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EXACT SOLUTIONS OF THE VARIANT BOUSSINESQ EQUATIONS

XUEQIN ZHAO, FANWEI MENG and CHENDONG TIAN

Department of Mathematics
Qufu Normal University
Qufu 273165, P. R. China
e-mail: qszhaoxueqin@163.com

Abstract

The tanh method is used to find travelling wave solutions of various nonlinear wave equations. In this paper, the extended tanh function method is further improved by picking up new solutions of an auxiliary ordinary differential equation and presenting a general ansätz. The BBM equation and the variant Boussinesq equation are chosen to illustrate the method. As a result, abundant new solitary wave solutions and periodic solutions are obtained.

1. Introduction

Nonlinear wave equations are related to nonlinear phenomena in physics, mechanics, biology, etc. To further explain some physical phenomena, seeking exact solutions of nonlinear wave equations is of great significance and has been a major subject. Many powerful methods have been developed such as inverse scattering method [1], Bäcklund transformation method [2] and Hirota's bilinear method [3]. In recent years, various direct methods

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were given such as homogeneous balance method [4], hyperbolic function expansion method [5, 6], sine-cosine method [7], Jacobian elliptic function expand method [8, 9], tanh method [10], the generalized Riccati equation method [11], the generalized projective Riccati equation method [12], and so on. This is due to the availability of symbolic computation systems like Maple or Mathematica which enable us to perform the complex and tedious computation on computer.

Recently, in [13, 14] starting from an auxiliary equation

$$\left(\frac{d\phi}{d\xi}\right)^2 = A\phi^2 + B\phi^3 + C\phi^4, \tag{1}$$

where A, B, C are real parameters, Sirendaoreji proposed an extended tanh function method (TFM). In [13, 14], Yomba further developed the work made in [8], they only found that equation (1) has two solutions as follows:

$$\phi_1(\xi) = \frac{2A\,\mathrm{sech}(\sqrt{A}\xi)}{\sqrt{B^2 - 4AC} - B\,\mathrm{sech}(\sqrt{A}\xi)}, \quad B^2 - 4AC > 0, \quad A > 0,$$

$$\phi_2(\xi) = \frac{2A\,\mathrm{sech}^2\left(\frac{\sqrt{A}}{2}\xi\right)}{B^2 - AC\left(1 - \tanh\left(\frac{\sqrt{A}}{2}\xi\right)\right)^2}, \quad A > 0.$$

Using their TFM, they only obtained several solitary wave solutions. As we know, when applying direct method, it is important to obtain the more new solutions of auxiliary equations and choose an appropriate ansätz.

Fortunately, by calculating, we find the following eight solutions to equation (1):

$$\phi_1(\xi) = -\frac{2A\,\mathrm{sech}(\sqrt{A}\xi)}{B\,\mathrm{sech}(\sqrt{A}\xi) - \sqrt{B^2 - 4CA}}, \quad A > 0, \quad B^2 - 4AC > 0, \tag{2a}$$

$$\phi_2(\xi) = -\frac{2A\,\mathrm{csch}(\sqrt{A}\xi)}{B\,\mathrm{csch}(\sqrt{A}\xi) + \sqrt{4CA - B^2}}, \quad A > 0, \quad B^2 - 4AC < 0, \tag{2b}$$

$$\phi_3(\xi) = -\frac{2A\sec(\sqrt{-A}\xi)}{B\sec(\sqrt{-A}\xi) - \sqrt{B^2 - 4CA}}, \quad A < 0, \quad B^2 - 4AC > 0, \tag{2c}$$

$$\phi_4(\xi) = -\frac{2A\csc(\sqrt{-A}\xi)}{B\csc(\sqrt{-A}\xi) - \sqrt{B^2 - 4CA}}, \quad A < 0, \quad B^2 - 4AC < 0, \tag{2d}$$

$$\phi_5(\xi) = -\frac{2A\operatorname{sech}(\sqrt{A}\xi)}{B\operatorname{sech}(\sqrt{A}\xi) + \sqrt{4A^2 + 4CA - B^2}\tanh(\sqrt{A}\xi) - 2A},$$

$$A > 0, \quad B^2 - 4A^2 - 4AC < 0, \tag{2e}$$

$$\phi_6(\xi) = -\frac{2A\operatorname{csch}(\sqrt{A}\xi)}{B\operatorname{csch}(\sqrt{A}\xi) - \sqrt{B^2 - 4CA + 4A^2}\coth(\sqrt{A}\xi) + 2A},$$

$$A > 0, \quad B^2 + 4A^2 - 4AC < 0, \tag{2f}$$

$$\phi_7(\xi) = -\frac{2A\sec(\sqrt{-A}\xi)}{B\sec(\sqrt{-A}\xi) + \sqrt{B^2 - 4A^2 - 4CA}\tan(\sqrt{-A}\xi) - 2A},$$

$$A < 0, \quad B^2 - 4A^2 - 4AC > 0, \tag{2g}$$

$$\phi_8(\xi) = -\frac{2A\csc(\sqrt{-A}\xi)}{B\csc(\sqrt{-A}\xi) - \sqrt{B^2 - 4A^2 - 4CA}\cot(\sqrt{-A}\xi) + 2A},$$

$$A < 0, \quad B^2 - 4A^2 - 4AC > 0. \tag{2h}$$

Along this way, in this paper, by introducing eight new solutions of the auxiliary equation and an appropriate anätz, we further improve the method such that it can be used to obtain more types of solutions which contain solitary wave solutions, singulary solitary wave solutions and periodic solutions.

2. Exact Solutions of the Variant Boussinesq Equations

We would like to seek the solutions of the variant Boussinesq equations:

$$h_t + (hu)_x + u_{xxx} = 0, \tag{3a}$$

$$u_t + h_x + uu_x = 0. \tag{3b}$$

The equations was introduced as models for water waves and called *variant Boussinesq* I [15, 16]. Their inverse transformation solutions, soliton solutions, symmetries and conservation laws have been obtained [16-19]. The proposed method gives more new travelling wave solutions for the two equations. By the travelling transformation

$$u = u(\xi), \quad h = h(\xi), \quad \xi = x + \lambda t, \tag{4}$$

equation (3) reduces to the following ordinary differential equations:

$$\lambda h' + u'h + uh' + u''' = 0, \tag{5a}$$

$$\lambda u' + h' + uu' = 0, \tag{5b}$$

where λ is a constant to be determined later, by the balance u''' with uh' in (5a), and balance h' with uu' in (5b), gives $n_1 = 2$, and $n_2 = 1$, so we can assume that

$$h = a_{10} + a_{11}\phi + a_{12}\phi^2 + \frac{c_{11}\sqrt{A\phi^2 + B\phi^3 + C\phi^4}}{\mu_1\phi + 1} + \frac{c_{12}(A\phi^2 + B\phi^3 + C\phi^4)}{(\mu_1\phi + 1)^2}, \tag{6a}$$

$$u = a_{20} + a_{21}\phi + \frac{c_{21}\sqrt{A\phi^2 + B\phi^3 + C\phi^4}}{\mu_1\phi + 1}. \tag{6b}$$

Substituting (6) with (1) into (5), and collecting coefficients of ϕ^i and $\phi^i(\sqrt{a\phi^2 + b\phi^3 + c\phi^4})^j$ $(i = 0, 1, 2,\ j = 0, 1)$, and setting it to be zero, it yields a system of algebraic equations, using the Maple solve the above over-determined equations, we get the following results:

Case 1.

$$C = a_{11} = a_{21} = a_{12} = c_{11} = 0, \quad c_{12} = -\frac{\mu_1^2}{2},$$
$$c_{21} = \mu_1, \quad B = \mu_1 A, \quad a_{10} = -A, \quad \lambda = -a_{20},$$

where A, μ_1, a_{20} are arbitrary constants.

Case 2.

$$c_{21} = c_{11} = a_{10} = 0, \quad \lambda = \frac{a_{21} - 2\mu_1 a_{20}}{2\mu_1}, \quad A = \frac{a_{21}^2}{4\mu_1^2}, \quad B = \frac{a_{21}^2}{2\mu_1},$$

$$C = \frac{a_{21}^2}{4}, \quad a_{12} = -\frac{a_{21}^2(2\mu_1^2 + c_{12})}{4\mu_1^2}, \quad a_{11} = -\frac{a_{21}^2}{2\mu_1},$$

where $\mu_1, c_{12}, a_{20}, a_{21}$ are arbitrary constants.

Substituting above two sets of solutions with (2) into (6), we obtain exact travelling wave solutions of the variant Boussinesq equations (3) as follows.

Family 1.

$$u_1 = a_{20} + \frac{2\mu_1 A^{3/2}\operatorname{sech}(\sqrt{A}\xi)\tanh(\sqrt{A}\xi)}{(\operatorname{sech}(\sqrt{A}\xi) - 1)[\mu_1 A\operatorname{sech}(\sqrt{A}\xi) + \mu_1 A]},$$

$$h_1 = -A - \frac{2\mu_1^2 A^3 \tanh^2(\sqrt{A}\xi)\operatorname{sech}^2(\sqrt{A}\xi)}{[\mu_1 A\operatorname{sech}(\sqrt{A}\xi) + \mu_1 A]^2(\operatorname{sech}(\sqrt{A}\xi) - 1)^2},$$

where $A > 0$.

Family 2.

$$u_2 = a_{20} - \frac{2\mu_1^2 A^2\sec(\sqrt{-A}\xi)\tan(\sqrt{-A}\xi)\sqrt{-A}}{(\sec(\sqrt{-A}\xi) - 1)[\mu_1 A\sec(\sqrt{-A}\xi) + \mu_1 A]},$$

$$h_2 = -A + \frac{2\mu_1^4 B^2 A^6\tan^2(\sqrt{-A}\xi)\sec^2(\sqrt{-A}\xi)}{(\sec(\sqrt{-A}\xi) - 1)^2[\mu_1 A\sec(\sqrt{-A}\xi) + \mu_1 A]^2},$$

where $A < 0$.

Family 3.

$$u_3 = a_{20} + 2\frac{\mu_1 A\csc(\sqrt{-A}\xi)\cot(\sqrt{-A}\xi)\sqrt{-A}\sqrt{B^2}}{(\csc(\sqrt{A}\xi) - 1)[\mu_1 A\csc(\sqrt{A}\xi) - \mu_1 A]},$$

$$h_3 = -A + 2\frac{\mu_1^2 A^3\cot^2(\sqrt{-A}\xi)\csc^2(\sqrt{-A}\xi)}{(\csc(\sqrt{-A}\xi) - 1)^2[\mu_1 A\csc(\sqrt{-A}\xi) - \mu_1 A]^2},$$

where $A < 0$.

Family 4.

$$u_4 = a_{20} + \frac{2\mu_1 A^{3/2}\operatorname{sech}(\sqrt{A}\xi)(A\sqrt{4-\mu_1^2} - 2A\tanh(\sqrt{A}\xi))}{\begin{array}{c}[-\mu_1 A\operatorname{sech}(\sqrt{A}\xi) + A\sqrt{4-\mu_1^2}\tanh(\sqrt{A}\xi) - 2A] \\ \times (B\operatorname{sech}(\sqrt{A}\xi) + A\sqrt{4-\mu_1^2}\tanh(\sqrt{A}\xi) - 2A)\end{array}},$$

$$h_4 = -A - \frac{2\mu_1^2 A^3\operatorname{sech}^2(\sqrt{A}\xi)(A\sqrt{4-\mu_1^2} - 2A\tanh(\sqrt{A}\xi))^2}{\begin{array}{c}[-\mu_1 A\operatorname{sech}(\sqrt{A}\xi) + A\sqrt{4-\mu_1^2}\tanh(\sqrt{A}\xi) - 2A]^2 \\ \times (B\operatorname{sech}(\sqrt{A}\xi) + A\sqrt{4-\mu_1^2}\tanh(\sqrt{A}\xi) - 2A)^2\end{array}},$$

where $A > 0,\ 4 - \mu_1^2 > 0.$

Family 5.

$$u_5 = a_{20} - \frac{2\mu_1 A^{3/2}\operatorname{csch}(\sqrt{A}\xi)(A\sqrt{4+\mu_1^2} - 2A\coth(\sqrt{A}\xi))}{\begin{array}{c}[\mu_1 A\operatorname{csch}(\sqrt{A}\xi) + A\sqrt{4+\mu_1^2}\coth(\sqrt{A}\xi) - 2A] \\ \times (-B\operatorname{csch}(\sqrt{A}\xi) A\sqrt{4+\mu_1^2}\coth(\sqrt{A}\xi) - 2A)\end{array}},$$

$$h_5 = -A - \frac{2\mu_1^2 A^3\operatorname{csch}^2(\sqrt{A}\xi)(A\sqrt{4+\mu_1^2} - 2A\coth(\sqrt{A}\xi))^2}{\begin{array}{c}[\mu_1 A\operatorname{csch}(\sqrt{A}\xi) + A\sqrt{4+\mu_1^2}\coth(\sqrt{A}\xi) - 2A]^2 \\ \times (-B\operatorname{csch}(\sqrt{A}\xi) + A\sqrt{4+\mu_1^2}\coth(\sqrt{A}\xi) - 2A)^2\end{array}},$$

where $A > 0.$

Family 6.

$$u_6 = a_{20} - \frac{2\mu_1 A\sqrt{-A}\sec(\sqrt{-A}\xi)(2A\tan(\sqrt{-A}\xi) - A\sqrt{\mu_1^2 - 4})}{\begin{array}{c}[\mu_1 A\sec(\sqrt{-A}\xi) + A\sqrt{\mu_1^2 - 4}\tan(\sqrt{-A}\xi) + 2A] \\ \times (B\sec(\sqrt{-A}\xi) - A\sqrt{\mu_1^2 - 4}\tan(\sqrt{-A}\xi) - 2A)\end{array}},$$

$$h_6 = -A + \frac{2\mu_1^2 A^3\sec^2(\sqrt{-A}\xi)(2A\tan(\sqrt{-A}\xi) - A\sqrt{\mu_1^2 - 4})^2}{\begin{array}{c}[\mu_1 A\sec(\sqrt{-A}\xi) + A\sqrt{\mu_1^2 - 4}\tan(\sqrt{-A}\xi) + 2A]^2 \\ \times (B\sec(\sqrt{-A}\xi) - A\sqrt{\mu_1^2 - 4}\tan(\sqrt{-A}\xi) - 2A)^2\end{array}},$$

where $A < 0,\ \mu_1^2 - 4 > 0.$

Family 7.

$$u_7 = a_{20} - \frac{2\mu_1 A\sqrt{-A}\csc(\sqrt{-A}\xi)(2A\cot(\sqrt{-A}\xi) - A\sqrt{\mu_1^2 - 4})}{[(2\mu_1 A - B)\csc(\sqrt{-A}\xi) - A\sqrt{\mu_1^2 - 4}\cot(\sqrt{-A}\xi) - 2A] \times (B\csc(\sqrt{-A}\xi) + A\sqrt{\mu_1^2 - 4}\cot(\sqrt{-A}\xi) + 2A)},$$

$$h_7 = -A + \frac{2\mu_1^2 A^3 \csc^2(\sqrt{-A}\xi)(2A\cot(\sqrt{-A}\xi) - A\sqrt{\mu_1^2 - 4})^2}{[(2\mu_1 A - B)\csc(\sqrt{-A}\xi) - A\sqrt{\mu_1^2 - 4}\cot(\sqrt{-A}\xi) - 2A]^2 \times (B\csc(\sqrt{-A}\xi) + A\sqrt{\mu_1^2 - 4}\cot(\sqrt{-A}\xi) + 2A)^2},$$

where $A < 0,\ \mu_1^2 - 4 > 0$.

In Family 1 ~ Family 7, $\xi = x - a_{20}t$.

Family 8.

$$u_8 = a_{20} - \frac{2a_{21}A\operatorname{sech}(\sqrt{A}\xi)}{B\operatorname{sech}(\sqrt{A}\xi) + \Phi\tanh(\sqrt{A}\xi) - 2A},$$

$$h_8 = \frac{a_{21}^2 A\operatorname{sech}(\sqrt{A}\xi)}{\mu_1(B\operatorname{sech}(\sqrt{A}\xi) + \Phi\tanh(\sqrt{A}\xi) - 2A)}$$

$$- \frac{a_{21}^2(2\mu_1^2 + c_{12})A^2\operatorname{sech}^2(\sqrt{A}\xi)}{\mu_1^2(B\operatorname{sech}(\sqrt{A}\xi) + \Phi\tanh(\sqrt{A}\xi) - 2A)^2}$$

$$+ \frac{4c_{12}A^3(2A\tanh(\sqrt{A}\xi) - \Phi)^2\operatorname{sech}^2(\sqrt{A}\xi)}{[(2\mu_1 A - B)\operatorname{sech}(\sqrt{A}\xi) - \Phi\tanh(\sqrt{A}\xi) + 2A]^2 \times (B\operatorname{sech}(\sqrt{A}\xi) + \Phi\tanh(\sqrt{A}\xi) - 2A)^2},$$

where $A > 0,\ a_{21} \neq 0$.

Family 9.

$$u_9 = a_{20} - \frac{2a_{21}A\operatorname{csch}(\sqrt{A}\xi)}{B\operatorname{csch}(\sqrt{A}\xi) - \Omega\coth(\sqrt{A}\xi) + 2A},$$

$$h_9 = \frac{a_{21}^2 A\operatorname{csch}(\sqrt{A}\xi)}{\mu_1(B\operatorname{csch}(\sqrt{A}\xi) - \Omega\coth(\sqrt{A}\xi) + 2A)}$$

$$- \frac{a_{21}^2(2\mu_1^2 + c_{12})A^2\operatorname{csch}^2(\sqrt{A}\xi)}{\mu_1^2(B\operatorname{csch}(\sqrt{A}\xi) - \Omega\coth(\sqrt{A}\xi) + 2A)^2}$$

$$+ \frac{4c_{12}A^3(2A\coth(\sqrt{A}\xi) - \Omega)^2\operatorname{csch}^3(\sqrt{A}\xi)}{[(2\mu_1 A - B)\operatorname{csch}(\sqrt{A}\xi) + \Omega\coth(\sqrt{A}\xi) - 2A]^2 \times (B\operatorname{csch}(\sqrt{A}\xi) - \Omega\coth(\sqrt{A}\xi) + 2A)^2},$$

where $A > 0$, $a_{21} \neq 0$.

In Families 8 and 9, $\xi = x + \dfrac{a_{21} - 2\mu_1 a_{20}}{2\mu_1}t.$

Remark. Among the above solutions, Families 1, 4 and 8 are solitary wave solutions, Families 5 and 9 are singulary solitary wave solutions, and Families 2, 3, 6 and 7 are periodic solutions. These nine families solutions are new, and cannot be obtained by other method such as tanh function method, various tanh function methods, and hyperbolic function method. These solutions have not been given in literature.

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