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THE FINITE ELEMENT APPROXIMATION OF PARABOLIC QUASI-VARIATIONAL INEQUALITIES RELATED TO IMPULSE CONTROL PROBLEM

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Abstract

The paper deals with the semi-implicit time scheme combined with a finite element spatial approximation for a class of parabolic quasi-variational inequalities with nonlinear source terms. The convergence of the iterative scheme is established and a quasi-optimal L^{∞}-asymptotic behavior is given.

1. Introduction

A great deal of work has been done since now three decades on questions

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of existence and uniqueness for parabolic and elliptic variational inequalities and quasi-variational. However, very much remains to be done on the numerical analysis side, especially error estimates for elliptic variational and quasi-variational inequalities in uniform norm (cf., e.g., [2-7, 10, 12, 18, 19]) and error estimate for parabolic variational inequalities (cf., e.g., [1, 11]). The existence, uniqueness and regularity of a solution to continuous parabolic variational inequalities have been intensively studied in the past years, see Boulbrachene [15] and Lions and Stampacchia [16] for details.

In this paper, we propose a discrete iterative algorithm to prove the existence and uniqueness and we devote the asymptotic behavior using the semi-implicit time scheme combined with a finite element spatial approximation for following parabolic quasi-variational inequalities. Find $u \in K(u)$ solution of

$$\frac{\partial u}{\partial t} + Au \leq f(u), \tag{1.1}$$

where Ω is convex domain in $\mathbb{R}^N$, with sufficiently smooth boundary Γ and Σ set in $\mathbb{R} \times \mathbb{R}^N$, $\Sigma = \Omega \times [0, T]$ with $T < +\infty$. $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$, A is an operator defined over $H^1(\Omega)$:

$$A(t)u = -\sum_{i, j=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(t, x) \frac{\partial u}{\partial x_j} + \sum_{j=1}^{N} b_j(t, x) \frac{\partial u}{\partial x_j} + a_0(t, x)u \tag{1.2}$$

and whose coefficients $a_{i, j}(t, x), b_j(t, x), a_0(t, x) \in L^\infty(\Omega) \cap C_2(\overline{\Omega})$, $x \in \overline{\Omega}$, $1 \leq i$, $j \leq N$ are sufficiently smooth coefficients and satisfy the following conditions:

$$a_{ij}(t, x) = a_{ji}(t, x); \quad a_0(t, x) \geq \beta > 0, \ \beta \text{ is a constant}, \tag{1.3}$$

$$\sum_{i, j=1}^{N} a_{ij}(t, x) \xi_i \xi_j \geq \gamma |\xi|^2; \ \xi \in \mathbb{R}^N, \gamma > 0, x \in \overline{\Omega} \tag{1.4}$$

and the bilinear forms associated with A, for $u, v \in H_0^1(\Omega)$,

$$a(u, v) = \int_\Omega \left(\sum_{j,k=1}^{N} a_{jk}(t, x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{j=1}^{N} b_k(t, x) \frac{\partial u}{\partial x_j} v + a_0(t, x) uv \right) dx. \quad (1.5)$$

Assume $K(u)$ is an implicit convex set defined as follows:

$$K(u) = \{u \in L^2(0, T, H_0^1(\Omega)), u \leq Mu, u(0, x) = u_0 \text{ in } \Omega\}, \quad (1.6)$$

where

$$M(u) = \psi + S(u)$$

with ψ is a smooth function and S is a nonlinear continuous operator $L^\infty(\Omega)$ into itself satisfying the following assumptions

$$S(u) \leq S(\tilde{u}) \text{ whenever } u \leq \tilde{u} \text{ a.e. in } \Omega$$

and

$$S(u + \delta) \leq S(u) + \delta \text{ for } \delta \text{ is a positive constant.}$$

$f(\cdot)$ is a second member nonlinear and Lipschitz with Lipschitz constant c satisfying the following conditions

$$f, \frac{\partial f}{\partial t} \in L^2(0, T, L^\infty(\Omega)) \cap C^1(0, T, H^1(\Omega)), \; f \geq 0 \quad (1.7)$$

and

$$c < \beta \quad (1.8)$$

and β is the constant defined in (1.3).

The class of parabolic quasi-variational inequalities (PQVIs) includes at least two well-known important problems: the variational inequality of obstacle type problems (VIs), (when S is identically equal to zero, cf. [3]), and the quasi-variational inequality of impulse control problems (when ψ is identically equal to zero and $S(u) = k + \inf(u + \xi)$, $x \in \Omega$, $\xi \geq 0$, $x + \xi \in \Omega$, cf. [3]).

The aim of the present paper is to show that the asymptotic behavior can be properly approximated by a semi-implicit time scheme combined with a finite element spatial using the discrete iterative algorithm, this method, we proceed in two steps: In the first step, we discretize in space, i.e., that we approach the space H_0^1 by a space discretization of finite dimensional $V^h \subset H_0^1$. In the second step, we discretize the problem with respect to time using the semi-implicit scheme time. Therefore, we search a sequence of elements $u_h^n \in V^h$ which approaches $u^n(t_n)$, $t_n = n\Delta t$, with initial data $u_h^0 = u_{0h}$. Our approach stands on a discrete stability result and error estimate for parabolic quasi-variational inequalities.

The paper is organized as follows: In Section 2, we lay down some fundamental definitions and theorem of PQVIs. In Section 3, we consider the discrete problem, discretize the iterative scheme by the standard finite element method, and we associate with the discrete PQVIs problem a fixed point mapping and we use that in proving the existence of unique discrete solution. In Section 4, we give the discrete iterative algorithm and prove its convergence. Finally, in Section 5, we give the main results of this paper which is the asymptotic behavior in uniform norm.

2. Parabolic Variational Inequalities

2.1. Strong variational inequality

Definition 1 (See [2])**.** We say that u is a *strong solution* of parabolic variational inequality if:

$$u \in L^2(0, T, H_0^1(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0, T, L^2(\Omega))$$

and it satisfies the following inequality:

$$\begin{cases} \left(\frac{\partial u}{\partial t}, v - u\right) + a(u, v - u) \geq (f, v - u), \ \forall v \in H_0^1(\Omega), \\ u \leq Mu, \ v \leq Mu \text{ in the } \Omega, \\ \frac{\partial u}{\partial \eta} = 0 \text{ or } u(\cdot, x) = 0 \text{ in } \Sigma, \\ u(x, T) = u(x, 0) = \overline{u}. \end{cases} \tag{2.1}$$

2.2. Weak variational inequality

For $v \in H_0^1(\Omega)$, consider the expression

$$\lambda = \int_0^T \left[-\left(\frac{\partial v}{\partial t}, v - u\right) + a(t; u, v - u) - (f, v - u) \right] dt$$

$$\lambda = \int_0^T \left[-\left(\frac{\partial u}{\partial t}, v - u\right) + a(t; u, v - u) - (f, v - u) \right] dt$$
$$+ \int_0^T -\left(\frac{\partial}{\partial t}(v - u), v - u\right). \tag{2.2}$$

On (2.2), the first integral of the second member is negative. Therefore,

$$\lambda \geq \int_0^T -\frac{1}{2}\frac{d}{dt}| v - u | dt,$$

thus

$$\lambda + \frac{1}{2}\frac{d}{dt}| v(T) - u(T) |^2 \geq | v(0) - u(0) |^2 \geq 0.$$

Replacing $u(T)$ by $\overline{u}$, we get the following definition:

Definition 2 (See [2]). We say that u is *weak solution* of the variational inequality if

$$u \in L^2(0, T; H_0^1(\Omega))$$

and

$$u \le Mu \text{ in } \Omega \times]0, T[$$

and it satisfies the following inequality:

$$\int_0^T \left[-\left(\frac{\partial v}{\partial t}, v - u \right) + a(t; u, v - u) - (f, v - u) \right] dt + \frac{1}{2} | v(T) - \bar{u} | \ge 0.$$

Theorem 1 (See [2])**.** *Under the previous assumptions, we have* $f \in L^2(0, T; H_0^1(\Omega))$ *with*:

$$\frac{\partial a_{ij}}{\partial t} \in L^\infty(\Omega)$$

and

$$Mu, \frac{\partial Mu}{\partial t} \in L^2(0, T; H_0^1(\Omega)), \; Mu \ge 0 \text{ in } \Sigma, \; \frac{\partial Mu}{\partial t} = 0 \text{ in } \Sigma,$$

$$\bar{u} \in H_0^1(\Omega), \; \bar{u} \le Mu(T).$$

Then the problem (1.1) admits a unique solution such that

$$\bar{u} \in L^\infty(0, T; H_0^1(\Omega)).$$

Notation 1. We specify the following notation:

$$\| \cdot \|_{L^2(\Omega)} = \| \cdot \|_2, \quad \| \cdot \|_1 = \| \cdot \|_{H_0^1(\Omega)} \text{ and } \| \cdot \|_{L^\infty(\Omega)} = \| \cdot \|_\infty.$$

Theorem 2 (Sobolev-Poincaré inequality)**.** *Let* Ω *be bounded open in* $\mathbb{R}^N$, *with sufficiently smooth boundary* Γ. *Then there exists a* C_* *such that*

$$\| u \|_2 \le C_* \| \nabla u \|_2, \quad v \in H_0^1(\Omega) \cap C^2(\overline{\Omega}), \quad \nabla = \sum_{i=1}^N \frac{\partial}{\partial x_i}. \tag{2.3}$$

Theorem 3. *Let us assume that the discrete bilinear form* $a(\cdot, \cdot)$, *weakly coercive in* $V^h \subset H_0^1(\Omega)$. *Then there exist two constants* $\alpha > 0$ *and* $\lambda > 0$

such that:

$$a(u_h, u_h) + \lambda \| u_h \|_2 \geq \alpha \| u_h \|_1,$$

where

$$\lambda = \left(\frac{\| b_j \|_\infty^2}{2\gamma} + \frac{\gamma}{2} \| a_0 \|_\infty \right), \ \alpha = \frac{\gamma}{2}.$$

Proof. The discrete bilinear form a $a(\cdot, \cdot)$ is defined by

$$a(u_h, u_h) = \int_\Omega \left(\sum_{i, j=1}^{n} a_{ij}(x) \frac{\partial u_h}{\partial x_i} \frac{\partial u_h}{\partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial u_h}{\partial x_j} u_h + a_0(x) u_h^2 \right) dx.$$

Under assumption (1.4), we have

$$\sum_{i, j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u_h}{\partial x_i} \frac{\partial u_h}{\partial x_j} > \gamma \sum_{i=1}^{n} \left(\int_\Omega \frac{\partial u_h}{\partial x_i} \right)^2 = \gamma \| \nabla u_h \|_2^2$$

and, moreover,

$$\left| \sum_{j=1}^{n} \int_\Omega b_j(x) \frac{\partial u_h}{\partial x_j} u_h \right| \leq \sup_j | b_j | \sum_{j=1}^{n} \int_\Omega \left| \frac{\partial u_h}{\partial x_j} u_h \right|$$

$$\leq \| b_j \|_\infty \| \nabla u_h \|_2 \| u_h \|_2.$$

We can easily show

$$ab \leq \frac{1}{2}(a^2 + b^2), \ \forall a, b \in \mathbb{R}, \ \forall \gamma > 0.$$

Choosing

$$\begin{cases} a = \| \nabla u_h \|_2 \cdot \sqrt{\gamma}, \\ b = \dfrac{\| b_j \|_\infty}{\sqrt{\gamma}} \| u_h \|_2. \end{cases}$$

Thus

$$\left| \sum_{j=1}^{n} \int_{\Omega} b_j(x) \frac{\partial u_h}{\partial x_j} u_h \right| \geq -\left(\frac{\gamma}{2} \| \nabla u_h \|_2^2 + \frac{\| b_j \|_\infty \cdot \| u_h \|_2}{2\gamma} \right).$$

So we get

$$a(u_h, u_h) \geq \gamma \| \nabla u_h \|_2^2 - \frac{\gamma}{2} \| \nabla u_h \|_2^2 + \frac{\| b_j \|_\infty}{2\gamma} \| u_h \|_2 - \| a_0 \|_\infty \| u_h \|_2^2.$$

It is easily verified that

$$a(u_h, u_h) \geq \frac{\gamma}{2} (\| \nabla u_h \|_2 + \| u_h \|_2^2) - \left(\frac{\| b_j \|_\infty^2}{2\gamma} + \frac{\gamma}{2} + \| a_0 \|_\infty \right) \| u_h \|_2^2.$$

We deduce that

$$a(u_h, u_h) + \lambda \| u_h \|_2^2 \geq \alpha \| u_h \|_1^2 \text{ such that}$$

$$\alpha = \frac{\gamma}{2}, \lambda = \left(\frac{\| b_j \|_\infty^2}{2\gamma} + \frac{\gamma}{2} + \| a_0 \|_\infty \right). \qquad \square$$

We can identify the following result on the time energy behavior

$$E_h(t) = \int_\Omega u_h^2 dx.$$

Setting $v = 0$ on (2.1) and after discretization by the finite element in the V^h, we have the semi-discretization problem

$$\left(\frac{\partial u_h}{\partial t}, u_h \right) + a(u_h, u_h) = \frac{1}{2} \int_\Omega \frac{\partial u_h^2}{\partial t} dx + a(u_h, u_h) \leq (f, u_h). \tag{2.4}$$

Using Theorem 2, we deduce that

$$\frac{1}{2} \int_\Omega \frac{du_h^2}{\partial t} dx + a(u_h, u_h) \geq \frac{1}{2} \frac{d}{dt} \int_\Omega u_h^2 dx + \alpha \| u_h \|_1^2 - \lambda \| u_h \|_2^2$$

$$= \frac{1}{2}\frac{d}{dt}\int_{\Omega} u_h^2 dx + \alpha \| u_h \|_2^2 + \alpha \| \nabla u_h \|_2^2 - \lambda \| u_h \|_2^2$$

$$= \frac{1}{2}\left(\frac{d}{dt} E_h(t) + 2(\alpha - \lambda) E_h(t) + 2\alpha \| \nabla u_h \|_2^2 dx\right)$$

$$\geq \frac{1}{2}\left(\frac{d}{dt} E_h(t) + 2(\alpha - \lambda) E_h(t) + \frac{2\alpha}{C_*^2}\int_{\Omega} u_h^2 dx\right).$$

Thus we have

$$\frac{d}{dt}\int_{\Omega} u_h^2 dx + \alpha \| u_h \|_1^2 - \lambda \| u_h \|_2^2 \geq \frac{d}{dt}(E_h(t)) + 2\left(\alpha - \lambda + \frac{\alpha}{C_*^2}\right) E_h(t).$$

Applying the Cauchy-Schwartz inequality on the right-hand side of (2.1), we find

$$(f, u_h) = \int_{\Omega} f(x, t) u_h(x, t) dx \leq \| f \|_2 \| u_h \|_2.$$

So that

$$\frac{d}{dt} E_h(t) + 2\left(\alpha - \lambda + \frac{\alpha}{C_*^2}\right) E_h(t) \leq 2 \| f \|_2 \| u_h \|_2.$$

Using the Young's inequality,

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \ \forall a, b \in \mathbb{R}, \ \forall \varepsilon > 0.$$

Thus we obtain

$$\frac{d}{dt} E_h(t) + 2\left(\alpha - \lambda + \frac{\alpha}{C_*^2}\right) E_h(t) \leq 2\varepsilon E_h(t) + \frac{1}{2\varepsilon} \| f \|_2^2,$$

taking $\eta = \alpha - \lambda + \frac{\alpha}{C_*^2}$, thus we have

$$\frac{d}{dt} E_h(t) + 2(\eta - \varepsilon) E_h(t) \leq \frac{1}{2\varepsilon} \| f \|_2^2,$$

or equivalently,

$$(e^{2(\eta-\varepsilon)t} E_h(t))' \leq \frac{1}{2\varepsilon} e^{2(\eta-\varepsilon)t} \int_\Omega (f(x, t))^2 dx.$$

Integrating the last inequality from 0 to t, we get

$$E_h(t) \leq e^{-2(\eta-\varepsilon)t} E_h(0) + \frac{1}{2\varepsilon} \int_0^t \left[e^{2(\eta-\varepsilon)(s-t)} \left(\int_\Omega (f(x, s))^2 dx \right) \right] ds. \quad (2.5)$$

Remark 1. In particular, when $f = 0$ and choosing $\varepsilon < \eta$. Then (2.5) shows that the energy $E(t)$ decreasing exponentially fast in time.

3. Existence and Uniqueness for Discrete Parabolic Variational Inequalities

3.1. Discretization

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i, $i = \{1, ..., m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$, where M_j is a summit of the considered triangulation. We introduce the following discrete spaces V^h of finite element:

$$V_h = \left\{ \begin{array}{l} v_h \in L^2(0, T, H_0^1(\Omega)) \cap C(0, T, H_0^1(\overline{\Omega})) \\ \text{such that } v_h|_K \in P_1, K \in \tau_h, \\ v_h \leq r_h M(v_h), v_h(\cdot, 0) = v_{h0} \text{ in } \Omega \end{array} \right\}. \quad (3.1)$$

We consider r_h as the usual interpolation operator defined by

$$v_h \in L^2(0, T, H_0^1(\Omega)) \cap C(0, T, H_0^1(\overline{\Omega})), \; r_h v = \sum_{i=1}^{m(h)} v(M_i) \varphi_i(x). \quad (3.2)$$

The discrete maximum principle assumption (cf. [4])**.** The matrix whose coefficients $a(\varphi_i, \varphi_j)$ are supposed to be *M*-matrix. For convenience in all the sequels, *C* will be a generic constant independent on *h*.

We discretize in space, i.e., that we approach the space H_0^1 by a space discretization of finite dimensional $V^h \subset H_0^1$. In the second step, we discretize the problem with respect to time using the semi-implicit time scheme. Therefore, we search a sequence of elements $u_h^n \in V^h$ which approaches $u^n(t_n)$, $t_n = n\Delta t$, with initial data $u_h^0 = u_{0h}$. Now we apply the semi-implicit time scheme on the following to the semi-discrete approximation

$$\begin{cases} \left(\dfrac{\partial u_h}{\partial t}, v_h - u_h\right) + a(u_h, v_h - u_h) \geq (f(u_h), v_h - u_h), \ \forall v_h \in V^h, \\ u_h(t, x) \leq r_h M(u_h), v_h \leq r_h M(u_h), \\ u_h(0, x) = 0. \end{cases} \tag{3.3}$$

Now we apply the semi-implicit time scheme in (3.3):

$$\begin{cases} \displaystyle\int_\Omega \frac{u_h^k - u_h^{k-1}}{\Delta t}(v_h - u_h^k)dx + a(u_h^k, v_h - u_h^k) \geq (f(u_h^{k-1}), v_h - u_h^k), \\ v \leq r_h M(u_h^{k-1}), u_h^k \leq r_h M(u_h^{k-1}); \ k = 1, \ldots, n, \end{cases} \tag{3.4}$$

thus we can rewrite (3.4) as for $u_h^k \in K(u_h^k)$,

$$\left(\frac{u_h^k}{\Delta t}, v_h - u_h^k\right) + a(u_h^k, v_h - u_h^k) \geq \left(f(u_h^{k-1}) + \frac{u_h^{k-1}}{\Delta t}, v_h - u_h^k\right). \tag{3.5}$$

The problem (3.5) is equivalent to

$$b(u_h^k, v_h - u_h^k) \geq (f(u_h^{k-1}) + \mu u_h^{k-1}, v_h - u_h^k), u_h^k \in V^h \tag{3.6}$$

such that

$$\begin{cases} b(u_h^k, v_h - u_h^k) = \mu(u_h^k, v_h - u_h^k) + a(u_h^k, v_h - u_h^k), u_h^k \in V^h, \\ \mu = \dfrac{1}{\Delta t} = \dfrac{T}{n}, k = 1, \ldots, n. \end{cases} \tag{3.7}$$

Notation 2. Let $F = f(w) + \mu\xi_h$, $\widetilde{F} = f(\widetilde{w}) + \mu\widetilde{\xi}_h, \in L^\infty(\Omega)$ and $\xi_h = \partial_h(F, r_h M\xi_h)$, $\widetilde{\xi}_h = \partial_h(\widetilde{F}, r_h M\widetilde{\xi}_h)$, where $Mu = k + \inf\limits_{\xi \geq 0, x+\xi \in \overline{\Omega}} u(x + \xi)$.

Lemma 1 (See [9]). *Under the discrete maximum principle assumption* (*dmp*), *we have if* $F \geqq \widetilde{F}$, *then* $\xi_h \geqq \widetilde{\xi}_h$.

3.2. A fixed point mapping associated with discrete problem

We consider the mapping

$$T_{h\mu} : L_+^\infty(\Omega) \to V_h$$

$$w \to T_{h\mu}(w) = \xi_{h\mu}, \tag{3.8}$$

where $\xi_{h\mu}$ is the unique solution of the following PQVI: Find $\xi_{h\mu} \in V^h$:

$$b(\xi_{h\mu}, v_h - \xi_{h\mu}) \geq (f(w) + \mu w, v_h - \xi_{h\mu}), \forall v \in V^h.$$

Proposition 1 (Cf. [9]). *The mapping* $T_{h\mu}$ *is a contraction in* $L_+^\infty(\Omega)$ *with rate of contraction* $\dfrac{c + \mu}{\beta + \mu}$. *Therefore*, $T_{h\mu}$ *admits a unique fixed point which coincides with the solution of PQVI* (3.3).

Theorem 4 (See [9]). *Under the previous assumptions and the maximum principle*, *there exists a constant C independent of h such that*

$$\| u^\infty - u_h^\infty \|_\infty \leq Ch^2 |\log h|^3,$$

where u^∞ *is the solution of the following asymptotic continue problem*

$$b(u^\infty, v - u^\infty) \geq (f + \mu u^\infty, v - u^\infty), v \in H_0^1(\Omega). \tag{3.9}$$

Notation 3. We note u_0 is the solution of the following variational equation:

$$a(u_{h0},\, v_h - u_{0h}) = (g,\, v_h - u_{0h}),\ \forall v_h \in V^h, \tag{3.10}$$

where g is a smooth function.

4. Iterative Discrete Algorithm

We choose $u_h^0 = u_{0h}$ (initial data) is a discrete of continuous solution on (3.10). Now we give our following discrete algorithm:

$$u_h^k = Tu_h^{k-1},\ k = 1, \ldots, n,$$

where u_h^k is the solution of the problem (3.6).

Proposition 2. *Under the previous hypotheses and notation, we have the following estimate of convergent:*

$$\| u_h^n - u_h^\infty \|_\infty \leq \left(\frac{1 + (\Delta t)c}{1 + (\Delta t)\beta}\right)^k \| u_{0h} - u_h^\infty \|_\infty.$$

Proof. We have

Step 1

$$u_h^\infty = T_h u_h^\infty;\ u_h^1 = T_h u_h^1,$$

thus

$$\begin{aligned} \| u_h^1 - u_h^\infty \|_\infty &= \| T_h u_h^0 - T_h u_h^\infty \|_\infty \\ &\leq \left(\frac{\mu + c}{\mu + \beta}\right) \| u_h^0 - u_h^\infty \|_\infty \\ &\leq \left(\frac{1 + (\Delta t)c}{1 + (\Delta t)\beta}\right) \| u_h^0 - u_h^\infty \|_\infty. \end{aligned}$$

We assume that

Step k

$$\| u_h^k - u_h^\infty \|_\infty \leq \left(\frac{\mu + c}{\mu + \beta}\right)^k \| u_h^0 - u_h^\infty \|_\infty,$$

then for step $k + 1$,

$$\| u_h^{k+1} - u_h^\infty \|_\infty = \| T_h u_h^k - T_h u_h^\infty \|_\infty \leq \left(\frac{\mu + c}{\mu + \beta}\right) \| u_h^k - u_h^\infty \|_\infty,$$

thus

$$\| u_h^{k+1} - u_h^\infty \|_\infty \leq \left(\frac{\mu + c}{\mu + \beta}\right)^{k+1} \| u_h^0 - u_h^\infty \|_\infty$$

$$= \left(\frac{1 + (\Delta t)c}{1 + (\Delta t)\beta}\right)^{k+1} \| u_h^0 - u_h^\infty \|_\infty. \qquad \square$$

5. Asymptotic Behavior for the PQVI

This section is devoted to the proof of main result of the present paper, where we prove the theorem of the asymptotic behavior in norm L^∞ of parabolic quasi-variational inequalities. We consider u^∞ and u_h^∞ are, respectively, stationary solutions to the continuous and discrete inequalities:

Now we evaluate the variation in L^∞-norm between $u_h(T, x)$, the discrete solution calculated at the moment $T = n\Delta t$ and u^∞, the asymptotic continuous solution of problem (3.9).

Theorem 5 (The main result). *Let u^∞ and u_h^n be the solutions of PQVI* (3.9) *and* (3.6), *respectively. Then under the previous hypotheses and notation, we have*

$$\| u_h(T, x) - u^\infty(x) \|_{L^\infty(\Omega)} \leq C\left[h^2 |\log h|^3 + \left(\frac{1 + (\Delta t)c}{1 + (\Delta t)\beta}\right)^n \right],$$

where C is a constant independent of h and k.

Proof. We have

$$u_h^k(x) = u_h(t, x) \text{ for } t \in \left](k-1)\Delta t; k\Delta t\right[$$

and

$$u_h^n(x) = u_h(T, x),$$

then

$$\| u_h(T, x) - u^\infty \|_{L^\infty(\Omega)} = \| u_h^n(x) - u^\infty \|_{L^\infty(\Omega)}$$

$$\leq \| u_h^n - u_h^\infty \|_{L^\infty(\Omega)} + \| u_h^\infty - u^\infty \|_{L^\infty(\Omega)}.$$

Using Theorem 4 and Proposition 2, It is easily verified that

$$\| u_h(T, x) - u^\infty(x) \|_{L^\infty(\Omega)} \leq C\left[h^2 |\log h|^3 + \left(\frac{1+(\Delta t)c}{1+(\Delta t)\beta}\right)^n\right]. \qquad \square$$

6. Conclusion

In this paper, we have a new approach for the finite element approximation of Parabolic Quasi-Variational Inequalities (PQVIs). We have established a convergence and asymptotic behavior, using the semi-implicit time scheme combined with a finite element spatial approximation of parabolic quasi-variational inequalities in the L^∞-norm. The type of estimation, which we get here, is important for the calculus of quasi-stationary state for the simulation of petroleum or gaseous deposit.

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