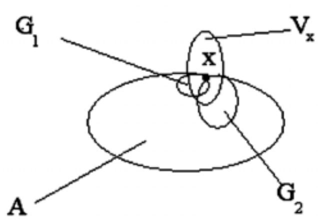

Figure 1. The figure represents a neighborhood V_x of the point x and two open sets $G_1, G_2 \in \mathcal{C}$ which cover $A \in V_x$. It is crucial that $A \cap V_x$ could be covered by a finite number of sets of $\mathcal{C}$ for any cover $\mathcal{C}$ of A.

In order to apply Theorem 1 to a concrete case, let us consider the interval $[-1, 1]$ with the overlapping interval topology introduced above.

Example 1. $X = [-1, 1]$ and the open sets are the sets of the kind $[-1, b)$, $(a, 1]$ and (a, b) with $b > 0$ and $a < 0$. Let us consider the subset $A = [-1/2, 1/2]$. As we showed above, A is not closed but it is compact. Now, we want to prove the compactness of A by using Theorem 1. In order to do this, it is sufficient to show that the hypothesis of the theorem is true. The smallest closed set containing A is X itself, so that $\overline{A} = X$. Let $x \in \overline{A} - A = [-1, -1/2) \cup (1/2, 1]$. In particular, assume $x \in (1/2, 1]$. Notice that, since $1/2 \in A$, any covering of A is of the form $\mathcal{C} = \{(a, 1] \cap A, G_l\}_{l \in L}$ or of the form $\widetilde{\mathcal{C}} = \{(a, b) \cap A, \widetilde{G}_l\}_{l \in L}$, with $b > 1/2$ and $-1/2 < a < 0$ (we are excluding the cases $A \in \mathcal{C}$ and $A \in \widetilde{\mathcal{C}}$). Now, let us consider the neighborhood $V_x = (a, 1]$ of x. We have

$$V_x \cap A = (a, 1] \cap A \in \mathcal{C},$$

$$V_x \cap A = (a, 1] \cap A = (a, 1/2] = (a, b) \cap A \in \mathcal{C}_1$$

so that the hypothesis of the theorem is satisfied in the case $x \in (1/2, 1]$. An analogous reasoning for the case $x \in [-1, -1/2)$ ends the proof.

Notice that the compactness of X in Theorem 1 is crucial for the validity of the theorem. Indeed, the following is an example in which X is not compact and there exists a subset A of X such that the hypothesis of the theorem is satisfied, in spite of the fact that A is not compact.

Example 2. We define a topology on the interval $X = [0, 1]$ by declaring open the set $\varnothing$ and any subset containing the point 0. Therefore, the only closed set containing 0 is X itself. The topological space X is not compact since there is no finite sub-cover of the cover $\mathcal{C} = \{G_1 = \{0, 1\}, \quad G_2 = [0, 1-1/2), \ldots, G_n = [0, 1-1/n), \ldots\}$. Now, let us consider the subset $A = [0, 1)$. In the relative topology, the sets $G_i \cap A = G_i$ are open. Therefore, A is not compact since there is no finite sub-cover of the cover $\mathcal{C} = \{G_2, \ldots, G_n, \ldots\}$. Now, we prove that the hypothesis of Theorem 1 is satisfied. The closure of A is [0, 1] so that $\overline{A} - A = \{1\}$. Moreover, every cover $\mathcal{C} = \{A_l\}_{l \in L}$ of A is such that $0 \in A_l$, $l \in L$. Let us consider the neighborhood $V_1 = \{0, 1\}$ of 1. We have

$$V_1 \cap A = \{0\} \subset A_l, \quad l \in L.$$

Thus, the hypothesis of Theorem 1 is satisfied although A is not compact.

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