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ASYMPTOTIC DICHOTOMY IN A CLASS OF THIRD-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract

Solutions of quite a few higher-order delay functional differential equations either oscillate or tend to $\pm\infty$ as $t \to +\infty$. In this paper, we obtain such a dichotomous criteria for a class of third-order neutral differential equations.

1. Introduction

It has been observed that the solutions of quite a few higher-order delay functional differential equations either oscillate or tend to $\pm\infty$ as $t \to +\infty$. (see, e.g., the recent paper [1] in which a third order nonlinear delay differential equation with damping is considered). Such a dichotomy may yield useful information in real problems (see, e.g., [2, 3] in which

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implications of this dichotomy are applied to the deflection of an elastic beam). Thus, it is of interest to see whether similar dichotomies occur in different types of functional differential equations.

But papers devoted to the study of asymptotic dichotomy in third-order neutral equations are quite rare. For this reason, we study here the third-order neutral differential equation of the form

$$\frac{d^3}{dt^3}\{x(t) - p(t)x(\sigma(t))\} + q(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where p and q are real continuous functions on $[t_0, +\infty)$ such that $p(t) > 0$, $q(t) > 0$; σ and τ are real continuous and strictly increasing functions on $[t_0, +\infty)$ such that $\lim_{t\to+\infty} \sigma(t) = +\infty$ and $\lim_{t\to+\infty} \tau(t) = +\infty$. We establish dichotomous criteria that guarantee solutions of (1) that are either oscillatory or tend to $\pm\infty$ as $t \to +\infty$.

By a solution of (1), we mean a real function $x(t)$ which is defined on $[T, +\infty)$ (where $T = \min\{t_0, \sigma(t_0), \tau(t_0)\}$) and if substituting it into (1) renders it into identity for each $t \in [t_0, +\infty)$.

A solution of (1) is said to be *non-oscillatory* if this solution is eventually positive or eventually negative. Otherwise, this solution is said to be *oscillatory*.

2. Main Results

Our main result of this paper is the following theorem:

Theorem 1. *Assume that* $p(t)$ *is bounded and* $\sigma(t) < \tau(t) < t$ *for* $t \geq t_0$. *Suppose further that there exists a real continuous function* $\rho(t)$ *defined on* $[t_0, +\infty)$ *such that* $\rho(t) > t$ *for* $t \geq t_0$,

$$\liminf_{t\to+\infty} \int_t^{\sigma^{-1}(\tau(t))} \int_u^{\rho(u)} (\xi - u)\frac{q(\xi)}{p(\sigma^{-1}(\tau(\xi)))}d\xi du > \frac{1}{e} \tag{2}$$

and

$$\lim_{t\to+\infty}\sup\int_{t}^{\tau^{-1}(t)}(\xi-t)^{2}q(\xi)d\xi > 2. \tag{3}$$

Then every solution of (1) *either oscillates or tends to* $\pm\infty$ *as* $t \to +\infty$.

Example 1. Consider the third-order neutral differential equation of the form

$$\frac{d^3}{dt^3}\left\{x(t)-\frac{37}{162}x\left(\frac{t}{2}\right)\right\}+x\left(\frac{2t}{3}\right)=0, \quad t \geq 1. \tag{4}$$

It is easy to see that (4) is the form (1) with $p(t)=\frac{37}{162}$, $q(t)=1$, $\sigma(t)=\frac{t}{2}$ and $\tau(t)=\frac{2t}{3}$ for $t \geq 1$. Getting $\rho(t)=2t$, by a direct calculation, we know that

$$\lim_{t\to+\infty}\inf\int_{t}^{\sigma^{-1}(\tau(t))}\int_{u}^{\rho(u)}(\xi-u)\frac{q(\xi)}{p(\sigma^{-1}(\tau(\xi)))}d\xi du$$

$$=\lim_{t\to+\infty}\inf\int_{t}^{\frac{4t}{3}}\int_{u}^{2u}(\xi-u)\frac{162}{37}d\xi du = +\infty$$

and

$$\lim_{t\to+\infty}\sup\int_{t}^{\tau^{-1}(t)}(\xi-t)^{2}q(\xi)d\xi = \lim_{t\to+\infty}\sup\int_{t}^{\frac{3t}{2}}(\xi-t)^{2}d\xi = +\infty.$$

That is, the conditions of Theorem 1 are satisfied. Thus, every solution of (4) either oscillates or tends to $\pm\infty$ as $t \to +\infty$.

In order to give the proof of Theorem 1, we first give the following lemma.

Lemma 1 ([4, 5])**.** *Assume that* q *is a real continuous function on* $[t_0, +\infty)$ *such that* $q(t) > 0$ *for* $t \in [t_0, +\infty)$, δ *is a real continuous function* *on* $[t_0, +\infty)$ *such that* $\delta(t) > t$ *for* $t \in [t_0, +\infty)$ *and* $\lim_{t\to+\infty} \delta(t) = +\infty.$ *Suppose further that*

$$\liminf_{t\to+\infty} \int_t^{\delta(t)} q(s)ds > \frac{1}{e}. \tag{5}$$

Then the differential inequality of the following form:

$$x'(t) - q(t)x(\delta(t)) \geq 0 \tag{6}$$

does not have any eventually positive solutions, and the differential inequality of the following form

$$x'(t) - q(t)x(\delta(t)) \leq 0 \tag{7}$$

does not have any eventually negative solutions.

Lemma 2. *Assume that* σ *is a real continuous and strictly increasing function on* $[t_0, +\infty)$ *such that* $\lim_{t\to+\infty} \sigma(t) = +\infty.$ *Then* $\lim_{t\to+\infty} \sigma^{-1}(t) = +\infty.$

Proof. Since σ is a real continuous and strictly increasing function on $[t_0, +\infty)$ such that $\lim_{t\to+\infty} \sigma(t) = +\infty,$ we know that $\sigma^{-1}(t)$ is also a continuous and strictly increasing function on $[\sigma(t_0), +\infty).$ Thus, $\lim_{t\to+\infty} \sigma^{-1}(t) = L,$ where $L = +\infty$ or $L \in R.$ If $L \in R,$ then we choose a real strictly increasing sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = +\infty$ and

$$t_0 \leq \sigma^{-1}(t_n) \leq L.$$

It is following that

$$\sigma(t_0) \leq t_n \leq \sigma(L), \quad n = 1, 2, \ldots,$$

and relations $\lim_{n\to\infty} t_n = +\infty,$ a contradiction is reached. The proof of Lemma 2 is completed.

We now turn to the proof of Theorem 1. If (1) has a nonoscillatory solution $x(t),$ then, without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1). Note that $\lim_{t\to+\infty}\sigma(t) = +\infty$ and $\lim_{t\to+\infty}\tau(t) = +\infty$, we know that there exists a $T_1 \geq t_0$ such that $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$ for $t \geq T_1$. Let

$$z(t) = x(t) - p(t)x(\sigma(t)). \tag{8}$$

By (1) and (8), we see that

$$z'''(t) < 0, \quad t \geq T_1. \tag{9}$$

Thus $z''(t)$ is strictly decreasing on $[T_1, +\infty)$. It follows that there exists some $T_2 \geq T_1$ such that one of the following statements hold:

(A) $z''(t) < 0$, $t \geq T_2$;

(B) $z''(t) > 0$, $t \geq T_2$.

If (A) holds, then Application Taylor formula with the Lagrange remainder and (9), we know that for $t \geq T_2$, there exists a $\xi : T_2 \leq \xi \leq t$ such that

$$z(t) = z(T_2) + z'(T_2)(t - T_2) + \frac{z''(T_2)}{2}(t - T_2)^2 + \frac{z'''(\xi)}{6}(t - T_2)^3$$

$$\leq z(T_2) + z'(T_2)(t - T_2) + \frac{z''(T_2)}{2}(t - T_2)^2 \to -\infty,\ t \to +\infty.$$

Thus $\lim_{t\to+\infty} z(t) = -\infty$. By (8), we have for $t \geq T_1$,

$$z(t) = x(t) - p(t)x(\sigma(t))$$

$$\geq -p(t)x(\sigma(t))$$

and noting $\lim_{t\to+\infty} z(t) = -\infty$, we see that $\lim_{t\to+\infty} p(t)x(\sigma(t)) = +\infty$. Furthermore, note that $p(t)$ is positive and bounded, we know that $\lim_{t\to+\infty} x(\sigma(t)) = +\infty$. It follows from $\sigma(t)$ is strictly increasing function and $\lim_{t\to+\infty}\sigma(t) = +\infty$ that $\lim_{t\to+\infty} x(t) = +\infty$. If (B) holds, then we must consider the following three cases:

(i) there is a $T_3 \geq T_2$ such that $z'''(t) < 0$, $z''(t) > 0$, $z'(t) > 0$ for $t \geq T_3$;

(ii) there is a $T_3 \geq T_2$ such that $z'''(t) < 0$, $z''(t) > 0$, $z'(t) < 0$, $z(t) > 0$ for $t \geq T_3$;

(iii) there is a $T_3 \geq T_2$ such that $z'''(t) < 0$, $z''(t) > 0$, $z'(t) < 0$, $z(t) < 0$ for $t \geq T_3$.

In case (i), we know that $z'(t) > 0$ for $t \geq T_3$ and $z(t)$ is strictly convex on $[T_3, +\infty)$, therefore for any $t \geq T_3$, we have

$$z(t) = z(T_3) + z'(T_3)(t - T_3) \to +\infty, \quad t \to +\infty,$$

and noting $z(t) < x(t)$ for $t \geq T_3$, we have $\lim_{t \to +\infty} x(t) = +\infty$.

In case (ii), application Taylor formula with integral remainder, we have for $s > t \geq T_3$,

$$\begin{aligned} z(t) &= z(s) + z'(s)(t - s) + \frac{z''(s)}{2}(t - s)^2 - \frac{1}{2}\int_t^s (\xi - t)^2 z'''(\xi)\,d\xi \\ &\geq -\frac{1}{2}\int_t^s (\xi - t)^2 z'''(\xi)\,d\xi \\ &= \frac{1}{2}\int_t^s (\xi - t)^2 q(\xi)x(\tau(\xi))\,d\xi \\ &\geq \frac{1}{2}\int_t^s (\xi - t)^2 q(\xi)z(\tau(\xi))\,d\xi. \end{aligned} \tag{10}$$

Let $s = \tau^{-1}(t)$. Then, from (10), we have

$$\begin{aligned} z(t) &\geq \frac{1}{2}\int_t^{\tau^{-1}(t)} (\xi - t)^2 q(\xi)z(\tau(\xi))\,d\xi \\ &\geq \frac{1}{2} z(t)\int_t^{\tau^{-1}(t)} (\xi - t)^2 q(\xi)\,d\xi. \end{aligned} \tag{11}$$

By (3) and (11), a contradiction is reached.

In case (iii), application Taylor formula with integral remainder, we have for $s > t \geq T_3$,

$$z'(t) = z'(s) - z''(s)(s-t) + \int_t^s (\xi - t) z'''(\xi) d\xi.$$

It follows that

$$z'(t) \leq \int_t^s (\xi - t) z'''(\xi) d\xi, \quad s > t \geq T_3. \tag{12}$$

Note that $z(t) < 0$ for $t \geq T_3$, thus

$$\frac{-z(t)}{p(t)} \leq x(\sigma(t)), \quad t \geq T_3. \tag{13}$$

From Lemma 2 and (13), we know that there is a $T_4 \geq T_3$ such that, for $t \geq T_4$,

$$\frac{-z(\sigma^{-1}(\tau(t)))}{p(\sigma^{-1}(\tau(t)))} \leq x(\tau(t)). \tag{14}$$

By (1), (12) and (14), we have for $s > t \geq T_4$,

$$\begin{aligned} z'(t) &\leq \int_t^s (\xi - t) z'''(\xi) d\xi \\ &= -\int_t^s (\xi - t) q(\xi) x(\tau(\xi)) d\xi \\ &\leq \int_t^s (\xi - t) q(\xi) \frac{z(\sigma^{-1}(\tau(\xi)))}{p(\sigma^{-1}(\tau(\xi)))} d\xi. \end{aligned} \tag{15}$$

By (15) and noting that $z(t)$ is strictly decreasing on $[T_3, +\infty)$, we have $t \geq T_4$,

$$z'(t) \leq z(\sigma^{-1}(\tau(t))) \int_t^{\rho(t)} (\xi - t) \frac{q(\xi)}{p(\sigma^{-1}(\tau(\xi)))} d\xi. \tag{16}$$

That is, $z(t)$ is an eventually positive solution of (16). On the other hand, by (2) and Lemma 1, we know that (16) does not have any eventually positive solutions, a contradiction is reached. The proof of Theorem 1 is completed.

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