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ON CHEBYSHEV VARIETIES, III: CHARACTERIZATION

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Abstract

We consider the problem whether or not the Chebyshev varieties V_n, introduced in Part I of this article, are characterized by their integral points. We show that for any $n \geq 2$, square-free polynomials of n variables which vanish on $V_n(\mathbf{Z})$ constitute a two-dimensional $\mathbf{Q}$-vector space generated by the defining equation $u(x_1, \ldots, x_n)$ of V_n together with $u(x_1, \ldots, x_{n-1}) - u(x_2, \ldots, x_n)$.

1. Introduction

This paper is a continuation of our previous papers [1, 2]. In Parts I and II, we investigated the arithmetic and geometry of a family of hypersurfaces $V_n \subset \mathbf{A}^n$, $n \geq 1$, called the *Chebyshev varieties* (of the second kind). We found, among other things, that there exists a finite number of linear subvarieties $L_i, 1 \leq i \leq m$, of dimension $\leq (n-1)/2$ such that $V_n(\mathbf{Z}) = \bigcup_{1 \leq i \leq m} L_i(\mathbf{Z})$. (See Theorem 2.5 for more precise statement.) For instance, $V_3(\mathbf{Z})$ turns out to be the union $L_1(\mathbf{Z}) \cup L_2(\mathbf{Z}) \cup \{P_1, P_2, P_3, P_4\}$ of two lines and four points, where $L_1 = \{x_1 = x_3 = 0\}$, $L_2 = \{x_1 + x_3 = x_2 = 0\}$, $\{P_1, P_2, P_3, P_4\} = \{(\pm 1, \pm 2, \pm 1), (\pm 2, \pm 1, \pm 2)\}$.

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In the present paper, we consider the problem whether or not the Chebyshev varieties are characterized by their integral points. We show that for any $n \geq 2$, square-free polynomials of n variables which vanish on $V_n(\mathbf{Z})$ constitute a two-dimensional $\mathbf{Q}$-vector space generated by the defining equation $u(x_1, \ldots, x_n)$ of V_n and $u(x_1, \ldots, x_{n-1}) - u(x_2, \ldots, x_n)$. When $n = 3$, our result specializes to the claim that if $f \in \mathbf{Q}[x_1, x_2, x_3]$ consists solely of square-free terms and the zero locus $V(f) = \{f = 0\} \subset \mathbf{A}^3$ contains two lines $\{x_1 = x_3 = 0\}$, $\{x_1 + x_3 = x_2 = 0\}$ and four points $(\pm 1, \pm 2, \pm 1)$, $(\pm 2, \pm 1, \pm 2)$, then f is a $\mathbf{Q}$-linear combination of $u(x_1, x_2, x_3) = x_1x_2x_3 - x_1 - x_3$ and $u(x_1, x_2) - u(x_2, x_3) = x_1x_2 - x_2x_3$. Thus our main result of this paper may be regarded as a natural generalization of this rather straightforward claim, and will provide one with an infinite family of varieties characterized by their integral points.

The plan of the paper is as follows. In Section 2 we recall the definition and some fundamental properties of the Chebyshev varieties. In Section 3, we check that $u(x_1, \ldots, x_{n-1}) - u(x_2, \ldots, x_n)$ as well as $u(x_1, \ldots, x_n)$ vanish on $V_n(\mathbf{Z})$ for any $n \geq 2$, and formulate the main theorem. In Section 4 we prove the theorem. Some polynomial identities satisfied by $u(x_1, \ldots, x_n)$, which is established in Section 2, play an essential role in the proof.

2. Chebyshev Varieties and their Integral Points

In this section, we recall the definition and some fundamental properties of Chebyshev varieties.

Let k be a field of arbitrary characteristic, and let $\mathbf{A}^n = \bar{k}^n$ be the affine space of dimension n over $\bar{k}$. For any n independent variables $x_1, x_2, \ldots, x_n$, let

$$U(x_1, x_2, \ldots, x_n) = \begin{pmatrix} x_1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & x_2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & x_3 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & x_{n-1} & -1 \\ 0 & 0 & 0 & \cdots & -1 & x_n \end{pmatrix},$$

and let

$$u(x_1, x_2, \ldots, x_n) = \det U(x_1, x_2, \ldots, x_n).$$

Let $V_n = \{u(x_1, x_2, \ldots, x_n) = 0\} \subset \mathbf{A}^n$, and call it to be the *Chebyshev variety* (of the second kind). We recall some properties of u.

Proposition 2.1 [1, Proposition 2.1]**.**

(i) $u(x_1, x_2, \ldots, x_n) = u(x_n, x_{n-1}, \ldots, x_1)$.

(ii) $u(x_1, x_2, \ldots, x_n) = x_1 u(x_2, \ldots, x_n) - u(x_3, \ldots, x_n)$.

(iii) $u(x_1, x_2, \ldots, x_n) = x_n u(x_1, \ldots, x_{n-1}) - u(x_1, \ldots, x_{n-2})$.

The symmetry stated in (i) will play a role when we characterize the Chebyshev varieties by their integral points. The following assertion follows easily from (ii) by induction on n.

Corollary 2.1.1. *The degree of* $u(x_1, x_2, \ldots, x_n)$ *is equal to n and the part of degree n of* $u(x_1, x_2, \ldots, x_n)$ *consists solely of* $x_1 x_2 \cdots x_n$. *Moreover every term of u is of degree congruent to n mod 2.*

The next proposition shows that there is a simple formula for the partial derivatives of u.

Proposition 2.2. *For any k with* $1 \leq k \leq n$, *we have*

$$\frac{\partial}{\partial x_k} u[1, n] = u[1, k-1]u[k+1, n]. \tag{2.1}$$

This implies that there is a strong restriction on the shape of $u(x_1, \ldots, x_n)$.

Corollary 2.2.1. *Every term of the polynomial* $u(x_1, \ldots, x_n)$ *is square-free.*

In the previous article [1], we introduced three families of maps between the Chebyshev varieties in order to investigate the set of integral points on them. We recall their definitions and basic properties. (We have changed the notation a little in order to indicate the dimension of their domains of definition.)

Theorem 2.3 [1, Definition 2.9, Theorem 2.11]**.** *For any* $n \geq 1$, *we define three maps*

$$(\textit{Blow-up}) \quad blup^n_{(i;\pm)} : \mathbf{A}^n \to \mathbf{A}^{n+1}, \ \ 1 \leq i \leq n+1,$$

$$(\textit{Splitting}) \quad split^n_{(i;c)} : \mathbf{A}^n \to \mathbf{A}^{n+2}, \ \ 1 \leq i \leq n,\ c \in k,$$

$$(\textit{Pasting}) \quad paste^n_{(\pm;c)} : \mathbf{A}^n \to A^{n+2}, \ \ c \in k,$$

by the following rules:

(i) $blup^n_{(i;\pm)}(x_1, \dots, x_n) = (x_1, \dots, x_{i-2}, x_{i-1} \pm 1, \pm 1, x_i \pm 1, x_{i+1}, \dots, x_n),$
$2 \leq i \leq n,$

$$blup^n_{(1;\pm)}(x_1, \dots, x_n) = (\pm 1, x_1 \pm 1, x_2, \dots, x_n),$$

$$blup^n_{(n+1;\pm)}(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n \pm 1, \pm 1),$$

(ii) $split^n_{(i;c)}(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_i - c, 0, c, x_{i+1}, \dots, x_n), 1 \leq i \leq n,$

(iii) $paste^n_{(-;c)}(x_1, \dots, x_n) = (0, c, x_1, \dots, x_n),$

$$paste^n_{(+;c)}(x_1, \dots, x_n) = (x_1, \dots, x_n, c, 0).$$

Then we have

(iv) $u(blup^n_{(i;\pm)}(x_1, \dots, x_n)) = \pm u(x_1, \dots, x_n), \ \ 1 \leq i \leq n+1,$

(v) $u(split^n_{(i;c)}(x_1, \dots, x_n)) = -u(x_1, \dots, x_n), \ \ 1 \leq i \leq n,\ c \in k,$

(vi) $u(paste^n_{(\pm;c)}(x_1, \dots, x_n)) = -u(x_1, \dots, x_n), \ \ c \in k.$

In particular, these families of maps restrict to morphisms between the Chebyshev varieties:

$$blup^n_{(i;\pm)} : V_n \to V_{n+1}, \ \ split^n_{(i;c)} : V_n \to V_{n+2}, \ \ paste^n_{(\pm;c)} : V_n \to V_{n+2}.$$

From now on, sometimes for ease of description, we write $[1, n]$ for $u(x_1, \dots, x_n)$.

Next we introduce another family of polynomials which satisfy transformation rules similar to those for $u(x_1, \ldots, x_n)$. They are defined by

$$v(x_1, \ldots, x_n) = u(x_1, \ldots, x_{n-1}) - u(x_2, \ldots, x_n)$$

for any $n \geq 2$. We show that they obey the following transformation rules.

Proposition 2.4.

(i) $$v(blup^n_{(i;\pm)}(x_1, \ldots, x_n)) = \begin{cases} \pm v(x_1, \ldots, x_n), & 2 \leq i \leq n, \\ \pm v(x_1, \ldots, x_n) - u(x_1, \ldots, x_n), & i = 1, \\ \pm v(x_1, \ldots, x_n) + u(x_1, \ldots, x_n), & i = n+1, \end{cases}$$

(ii) $v(split^n_{(i;c)}(x_1, \ldots, x_n)) = -v(x_1, \ldots, x_n)$, $1 \leq i \leq n$, $c \in k$,

(iii) $v(paste^n_{(-;c)}(x_1, \ldots, x_n)) = -v(x_1, \ldots, x_n) - cu(x_1, \ldots, x_n)$, $c \in k$,

$v(paste^n_{(+;c)}(x_1, \ldots, x_n)) = -v(x_1, \ldots, x_n) + cu(x_1, \ldots, x_n)$, $c \in k$.

Proof. (i) We give a proof of the transformation formula for $blup^n_{(i;+)}$, since that for $blup^n_{(i;-)}$ can be proved similarly. When $2 \leq i \leq n$, the assertion follows from Theorem 2.3(iv). When $i = 1$, we compute as follows by using Theorem 2.3(iv) together with Proposition 2.1,

$$\begin{aligned} & v(blup^n_{(1;+)}(x_1, \ldots, x_n)) \\ &= v(1, x_1 + 1, \ldots, x_n) \\ &= u(1, x_1 + 1, \ldots, x_{n-1}) - u(x_1 + 1, \ldots, x_n) \\ &= u(x_1, \ldots, x_{n-1}) - ((x_1 + 1)u(x_2, \ldots, x_n) - u(x_3, \ldots, x_n)) \\ &= (u(x_1, \ldots, x_{n-1}) - u(x_2, \ldots, x_n)) - (x_1 u(x_2, \ldots, x_n) - u(x_3, \ldots, x_n)) \\ &= v(x_1, \ldots, x_n) - u(x_1, \ldots, x_n). \end{aligned}$$

The case $i = n + 1$ can be treated similarly.

(ii) When $2 \leq i \leq n-1$, the assertion follows from Theorem 2.3(v). When $i = 1$, we can compute as follows:

$$\begin{aligned} v(split^n_{(1;c)}(x_1, \ldots, x_n)) &= v(x_1 - c, 0, c, x_2, \ldots, x_n) \\ &= u(x_1 - c, 0, c, x_2, \ldots, x_{n-1}) - u(0, c, x_2, \ldots, x_n) \\ &= -u(x_1, x_2, \ldots, x_{n-1}) + u(x_2, \ldots, x_n) \\ &\qquad \text{(by Theorem 2.3(v) and (vi))} \\ &= -v(x_1, x_2, \ldots, x_n). \end{aligned}$$

When $i = n$, we compute as follows:

$$\begin{aligned} v(split^n_{(n;c)}(x_1, \ldots, x_n)) &= v(x_1, \ldots, x_n - c, 0, c) \\ &= u(x_1, \ldots, x_n - c, 0) - u(x_2, \ldots, x_n - c, 0, c) \\ &= -u(x_1, x_2, \ldots, x_{n-1}) + u(x_2, \ldots, x_n) \\ &\qquad \text{(by Proposition 2.1(iii) and Theorem 2.3(v))} \\ &= -v(x_1, x_2, \ldots, x_n). \end{aligned}$$

(iii) The first assertion is assured by the following computation:

$$\begin{aligned} v(paste^n_{(-;c)}(x_1, \ldots, x_n)) &= v(0, c, x_1, \ldots, x_n) \\ &= u(0, c, x_1, \ldots, x_{n-1}) - u(c, x_1, \ldots, x_n) \\ &= -u(x_1, \ldots, x_{n-1}) - (cu(x_1, \ldots, x_n) - u(x_2, \ldots, x_n)) \\ &= -v(x_1, \ldots, x_n) - cu(x_1, \ldots, x_n). \end{aligned}$$

For the second assertion we compute as follows:

$$\begin{aligned} v(paste^n_{(+;c)}(x_1, \ldots, x_n)) &= v(x_1, \ldots, x_n, c, 0) \\ &= u(x_1, \ldots, x_n, c) - u(x_2, \ldots, x_n, c, 0) \\ &= (cu(x_1, \ldots, x_n) - u(x_1, \ldots, x_{n-1})) + u(x_2, \ldots, x_n) \\ &= -v(x_1, \ldots, x_n) + cu(x_1, \ldots, x_n). \end{aligned}$$

This completes the proof of Proposition 2.4.

When the base field is $k = \mathbf{Q}$, the set of integral points on V_n is determined in [1] as follows. For any variety V defined over $\mathbf{Q}$, we denote the set of integral points on V by $V(\mathbf{Z})$.

Theorem 2.5 [1, Theorem 3.3]**.** *The set of integral points on the Chebyshev variety V_n is given by the following*:

(i) $V_1(\mathbf{Z}) = \{0\}$,

(ii) $V_2(\mathbf{Z}) = \{(1, 1), (-1, -1)\}$,

(iii) $$V_n(\mathbf{Z}) = \left(\bigcup_{1 \le i \le n} blup_{(i;\pm)}^{n-1}(V_{n-1}(\mathbf{Z}))\right) \cup \left(\bigcup_{\substack{1 \le i \le n-2 \\ c \in \mathbf{Z}}} split_{(i;c)}^{n-2}(V_{n-2}(\mathbf{Z}))\right)$$
$$\cup \left(\bigcup_{c \in \mathbf{Z}} paste_{(\pm;c)}^{n-2}(V_{n-2}(\mathbf{Z}))\right),$$

for any $n \ge 3$. In particular, $V_n(\mathbf{Z})$ is stable under the map $r : (x_1, x_2, \ldots, x_n) \mapsto (x_n, x_{n-1}, \ldots, x_1)$.

Together with Proposition 2.4 this implies the following by induction on n.

Theorem 2.6. *For any $n \ge 2$, the polynomial $v(x_1, \ldots, x_n)$ vanishes on $V_n(\mathbf{Z})$.*

One of the main purposes of this paper is to show that under some condition u and v are the only polynomials which vanish on $V_n(\mathbf{Z})$.

3. Family of Varieties Determined by their Integral Points

In this section, we discuss the question: are the Chebyshev varieties characterized by the integral points on them?

In order to formulate our result we introduce some notion. For any integers $n \ge 1$, $d \ge 0$, let

$$\mathbf{P}_{sqf}(n, d) = \{f \in \mathbf{Q}[x_1, \ldots, x_n];\ \deg f = d, \text{ each term of } f \text{ is square-free}\},$$

and let

$$\mathbf{P}_{sqf}(n) = \bigcup_{d \geq 0} \mathbf{P}_{sqf}(n, d) \left(= \bigcup_{0 \leq d \leq n} \mathbf{P}_{sqf}(n, d) \right).$$

A polynomial $f \in \mathbf{Q}[x_1, \ldots, x_n]$ is said to be *consistent* if

$$f(-x_1, \ldots, -x_n) = (-1)^n f(x_1, \ldots, x_n),$$

and is said to be *anti-consistent* if

$$f(-x_1, \ldots, -x_n) = (-1)^{n-1} f(x_1, \ldots, x_n).$$

In other words, $f \in \mathbf{Q}[x_1, \ldots, x_n]$ is *consistent* (resp. *anti-consistent*) if it has the same (resp. opposite) parity as $x_1 \cdots x_n$. Furthermore, let

$$\mathbf{I}(n) = \{f \in \mathbf{Q}[x_1, \ldots, x_n];\ f(P) = 0 \text{ for any } P \in V_n(\mathbf{Z})\},$$

$$\mathbf{I}_{sqf}(n) = \mathbf{I}(n) \cap \mathbf{P}_{sqf}(n),$$

$$\mathbf{I}_{sqf}(n)_{cons} = \mathbf{I}_{sqf}(n) \cap \{f;\ f \text{ is consistent}\},$$

$$\mathbf{I}_{sqf}(n)_{anti} = \mathbf{I}_{sqf}(n) \cap \{f;\ f \text{ is anti-consistent}\}.$$

Our main result in this paper is formulated as follows:

Theorem 3.1. *For any* $n \geq 2$, $\mathbf{I}_{sqf}(n)$ *is a two-dimensional* $\mathbf{Q}$*-vector space generated by* $[1, n]$ *and* $[1, n-1] - [2, n]$. *More precisely, we have*

$$\mathbf{I}_{sqf}(n)_{cons} = \mathbf{Q} \cdot [1, n],$$

$$\mathbf{I}_{sqf}(n)_{anti} = \mathbf{Q} \cdot ([1, n-1] - [2, n]).$$

The next section is devoted to prove this theorem.

4. Proof of Theorem 3.1

In this section, we give a proof of Theorem 3.1 by induction on n.

Case 1. $n = 2$. The most general element f of $\mathbf{P}_{sqf}(2)$ is written as

$$f = a_0 + a_1 x_1 + a_2 x_2 + a_{12} x_1 x_2.$$

In order for f to belong to $\mathbf{I}(2)$, it must satisfy $f(1, 1) = f(-1, -1) = 0$, since $V_2(\mathbf{Z}) = \{(1, 1), (-1, -1)\}$ by Theorem 2.5(ii). It follows that

$$f(1, 1) = a_0 + a_1 + a_2 + a_{12} = 0,$$

$$f(-1, -1) = a_0 - a_1 - a_2 + a_{12} = 0.$$

Therefore we see that any element of $\mathbf{I}_{sqf}(2)$ is a linear combination of $[1, 2] = x_1x_2 - 1$ and $[1] - [2] = x_1 - x_2$. Noticing that $[1, 2] \in \mathbf{I}_{sqf}(n)_{cons}$ and $[1] - [2] \in \mathbf{I}_{sqf}(n)_{anti}$, we see that Theorem 3.1 holds for $n = 2$.

Remark. If we drop the assumption that every term of f is square-free, then many other polynomials like $x_1^2x_2^4 - 1$, $x_1^9 - x_2^5$ belong to $\mathbf{I}(2)$. For this reason we restrict our attention on square-free polynomials.

Case 2. $n = 3$. Let $f(x_1, x_2, x_3) \in \mathbf{I}_{sqf}(3)$ and put

$$f = a_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3 + a_{123}x_1x_2x_3.$$

Since $(0, t, 0), (-t, 0, t) \in V_3(\mathbf{Z})$ for any $t \in \mathbf{Z}$ by Theorem 2.5(iii), we see that

$$a_0 + a_2t = 0, \text{ hence } a_0 = a_2 = 0,$$

$$-a_1t + a_3t - a_{13}t^2 = 0, \text{ hence } a_1 = a_3 \text{ and } a_{13} = 0.$$

Hence we have

$$f = a_1(x_1 + x_3) + a_{12}x_1x_2 + a_{23}x_2x_3 + a_{123}x_1x_2x_3.$$

Furthermore, since $(1, 2, 1), (-1, -2, -1) \in V_3(\mathbf{Z})$ by Theorem 2.5(iii), we have

$$2a_1 + 2a_{12} + 2a_{23} + 2a_{123} = 0,$$

$$-2a_1 + 2a_{12} + 2a_{23} - 2a_{123} = 0,$$

which imply that $a_1 = -a_{123}$, $a_{23} = -a_{12}$, and hence

$$f = a_{123}(x_1x_2x_3 - x_1 - x_3) + a_{12}(x_1x_2 - x_2x_3) = a_{123}[1, 3] + a_{12}([1, 2] - [2, 3]).$$

Noticing that $[1, 3] \in \mathbf{I}_{sqf}(3)_{cons}$ and $[1, 2] - [2, 3] \in \mathbf{I}_{sqf}(3)_{anti}$, we see that Theorem 3.1 holds for $n = 3$.

Case 3. $n \geq 4$. We assume that Theorem 3.1 holds for $k < n$ and prove it by induction. Let $f(x_1, \ldots, x_n) \in \mathbf{I}_{sqf}(n)$. Then there exist $f_1, f_2 \in \mathbf{P}_{sqf}(n-2)$ and $f_3 \in \mathbf{P}_{sqf}(n-1)$ such that

$$f = f_1 + x_{n-1} f_2 + x_n f_3. \tag{4.1}$$

Recall that for any $(a_1, \ldots, a_{n-2}) \in V_{n-2}(\mathbf{Z})$, we have $paste_{(+;t)}(a_1, \ldots, a_{n-2}) \in V_n(\mathbf{Z})$ with $t \in \mathbf{Z}$. Hence

$$0 = f(a_1, \ldots, a_{n-2}, t, 0) = f_1(a_1, \ldots, a_{n-2}) + t f_2(a_1, \ldots, a_{n-2}), \tag{4.2}$$

which implies that $f_1, f_2 \in \mathbf{I}_{sqf}(n-2)$. Note that any polynomial in $\mathbf{P}_{sqf}(n)$ can be written as $f = f' + f''$ with f' consistent and f'' anti-consistent. The following lemma shows that the $\mathbf{I}_{sqf}(n)$ is stable under this decomposition.

Lemma 4.1. $\mathbf{I}_{sqf}(n) = \mathbf{I}_{sqf}(n)_{cons} \oplus \mathbf{I}_{sqf}(n)_{anti}$.

Proof of Lemma 4.1. Let $f \in \mathbf{I}_{sqf}(n)$ and write it as $f = f' + f''$ with f' consistent and f'' anti-consistent. Since $V_n(\mathbf{Z})$ is, by definition, the set of integral zeros of the *consistent* polynomial $[1, n]$, it is stable under the map $\iota : (x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_n)$. Therefore for any $(a_1, \ldots, a_n) \in V_n(\mathbf{Z})$, we have

$$0 = f(a_1, \ldots, a_n) = f'(a_1, \ldots, a_n) + f''(a_1, \ldots, a_n) \tag{4.3}$$

and

$$\begin{aligned} 0 &= f(-a_1, \ldots, -a_n) \\ &= f'(-a_1, \ldots, -a_n) + f''(-a_1, \ldots, -a_n) \\ &= (-1)^n f'(a_1, \ldots, a_n) + (-1)^{n-1} f''(a_1, \ldots, a_n). \end{aligned} \tag{4.4}$$

Comparing the right hand sides of (4.3) and (4.4), we see that $f'(a_1, \ldots, a_n) = f''(a_1, \ldots, a_n) = 0$ for any $(a_1, \ldots, a_n) \in V_n(\mathbf{Z})$, hence the assertion of Lemma 4.1 follows.

By this lemma, we can divide our argument into two cases where

$$f \in \mathbf{I}_{sqf}(n)_{cons} \quad \text{and} \quad f \in \mathbf{I}_{sqf}(n)_{anti}.$$

Case 3.1. Let $f \in \mathbf{I}_{sqf}(n)_{cons}$. Then we see that $f_1 \in \mathbf{I}_{sqf}(n-2)_{cons}$ and $f_2 \in \mathbf{I}_{sqf}(n-2)_{anti}$ in (4.1). Therefore by induction hypothesis, we see that

$$f_1 = c_1[1, n-2]$$

$$f_2 = c_2([1, n-3] - [2, n-2])$$

hold for some $c_1, c_2 \in \mathbf{Q}$, hence we have

$$f = c_1[1, n-2] + c_2([1, n-3] - [2, n-2])x_{n-1} + f_3 x_n. \tag{4.5}$$

On the other hand, by using $paste^{n-2}_{(-;t)}$ in the derivation of (4.1) and (4.2) instead of $paste^{n-2}_{(+;t)}$, we see that

$$f = d_1[3, n] + d_2([3, n-1] - [4, n])x_2 + g_3 x_1 \tag{4.6}$$

holds for some $d_1, d_2 \in \mathbf{Q}$, $g_3 \in \mathbf{Q}[x_2, ..., x_n]$. By letting $x_1 = x_n = 0$ in (4.5) and (4.6), we have

$$c_1([0, x_2, ..., x_{n-2}]) + c_2([0, x_2, ..., x_{n-3}] - [x_2, ..., x_{n-2}])x_{n-1}$$
$$= d_1[x_3, ..., x_{n-1}, 0] + d_2([x_3, ..., x_{n-1}] - [x_4, ..., x_{n-1}, 0])x_2.$$

By Proposition 2.1, this is equivalent to

$$-c_1[x_3, ..., x_{n-2}] + c_2(-[x_3, ..., x_{n-3}] - [x_2, ..., x_{n-2}])x_{n-1}$$
$$+ d_1[x_3, ..., x_{n-2}] - d_2([x_3, ..., x_{n-1}] + [x_4, ..., x_{n-2}])x_2 = 0.$$

Since the coefficient of $x_2 \cdots x_{n-1}$ on the left hand side is $-c_2 - d_2$, we find that $d_2 = -c_2$. Hence the coefficient of x_{n-1} on the left hand side is equal to

$$c_2(-[x_3, ..., x_{n-3}] - [x_2, ..., x_{n-2}] + [x_3, ..., x_{n-2}]x_2)$$
$$= c_2(-[x_3, ..., x_{n-3}] - ([x_2, ..., x_{n-2}] - [x_3, ..., x_{n-2}]x_2))$$
$$= c_2(-[x_3, ..., x_{n-3}] + [x_4, ..., x_{n-2}]).$$

Therefore we must have $c_2 = 0$. Thus (4.5) simplifies to

$$f = c_1[1, n-2] + f_3 x_n. \tag{4.7}$$

Furthermore it follows from Theorem 2.3(iv) that $blup^{n-1}_{(n;+)}(a_1, ..., a_{n-1}) \in V_n(\mathbf{Z})$ for any $(a_1, ..., a_{n-1}) \in V_{n-1}(\mathbf{Z})$. Hence we have

$$0 = f(a_1, ..., a_{n-1} + 1, 1) = c_1[a_1, ..., a_{n-2}] + f_3(a_1, ..., a_{n-1} + 1)$$

for any $(a_1, ..., a_{n-1}) \in V_{n-1}(\mathbf{Z})$. Hence if we put $h(x_1, ..., x_{n-1}) = f_3(x_1, ..., x_{n-1} + 1)$, then $h(x_1, ..., x_{n-1}) + c_1[x_1, ..., x_{n-2}] \in \mathbf{I}_{sqf}(n-1)$. Therefore by induction hypothesis there exist $A, B \in \mathbf{Q}$ such that

$$h(x_1, ..., x_{n-1}) + c_1[1, n-2] = A[1, n-1] + B([1, n-2] - [2, n-1]).$$

Hence we have

$$\begin{aligned}
& f_3(x_1, ..., x_{n-1}) \\
&= h(x_1, ..., x_{n-1} - 1) \\
&= A[x_1, ..., x_{n-1} - 1] + B([x_1, ..., x_{n-2}] - [x_2, ..., x_{n-1} - 1]) - c_1[x_1, ..., x_{n-2}] \\
&= A([x_1, ..., x_{n-1}] - [x_1, ..., x_{n-2}]) \\
&\quad + B([x_1, ..., x_{n-2}] - [x_2, ..., x_{n-1}] + [x_2, ..., x_{n-2}]) - c_1[x_1, ..., x_{n-2}] \\
&\qquad \text{(by the equality } [x_1, ..., x_{n-1} - 1] = [x_1, ..., x_{n-1}] - [x_1, ..., x_{n-2}] \\
&\qquad \text{and the like)} \\
&= A[1, n-1] + (-A + B - c_1)[1, n-2] - B([2, n-1] - [2, n-2]). \qquad (4.8)
\end{aligned}$$

Note that our assumption $f \in \mathbf{I}_{sqf}(n)_{cons}$ implies by (4.7) that f_3 is consistent. Hence it follows from (4.8) that $f_3 = A[1, n-1]$, which implies

$$f = c_1[1, n-2] + A[1, n-1]x_n. \tag{4.9}$$

By symmetry, mentioned in Theorem 2.5, we see that there exist $c_1', A' \in \mathbf{Q}$ such that

$$f = c_1'[3, n] + A'[2, n]x_1. \tag{4.10}$$

Comparing the coefficients of $x_1x_2 \cdots x_n$ on the right hand sides of (4.9) and (4.10), we have $A = A'$. Moreover, since the right hand side of (4.10) is transformed as

$$c_1'[3, n] + A[2, n]x_1 = c_1'[3, n] + A([2, n-1]x_n - [2, n-2])x_1,$$

we have $c_1 = -A$ by comparing the coefficients of $x_1x_2 \cdots x_{n-2}$. Thus we see that $f = -A[1, n-2] + A[1, n-1]x_n = A[1, n]$, which completes the proof of Theorem 3.1 when f is consistent.

Case 3.2. Let $f \in \mathbf{I}_{sqf}(n)_{anti}$. Then we see that there exist $f_1 \in \mathbf{I}_{sqf}(n-2)_{anti}$, $f_2 \in \mathbf{I}_{sqf}(n-2)_{cons}$ and $f_3 \in \mathbf{P}_{sqf}(n-1)$ such that

$$f = f_1 + x_{n-1}f_2 + x_nf_3. \tag{4.11}$$

By a similar argument to the one for Case 3.1, we see that

$$f_1 = c_1([1, n-3] - [2, n-2]),$$

$$f_2 = c_2[1, n-2]$$

for some $c_1, c_2 \in \mathbf{Q}$, hence we have

$$f = c_1([1, n-3] - [2, n-2]) + c_2[1, n-2]x_{n-1} + f_3x_n. \tag{4.12}$$

By symmetry, we see that there exist $d_1, d_2 \in \mathbf{Q}$, $g_3 \in \mathbf{Q}[x_2, ..., x_n]$ such that

$$f = d_1([3, n-1] - [4, n]) + d_2[3, n]x_2 + g_3x_1. \tag{4.13}$$

By putting $x_1 = x_n = 0$ in (4.12) and (4.13), we have

$$c_1([0, x_2, ..., x_{n-3}] - [x_2, ..., x_{n-2}]) + c_2[0, x_2, ..., x_{n-2}]x_{n-1}$$
$$= d_1([x_3, ..., x_{n-1}] - [x_4, ..., x_{n-1}, 0]) + d_2[x_3, ..., x_{n-1}, 0]x_2.$$

By Proposition 2.1, this is equivalent to

$$c_1(-[x_3, ..., x_{n-3}] - [x_2, ..., x_{n-2}]) - c_2[x_3, ..., x_{n-2}]x_{n-1}$$
$$- d_1([x_3, ..., x_{n-1}] + [x_4, ..., x_{n-2}]) + d_2[x_3, ..., x_{n-2}]x_2 = 0.$$

Looking at the coefficients of $x_2 \cdots x_{n-2}$ and $x_3 \cdots x_{n-1}$, we have

$$-c_1 + d_2 = 0,$$

$$-c_2 - d_1 = 0.$$

Hence the left hand side becomes

$$\begin{aligned}
&c_1(-[3, n-3] - [2, n-2] + x_2[3, n-2]) \\
&\quad + c_2([3, n-1] + [4, n-2] - [3, n-2]x_{n-1}) \\
&= c_1(-[3, n-3] + [4, n-2]) + c_2(-[3, n-3] + [4, n-2]) \\
&= (c_1 + c_2)(-[3, n-3] + [4, n-2]).
\end{aligned}$$

Hence we have $c_2 = -c_1$. Thus we obtain

$$f = c_1([1, n-3] - [2, n-2] - [1, n-2]x_{n-1}) + f_3 x_n, \tag{4.14}$$

and hence

$$f = c_1(-[1, n-1] - [2, n-2]) + f_3 x_n. \tag{4.15}$$

Since $blup^{n-1}_{(n;+)}(a_1, ..., a_{n-1}) \in V_n(\mathbf{Z})$ for any $(a_1, ..., a_{n-1}) \in V_{n-1}(\mathbf{Z})$, (4.15) implies that

$$\begin{aligned}
0 &= c_1(-[a_1, ..., a_{n-1} + 1] - [a_2, ..., a_{n-2}]) + f_3(a_1, ..., a_{n-1} + 1) \\
&= c_1(-[a_1, ..., a_{n-1}] - [a_1, ..., a_{n-2}] - [a_2, ..., a_{n-2}]) + f_3(a_1, ..., a_{n-1} + 1).
\end{aligned}$$

Hence $f_3(x_1, ..., x_{n-1} + 1) - c_1([1, n-1] + [1, n-2] + [2, n-2]) \in \mathbf{I}_{sqf}(n-1)$, and it follows from induction hypothesis that there exist $A, B \in \mathbf{Q}$ such that

$$\begin{aligned}
&f_3(x_1, ..., x_{n-1} + 1) - c_1([1, n-1] + [1, n-2] + [2, n-2]) \\
&= A[1, n-1] + B([1, n-2] - [2, n-1]).
\end{aligned}$$

Hence we see that

$$\begin{aligned}
f_3(x_1, ..., x_{n-1}) = {} & c_1(([1, n-1] - [1, n-2]) + [1, n-2] + [2, n-2]) \\
& + A([1, n-1] - [1, n-2]) \\
& + B([1, n-2] - [2, n-1] + [2, n-2])
\end{aligned}$$

$$= (A + c_1)[1, n-1] + (B - A)[1, n-2]$$
$$- B[2, n-1] + (c_1 + B)[2, n-2].$$

Since f_3 is anti-consistent, we must have

$$A + c_1 = 0, \ \ c_1 + B = 0.$$

Hence $A = B$ and we have

$$f_3(x_1, \ldots, x_{n-1}) = -B[2, n-1].$$

It follows from (4.14) that

$$\begin{aligned} f &= c_1([1, n-3] - [2, n-2] - [1, n-2]x_{n-1}) + c_1[2, n-1]x_n \\ &= c_1([1, n-3] - [2, n-2] - [1, n-2]x_{n-1} + [2, n-1]x_n) \\ &= -c_1((-[1, n-3] + [1, n-2]x_{n-1}) - (-[2, n-2] + [2, n-1]x_n)) \\ &= -c_1([1, n-1] - [2, n]). \end{aligned}$$

Thus we see that Theorem 3.1 holds when f is anti-consistent. This completes the proof of Theorem 3.1.

Remark. As is seen in the proof, we do not need the whole $V_n(\mathbf{Z})$ to characterize the two-dimensional vector space generated by $[1, n]$ and $[1, n-1] - [2, n]$. Indeed, the subset obtained from successive applications of $blup^k_{(1;\pm)}, blup^k_{(k+1;\pm)}, paste^k_{(\pm, c)}$ $(k \geq 1)$ to $\{0\} \subset V_1(\mathbf{Z})$ is enough to do the same task.

References

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