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STEADY STATE SOLUTION OF OPTIMAL PRODUCTION CONTROL OF A DYNAMIC FIVE-PRODUCT MANUFACTURING SYSTEM WITH SETUP COSTS AND SETUP TIME

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Abstract

The steady state solution of optimal control of a one-machine five-product manufacturing system with setup changes, operating in a continuous time dynamic environment is considered. The system is deterministic. When production is switched from one product to the other, a known constant setup time and a setup cost are incurred. Each product has specified constant processing time and constant demand rate, as well as a reasonably long supply of raw materials. The problem is formulated as a feedback control problem. The aim is to minimize the total backlog,

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inventory and setup costs incurred over a finite horizon. The optimal solution provides the optimal production rate and setup switching epochs as a function of the state of the system (backlog and inventory levels).

1. Introduction

The idea of a machine that automatically carries out pre assigned task is now common in today's manufacturing [3, 6, 15, 16]. The means by which such system is being controlled by the laws that govern their behavior must be looked into with greater care. These laws sometimes control variables whose values can be changed by someone acting outside and independent of the system itself. There are mainly two kinds of variables - state variables that define precisely what the system is doing at a time t and the control variables that can be used to modify the subsequent behavior of the system [5, 7, 13].

The aim of the Production and Setup Scheduling Problem (PSSP) is to determine the optimal production rates and setup epochs of several products on a single machine [17]. The system is believed to have controllable production rates. There are known constant demand rate and processing time for each product [14]. Any time there is a production switch from one product to the next, a constraint setup time as well as a fixed setup cost are incurred [7, 8, 15, 16]. The objective here is to control the rate of production of each product as well as to control the setup change epochs so as to minimize the total setup, inventory and backlog costs over a finite or infinite planning horizon. The problem therefore, reduces to the Economic Lot Scheduling Problem (ELSP) [11, 15, 16]. When the planning horizon is infinite, the system is in steady state, the machine has fixed production rates, and the objective is to determine lot sizes that minimize the average set up and inventory holding costs per unit time.

A case study of a bottleneck machine is used for the production planning of several products, periodically (say quarterly). The demand rate at each period is forecasted at the beginning of that period and should be satisfied during the planning period. The system is assumed also to have enough capacity to meet the forecasted demand for each product and is fast enough to bring the inventory/backlog of all products to a steady state (which constitutes the most economical way of operating the system). It is clear that the output (inventory/backlog) of the previous period constitute the input for the current period and hence new initial conditions as well as new parameters are to be used. The PSSP is designed as a feedback control problem. The control here must respond to certain criterion. Elhafsi and Bai [8] considered a dynamic two-product manufacturing system with setup costs and setup time. Onanaye and Odekunle [15, 16] considered steady solution of a dynamic three-product and four-product manufacturing systems with setup costs and setup time, respectively. Eugene [10] worked on the problem of detailed scheduling of complex flexible manufacturing systems using optimal flow control. A model problem of scheduling parallel machines was considered to obtain necessary setup conditions and this resulted in a new solution approach that took advantage of a juggling analogy of the production setup scheduling.

Hu and Caramanis [12] solved the three-part type setup problem numerically and deduced structural properties of the optimal policies. Based on the numerical results, they proposed near-optimal policies. Akella and Kumar [1] dealt with a single machine (with two states: up and down), single product problem. They obtained an explicit solution for the threshold inventory level. The preceding sections give details of the preliminary of the study; problem formulation and design; optimal steady state solution; and a solved numerical example.

2. Preliminary

It was shown by Bai and Elhafsi [4], Onanaye and Odekunle [15, 16] that the optimal solution to the earlier formulated problem can be obtained in two parts consisting a transient period and a steady state period. The steady state corresponds to the case where the state of the system (inventory/backlog) has already reached a cyclic schedule and where the produced lots of each product type are of constant size over time. Let t_s be the time instant, the system reaches the steady state. The total cost can then be written as

$$J_\mu(x(t), t) = \int_t^{t_f} g(k(s), \sigma(s))ds + \int_{t_s}^{t_f} g(x(s), \sigma(s))ds$$

$$= J_\mu^T[(x(t), t) + (t_f - t_s)]_\mu^s(x(t_s), t_s),$$

where $J_\mu^T(x(t), t)$ is referred to as the transient cost and $J_\mu^s(x(t), t)$ is the average steady state cost.

3. Problem Formulation and Design

Consider a one-machine manufacturing system producing at least four products, each has a constant demand rate d_i $(i = 1, 2, 3, 4, 5)$ when production is switched from product type *j* to product type *i* $(j \neq i \neq l \neq m \neq n)$, given that constant setup time δ_i and setup cost k_i $(i = 1, 2, 3, 4, 5)$ are incurred. This formulation follows the general framework introduced by Kimemia and Gershwin [13], where the production flow is modeled as continuous rather than discrete. Let $x_i(t)$ be the production surplus of product type i $(i = 1, 2, 3, 4, 5)$ at time *t*; a positive value of $x_i(t)$ represents inventory, while a negative value represents backlog. Let $u_i(t)$ be the controlled production rate of machine producing type *i* product at time *t*. Let

$$\sigma(t) = \begin{Bmatrix} \sigma_1(t),\ \sigma_2(t),\ \sigma_3(t),\ \sigma_4(t),\ \sigma_5(t),\ \sigma_{12345}(t),\ \sigma_{12435}(t), \\ \sigma_{12453}(t),\ \sigma_{12543}(t),\ \sigma_{12534}(t),\ \sigma_{12354}(t), \\ \sigma_{13245}(t),\ \sigma_{13254}(t),\ \sigma_{13425}(t),\ \sigma_{13542}(t),\ \sigma_{13524}(t), \\ \sigma_{13452}(t),\ \sigma_{14352}(t),\ \sigma_{14235}(t),\ \sigma_{14253}(t), \\ \sigma_{14523}(t),\ \sigma_{14532}(t),\ \sigma_{14253}(t),\ \sigma_{15243}(t),\ \sigma_{15234}(t), \\ \sigma_{15432}(t),\ \sigma_{15423}(t),\ \sigma_{15324}(t),\ \ldots,\ \sigma_{54213}(t), \\ \sigma_{54132}(t),\ \sigma_{54123}(t),\ \sigma_{54312}(t),\ \sigma_{54321}(t),\ \sigma_{54231}(t) \end{Bmatrix} \tag{1}$$

be the set of state vectors of the machine at time *t*. Then

$$\sigma_i(t),\ \sigma_{ijlmn}(t),\ [i \neq\ j \neq 1 \neq m, \neq n,\ i = 1, 2, 3, 4, 5,\ j = 1, 2, 3, 4, 5$$
$$l = 1, 2, 3, 4, 5,\ m = 1, 2, 3, 4, 5,\ n = 1, 2, 3, 4, 5]$$

are right continuous binary functions of *t* such that $\sigma_i(t) = 1$ when the machine is ready to produce type *i* product and $\sigma_i(t) = 0$ otherwise; $\sigma_{ijlmn}(t) = 1$ when the machine is undergoing a setup change from product type *j* to product type *i* and from type *i* to type *l* and from type *l* to *m* and from type *m* to *n* and $\sigma_{ijlmn}(t) = 0$ otherwise. Let $s(t)$ be a non-negative right continuous function of *t* which takes on the value δ_i at the beginning of each setup change to type i $(i = 1, 2, 3, 4, 5)$ and decreases with time. $s(t)$ indicates whether a setup is completed or not. It is assumed initially that, the machine is not setup for either product type.

The dynamics of the system can be described by:

$$\frac{dx_i(t)}{dt} = u_i(t) - d_i,\ i = 1, 2, 3, 4, 5, \tag{2}$$

$$0 \leq u_i(t) \leq U_i\sigma_i(t),\ i = 1, 2, 3, 4, 5, \tag{3}$$

where U_i is the maximum production rate of the machine when it is producing type i products. The setup states of the machine obey the following set of constraints:

$$\left\{\begin{array}{l} \sigma_1(t) + \sigma_2(t) + \sigma_3(t) + \sigma_4(t) + \sigma_5(t) + \sigma_{12345}(t) + \sigma_{12435}(t) \\ + \sigma_{12453}(t) + \sigma_{12543}(t) + \sigma_{12534}(t) \\ + \sigma_{12354}(t) + \sigma_{13245}(t) + \sigma_{13254}(t) + \sigma_{13425}(t) + \sigma_{13542}(t) \\ + \sigma_{13524}(t) + \sigma_{13452}(t) + \sigma_{14352}(t) \\ + \sigma_{14235}(t) + \sigma_{14253}(t) + \sigma_{14523}(t) + \sigma_{14532}(t) + \sigma_{14253}(t) \\ + \sigma_{15243}(t) + \sigma_{15234}(t) + \sigma_{15432}(t) \\ + \sigma_{15423}(t) + \sigma_{15324}(t) + \cdots + \sigma_{54213}(t) + \sigma_{54132}(t) + \sigma_{54123}(t) \\ + \sigma_{54312}(t) + \sigma_{54321}(t) + \sigma_{54231}(t) \end{array}\right\}. \tag{4}$$

If $\sigma_i(t^-) = 1$ and $\sigma_i(t) = 0$, then $s(t) = \delta_j$ and $\sigma_{ijlmn}(t) = 1$, (5)

If $s(t^-) > 0$ and $\sigma_{ijlmn}(t^-) = 1$, then $\dot{s}(t) = -1$ and $\sigma_{ijlmn}(t) = 1$, (6)

If $s(t^-) = 0$ and $\sigma_{ijlmn}(t^-) = 1$, then $\sigma_{ijlmn}(t) = 0$ and $s(t) = 0$ and

$\sigma_j(t) = 1$, (7)

for $i = 1, 2, 3, 4, 5$; $j = 1, 2, 3, 4, 5$; $i \neq j \neq l \neq m \neq n$, where $\dot{s}(t)$ denotes the time derivative of $s(t)$. For mathematical convenience, it is assumed that setup costs are incurred at a constant rate [2]:

$$x_i = k_i / d_i,\ i = 1, 2, 3, 4, 5$$

money unit per unit time, during a setup change. Hence, at the end of a setup change of product type *i*, having a total cost of k_i, the instantaneous cost which penalizes the production for being ahead $(\text{i.e., } x_i > 0)$ or being behind $(\text{i.e., } x_i < 0)$. The demand is given by

$$h(x) = \sum_{i=1}^{i=5} [c_i^+ x_i^+(t) + c_i^- x_i^-(t)],$$

where c_i^+ and c_i^- are instantaneous inventory holding per unit cost and instantaneous backlog cost, respectively, $x_i^+(t) = \max\{x_i(t), 0\}$ and $x_i^-(t) = \max\{-x_i(t), 0\}$. The total instantaneous cost is given by:

$$g(x, \sigma) = \sum_{i=1,\, j \neq i \neq l \neq m \neq n}^{i=5} \left[c_i^+ x_i^+(t) + c_i^- x_i^-(t) + \frac{k_i}{\delta_i} \sigma_{ijlmn}(t) \right].$$

The state variable of the system is given by the vector $x(t) = [x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)]$. The variables

$$u(t) = [u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)]$$

and equation (1) are the complete control variables denoted by (σ, u). The capacity set represents the set of feasible production rates [9, 15, 16]. The setup state is $\sigma(t)$ at time *t*, it is given by

$$\Omega[\sigma(t)] = \{u(t) | 0 \leq u_i(t) \leq U_i \sigma_i(t),\ i = 1, 2, 3, 4, 5\}.$$

The setup constraints set is the set of all possible setup vectors (1) satisfying constraints (3) to (7). We shall denote this set by ϕ. Denote by $\psi(\phi, \Omega)$ the set of feasible controls, which depends on ϕ and Ω. The set of admissible control policies β, is the set of all mappings $\mu : R^5 \to \psi(\Omega, \phi)$ which satisfy $\mu(x) = (\sigma, u)$ and which are piecewise continuously differentiable. These admissible control policies are feedback controls that specify the control actions (setup and production rate of the machine) to be taken, given the state of the system. The objective is to determine an optimal control policy $\mu^* \in \beta$, corresponding to a setup control

$$\sigma^*(t) = \left\{\begin{matrix} \sigma_1^*(t),\ \sigma_2^*(t),\ \sigma_3^*(t),\ \sigma_4^*(t),\ \sigma_5^*(t),\ \sigma_{12345}^*(t),\ \sigma_{12435}^*(t),\ \sigma_{12453}^*(t), \\ \sigma_{12543}^*(t),\ \sigma_{12534}^*(t),\ \sigma_{12354}^*(t), \\ \sigma_{13245}^*(t),\ \sigma_{13254}^*(t),\ \sigma_{13425}^*(t),\ \sigma_{13542}^*(t),\ \sigma_{13524}^*(t), \\ \sigma_{13452}^*(t),\ \sigma_{14352}^*(t),\ \sigma_{14235}^*(t),\ \sigma_{14253}^*(t), \\ \sigma_{14523}^*(t),\ \sigma_{14532}^*(t),\ \sigma_{14253}^*(t),\ \sigma_{15243}^*(t),\ \sigma_{15234}^*(t), \\ \sigma_{15432}^*(t),\ \sigma_{15423}^*(t),\ \sigma_{15324}^*(t),\ \ldots,\ \sigma_{54213}^*(t), \\ \sigma_{54132}^*(t),\ \sigma_{54123}^*(t),\ \sigma_{54312}^*(t),\ \sigma_{54321}^*(t),\ \sigma_{54231}^*(t) \end{matrix}\right\}$$

and *a* production rate control

$$u^* = (u_1^*,\ u_2^*,\ u_3^*,\ u_4^*,\ u_5^*)$$

that minimizes for each initial state $x(t)$ the cost function

$$J_\mu(x(t),\ t) = \int_t^{t_f} g(x(s),\ \sigma(s))\,ds,$$

where the minimization is over all functions $\mu[x(\tau)] = [\sigma(\tau),\ u(\tau)]$ such that $x(\tau)$, $\sigma(\tau)$ and $\mu(\tau)$ satisfy constraint (3) and $\sigma(\tau),\ \mu(\tau) \in \Psi(\varphi,\ \Omega)$ for $t \le \tau \le t_f$, where t_f is assumed to be sufficiently large.

3.1. Assumption

Throughout this work, it shall be assumed that $t_f - t$, the planning horizon is long enough so that the system reaches the steady state and stays there for a long period.

4. Optimal Steady State Solution

We first introduced the following notation for product i $(i = 1,\ 2,\ 3,\ 4,\ 5)$:

t_i is time-spent producing at maximum rate within a cyclic schedule

Y_i is time spent producing at demand rate within a cyclic schedule

S_i is maximum inventory level

s_i is maximum backlog level

$\gamma_i = c_i^+ c_i^- / (c_i^+ + c_i^-)$ cost factor

$\ell_i = d_i / U_i$ utilization factor of the machine by product i

$A_i = \gamma_i / 2d_i (1 - \ell_i)$

$T = \sum_{i=1}^{i=4} (t_i + Y_i + \delta_i)$ length of the cyclic schedule

$\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5$ total setup time during T

$k = k_1 + k_2 + k_3 + k_4 + k_5$ total setup cost during T

$\ell = \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5$ total utilization factor of the machine

$\alpha_i = (1 - \ell_i)/(1 - \ell)$.

Now, consider the following optimization problem:

Minimize

$$F(S, Y) = \sum_{i=1}^{i=5} (k_i + (1/2\, d_i (i - e_i))) \left(c_i^+ S_i^2 + c_i^- (S_i - Q_i^2) \Big/ \sum_{i=1}^{i=5} (Q_i / d_i) - \delta \right)$$

subject to

$$Q_i = q_{i\left(1 + \sum_{i=1}^{i=4} (1 - \ell j) Y_j / \delta (1 - \ell) Y_i / \delta\right)}, \; i = 1, 2, 3, 4, 5$$

$$\delta_i \geq 0, \; Y_i \geq 0, \; Q_i \geq 0, \; i = 1, 2, 3, 4, 5$$

$$Q_i = S_i - s_i$$

and

$$q_i = d_i \delta (1 - \ell_i)/(1 - \ell) \text{ for } i = 1, 2, 3, 4, 5$$

The optimal solution to this optimization problem is

$$S_i = Q_i c_i^- / (c_i^+ + c_i^-), \; i = 1, 2, 3, 4, 5$$

$$s_i = -Q_i c_i^- / (c_i^+ + c_i^-), \; i = 1, 2, 3, 4, 5.$$

Substituting S_i in $F(S, Y)$ gives

$$F(S, Y) = \frac{[k + A_1Q_1^2 + A_2Q_2^2 + A_3Q_3^2 + A_4Q_4^2 + A_5Q_5^2]}{[(Q_1/d_1) + (Q_2/d_2) + (Q_3/d_3) + (Q_4/d_4) + (Q_5/d_5) - \delta]}, \tag{8}$$

where

$$k = k_1 + k_2 + k_3 + k_4 + k_5 \text{ (total setup cost during time } T\text{)}$$

The quantities $Q_1, Q_2, Q_3, Q_4, Q_5, Y_1, Y_2, Y_3, Y_4, Y_5$ are calculated as

$$Q_1^u = q_1d_5\gamma_5(1 + \sqrt{(1 + 2k(1-\ell)(\alpha_1/\gamma_1d_1 + \alpha_5/\gamma_5d_5)/\delta^2)/(\alpha_5d_1\gamma_1 + \alpha_1d_5\gamma_5)})$$

$$Q_2^u = q_2d_1\gamma_1(1 + \sqrt{(1 + 2k(1-\ell)(\alpha_1/\gamma_1d_1 + \alpha_2/\gamma_2d_2)/\delta^2)/(\alpha_2d_1\gamma_1 + \alpha_1d_2\gamma_3)})$$

$$Q_3^u = q_3d_2\gamma_2(1 + \sqrt{(1 + 2k(1-\ell)(\alpha_2/\gamma_2d_2 + \alpha_3/\gamma_3d_3)/\delta^2)/(\alpha_3d_2\gamma_2 + \alpha_2d_3\gamma_3)})$$

$$Q_4^u = q_4d_3\gamma_3(1 + \sqrt{(1 + 2k(1-\ell)(\alpha_3/\gamma_3d_3 + \alpha_4/\gamma_4d_4)/\delta^2)/(\alpha_3d_4\gamma_4 + \alpha_4d_3\gamma_3)})$$

$$Q_5^u = q_5d_4\gamma_4(1 + \sqrt{(1 + 2k(1-\ell)(\alpha_4/\gamma_4d_4 + \alpha_5/\gamma_5d_5)/\delta^2)/(\alpha_4d_5\gamma_5 + \alpha_5d_4\gamma_4)})$$

$$Y_1^\mu = Q_3^\mu/d_3 - Q_1^\mu\,\ell_1/(1-\ell_1)d_1 - \delta \tag{9}$$

$$Y_2^\mu = Q_1^\mu/d_1 - Q_2^\mu\,\ell_2/(1-\ell_2)d_2 - \delta \tag{10}$$

$$Y_3^\mu = Q_2^\mu/d_2 - Q_3^\mu\,\ell_3/(1-\ell_3)d_3 - \delta \tag{11}$$

$$Y_4^\mu = Q_3^\mu/d_3 - Q_4^\mu\,\ell_4/(1-\ell_4)d_4 - \delta \tag{12}$$

$$Y_5^\mu = Q_4^\mu/d_4 - Q_5^\mu\,\ell_5/(1-\ell_5)d_5 - \delta \tag{13}$$

If $Y_1^\mu \geq 0, Y_2^\mu \geq 0, Y_3^\mu \geq 0, Y_4^\mu \geq 0$ and $Y_5^\mu \geq 0$ THEN

$$Y_1^* = Y_1^\mu,\ Y_2^* = Y_2^\mu,\ Y_3^* = Y_3^\mu,\ Y_4^* = Y_4^\mu,\ Y_5^* = Y_5^\mu,$$

$$Q_1^* = Q_1^\mu,\ Q_2^* = Q_2^\mu,\ Q_3^* = Q_3^\mu,\ Q_4^* = Q_4^\mu,\ Q_5^* = Q_5^\mu$$

STOP

ELSE

$$Q_1^{\mu} = (1-\ell_1)d_1\sqrt{((k+A_5(\delta d_5)^2)/(d_1^2(1-\ell_1)^2A_1+d_2^2\ell_1^2A_2+d_3^2\ell_1^2A_3+d_4^2\ell_1^2A_4+d_5^2\ell_1^2A_5))}$$

$$Q_2^{\mu} = d_2\delta+\ell_1 d_2\sqrt{((k+A_5(\delta d_5)^2)/(d_1^2(1-\ell_1)^2A_1+d_2^2\ell_1^2A_2+d_3^2\ell_1^2A_3)+d_4^2\ell_1^2A_4+d_5^2\ell_1^2A_5)}$$

$$Q_3^{\mu} = d_3\delta+\ell_1 d_3\sqrt{((k+A_5(\delta d_5)^2)/(d_1^2(1-\ell_1)^2A_1+d_2^2\ell_1^2A_2+d_3^2\ell_1^2A_3+d_4^2\ell_1^2A_4+d_5^2\ell_1^2A_5))}$$

$$Q_4^{\mu} = d_4\delta+\ell_1 d_4\sqrt{((k+A_5(\delta d_5)^2)/(d_1^2(1-\ell_1)^2A_1+d_2^2\ell_1^2A_2+d_3^2\ell_1^2A_3+d_4^2\ell_1^2A_4+d_5^2\ell_1^2A_5))}$$

$$Q_5^{\mu} = d_5\delta+\ell_1 d_5\sqrt{((k+A_5(\delta d_5)^2)/(d_1^2(1-\ell_1)^2A_1+d_2^2\ell_1^2A_2+d_3^2\ell_1^2A_3+d_4^2\ell_1^2A_4+d_5^2\ell_1^2A_5))}$$

Compute $Y_1^{\mu}, Y_2^{\mu}, Y_3^{\mu}, Y_4^{\mu}$ and Y_5^{μ} (use (9), (10), (11), (12) and (13), respectively)

If $Y_5^{\mu} > 0$ THEN

Calculate $C5 = F(Y_1^{\mu}, Y_2^{\mu}, Y_3^{\mu}, Y_4^{\mu}, Y_5^{\mu})$ (use (8))

ELSE

$C5 = \infty$

END

LET

$$Q_1^{\mu} = d_1\delta+\ell_2 d_1\sqrt{((k+A_4(\delta d_4)^2)/(d_2^2(1-\ell_2)^2A_2+d_1^2\ell_2^2A_1+d_3^2\ell_2^2A_3+d_4^2\ell_2^2A_4+d_5^2\ell_2^2A_5))}$$

$$Q_2^{\mu} = (1-\ell_2)d_2\sqrt{((k+A_4(\delta d_4)^2)/(d_2^2(1-\ell_2)^2A_2+d_1^2\ell_2^2A_1+d_3^2\ell_2^2A_3+d_4^2\ell_2^2A_4+d_5^2\ell_2^2A_5))}$$

$$Q_3^{\mu} = d_3\delta+\ell_2 d_1\sqrt{((k+A_4(\delta d_4)^2)/(d_2^2(1-\ell_2)^2A_2+d_1^2\ell_2^2A_1+d_3^2\ell_2^2A_3+d_4^2\ell_2^2A_4+d_5^2\ell_2^2A_5))}$$

$$Q_4^{\mu} = d_4\delta+\ell_2 d_1\sqrt{((k+A_4(\delta d_4)^2)/(d_2^2(1-\ell_2)^2A_2+d_1^2\ell_2^2A_1+d_3^2\ell_2^2A_3+d_4^2\ell_2^2A_4+d_5^2\ell_2^2A_5))}$$

$$Q_5^{\mu} = d_4\delta+\ell_2 d_1\sqrt{((k+A_4(\delta d_4)^2)/(d_2^2(1-\ell_2)^2A_2+d_1^2\ell_2^2A_1+d_3^2\ell_2^2A_3+d_4^2\ell_2^2A_4+d_5^2\ell_2^2A_5))}$$

Recompute $Y_1^{\mu}, Y_2^{\mu}, Y_3^{\mu}, Y_4^{\mu}$ and Y_5^{μ} (use (9), (10), (11), (12) and (13), respectively)

If $Y_4^{\mu} > 0$ THEN

Calculate $C4 = F(Y_1^{\mu}, Y_2^{\mu}, Y_3^{\mu}, Y_4^{\mu}, Y_5^{\mu})$ (use (8))

ELSE

$C4 = \infty$

END

LET

$$Q_1^{\mu} = d_1\delta + \ell_3 d_1 \sqrt{((k + A_3(\delta d_3)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + d_4^2\ell_3^2 A_4) + d_5^2\ell_3^2 A_5))}$$

$$Q_2^{\mu} = d_3\delta + \ell_3 d_3 \sqrt{((k + A_3(\delta d_3)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + d_4^2\ell_3^2 A_4 + d_5^2\ell_3^2 A_5))}$$

$$Q_3^{\mu} = (1+\ell_3)\, d_3 \sqrt{((k + A_3(\delta d_3)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + d_4^2\ell_3^2 A_4 + d_5^2\ell_3^2 A_5))}$$

$$Q_4^{\mu} = d_4\delta + \ell_3 d_4 \sqrt{((k + A_3(\delta d_3)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + d_4^2\ell_3^2 A_4 + d_5^2\ell_3^2 A_5))}$$

$$Q_5^{\mu} = d_5\delta + \ell_4 d_5 \sqrt{((k + A_3(\delta d_3)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + d_4^2\ell_3^2 A_4 + d_5^2\ell_3^2 A_5))}$$

Recompute $Y_1^{\mu}, Y_2^{\mu}, Y_3^{\mu}, Y_4^{\mu}$ and Y_5^{μ} (use (9), (10), (11), (12) and (13), respectively)

If $Y_3^{\mu} > 0$ THEN

Calculate $C3 = F(Y_1^{\mu}, Y_2^{\mu}, Y_3^{\mu}, Y_4^{\mu}, Y_5^{\mu})$ (use (8))

ELSE

$C3 = \infty$

END

LET

$$Q_1^{\mu} = d_1\delta + \ell_4 d_1 \sqrt{((k + A_2(\delta d_2)^2)/(d_4^2(1-\ell_4)^2 A_4 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_2 + d_3^2\ell_4^2 A_3 + d_5^2\ell_4^2 A_5))}$$

$$Q_2^{\mu} = d_2\delta + \ell_4 d_2 \sqrt{((k + A_2(\delta d_2)^2)/(d_4^2(1-\ell_4)^2 A_4 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_2 + d_3^2\ell_4^2 A_3 + d_5^2\ell_4^2 A_5))}$$

$$Q_3^{\mu} = d_3\delta + \ell_4 d_3 \sqrt{((k + A_2(\delta d_2)^2)/(d_4^2(1-\ell_4)^2 A_4 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_2 + d_3^2\ell_4^2 A_3 + d_5^2\ell_4^2 A_5))}$$

$$Q_4^{\mu} = (1+\ell_4)\, d_4 \sqrt{((k + A_2(\delta d_2)^2)/(d_4^2(1-\ell_4)^2 A_4 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_2 + d_3^2\ell_4^2 A_3 + d_5^2\ell_4^2 A_5))}$$

$$Q_5^{\mu} = d_5\delta + \ell_4 d_4 \sqrt{((k + A_2(\delta d_2)^2)/(d_4^2(1-\ell_4)^2 A_4 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_2 + d_3^2\ell_4^2 A_3 + d_5^2\ell_4^2 A_5))}$$

Recompute $Y_1^\mu, Y_2^\mu, Y_3^\mu, Y_4^\mu$ and Y_5^μ (use (9), (10), (11), (12) and (13), respectively)

If $Y_2^\mu > 0$ THEN

Calculate $C2 = F(Y_1^\mu, Y_2^\mu, Y_3^\mu, Y_4^\mu, Y_5^\mu)$ (use (8))

ELSE

$C2 = \infty$

END

LET

$$Q_1^\mu = d_1\delta + \ell_5 d_1 \sqrt{((k + A_1(\delta d_1)^2)/(d_5^2(1-\ell_5)^2 A_5 + d_1^2\ell_5^2 A_1 + d_2^2\ell_5^2 A_3 + d_3^2\ell_5^2 A_3 + d_4^2\ell_5^2 A_4))}$$

$$Q_2^\mu = d_2\delta + \ell_5 d_2 \sqrt{((k + A_1(\delta d_1)^2)/(d_5^2(1-\ell_5)^2 A_5 + d_1^2\ell_5^2 A_1 + d_2^2\ell_5^2 A_3 + d_3^2\ell_5^2 A_3 + d_4^2\ell_5^2 A_4))}$$

$$Q_3^\mu = d_3\delta + \ell_5 d_3 \sqrt{((k + A_1(\delta d_1)^2)/(d_5^2(1-\ell_5)^2 A_5 + d_1^2\ell_5^2 A_1 + d_2^2\ell_5^2 A_3 + d_3^2\ell_5^2 A_3 + d_4^2\ell_5^2 A_4))}$$

$$Q_4^\mu = d_4\delta + \ell_5 d_4 \sqrt{((k + A_1(\delta d_1)^2)/(d_5^2(1-\ell_5)^2 A_5 + d_1^2\ell_5^2 A_1 + d_2^2\ell_5^2 A_3 + d_3^2\ell_5^2 A_3 + d_4^2\ell_5^2 A_4))}$$

$$Q_5^\mu = (1-\ell_5) d_5 \sqrt{((k + A_1(\delta d_1)^2)/(d_5^2(1-\ell_5)^2 A_5 + d_1^2\ell_5^2 A_1 + d_2^2\ell_5^2 A_3 + d_3^2\ell_5^2 A_3 + d_4^2\ell_5^2 A_4))}$$

Recompute $Y_1^\mu, Y_2^\mu, Y_3^\mu, Y_4^\mu$ and Y_5^μ (use (9), (10), (11), (12) and (13), respectively)

If $Y_1^\mu > 0$ THEN

Calculate $C1 = F(Y_1^\mu, Y_2^\mu, Y_3^\mu, Y_4^\mu, Y_5^\mu)$ (use (8))

LET

$$Q_1^\mu = q_1, Q_2^\mu = q_2, Q_3^\mu = q_3, Q_4^\mu = q_4$$

and

$$Q_5^\mu = q_5 \to Y_1^\mu = 0, Y_2^\mu = 0, Y_3^\mu = 0, Y_4^\mu \text{ and } Y_5^\mu = 0$$

Calculate $C0 = F(Y_1^{\mu}, Y_2^{\mu}, Y_3^{\mu}, Y_4^{\mu}, Y_5^{\mu})$ (use (8))

$$(Y_1^*, Y_2^*, Y_3^*, Y_4^*, Y_5^*) : (Q_1^*, Q_2^*, Q_3^*, Q_4^*, Q_5^*)$$

$$= \arg\min(Y_1^{\mu}, Y_2^{\mu}, Y_3^{\mu}, Y_4^{\mu}, Y_5^{\mu}),$$

$$(Q_1^{\mu}, Q_2^{\mu}, Q_3^{\mu}, Q_4^{\mu}, Q_5^{\mu})\{C0, C1, C2, C3, C4, C5\}$$

END IF

5. Numerical Example

Consider the following system:

Product Type 1:

$c_1^+ = 5/\text{Unit}/\text{day}$, $c_1^- = 25/\text{Unit}/\text{day}$, $U_1 = 10/\text{day}$, $d_1 = 3.125/\text{day}$,

$\delta_1 = 1\text{day}$ and $k_1 = 250.00$.

Product Type 2:

$c_2^+ = 5/\text{Unit}/\text{day}$, $c_2^- = 25/\text{Unit}/\text{day}$, $U_2 = 12/\text{day}$, $d_2 = 3.6/\text{day}$,

$\delta_2 = 1\text{day}$ and $k_2 = 300.00$.

Product Type 3:

$c_3^+ = 5/\text{Unit}/\text{day}$, $c_3^- = 25/\text{Unit}/\text{day}$, $U_3 = 14/\text{day}$, $d_3 = 4.075/\text{day}$,

$\delta_3 = 1\text{day}$ and $k_3 = 350.00$.

Product Type 4:

$c_4^+ = 5/\text{Unit}/\text{day}$, $c_4^- = 25/\text{Unit}/\text{day}$, $U_4 = 16/\text{day}$, $d_4 = 6.613/\text{day}$,

$\delta_4 = 1\text{day}$ and $k_4 = N400.00$.

Product Type 5:

$c_5^+ = 5/\text{Unit}/\text{day}$, $c_5^- = 25/\text{Unit}/\text{day}$, $U_5 = 18/\text{day}$, $d_4 = 5.222/\text{day}$,

$\delta_5 = 1\text{day}$ and $k_5 = N450.00$.

The optimal production and setup planning for this system can be described as follows (after rounding up):

Setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 17 Units; setup the machine and produce product type1 at the rate of 10 Units/day until its surplus level reaches 13 Units; setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 18 Units; setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 20 Units. Switch the control action to the cyclic schedule given as follows: setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 0; continue producing product type 2 at the demand rate of 5 Units/day until the surplus level of product Type 1 drops to 8 Units; at this moment, increase the production rate of product type 2 to 12 Units/day until its surplus level reaches 19 Units. At this point, setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 20 Units; setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 17 Units; setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 21 Units; setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 18 Units. Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 0; continue producing product type 3 at the demand rate of 4.5 Units/day until the surplus level of product type 2 drops to 11 Units; at this moment, increase the production rate of product type 3 to 14 Units/day until its surplus level reaches 21 Units. Again, we now, setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 13 Units; setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 20 Units; setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 20 Units, setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 21 Units. Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 0; continue producing product type 1 at the demand rate of 4 Units/day until the surplus level of product type 3 drops to 14 Units; at this moment, increase the production rate of product type 1 to 10 Units/day until its surplus level reaches 20 Units. Again, we now, setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 13 Units; setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 23 Units; setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 20 Units, setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 32 Units. Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 0; continue producing product type 1 at the demand rate of 3.5 Units/day until the surplus level of product type 4 drops to 21 Units; at this moment, increase the production rate of product type 1 to 10 Units/day until its surplus level reaches 20 Units. Setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 17 Units; setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 23 Units; setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 18 Units; setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 24 Units. Switch the control action to the cyclic schedule given as follows: setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 0; continue producing product type 2 at the demand rate of 3 Units/day until the surplus level of product type 4 drops to 13 Units; at this moment, increase the production rate of product type 2 to 12 Units/day until its surplus level reaches 19 Units. Again, we now, setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 20 Units; setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 23 Units; setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 21 Units, setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 24 Units. Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 0; continue producing product type 3 at the demand rate of 2.5 Units/day until the surplus level of product type 4 drops to 15 Units; at this moment, increase the production rate of product type 3 to 14 Units/day until its surplus level reaches 21 Units. Again, we now, setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 15 Units; setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 26 Units; setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches

23 Units, setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 36 Units. Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 0; continue producing product type 1 at the demand rate of 3.5 Units/day until the surplus level of product type 5 drops to 24 Units; at this moment, increase the production rate of product type 1 to 10 Units/day until its surplus level reaches 23 Units. Setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 20 Units; setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 26 Units; setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 21 Units; setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 27 Units. Switch the control action to the cyclic schedule given as follows: setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 0; continue producing product type 2 at the demand rate of 3 Units/day until the surplus level of product type 5 drops to 15 Units; at this moment, increase the production rate of product type 2 to 12 Units/day until its surplus level reaches 22 Units. Again, we now, setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 23 Units; setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 26 Units; setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 22 Units, setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 27 Units. Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 0; continue producing product type 3 at the demand rate of 2.5 Units/day until the surplus level of product type 5 drops to 17 Units; at this moment, increase the production rate of product type 3 to 14 Units/day until its surplus level reaches 24 Units. Again, we now, setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 26 Units; setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 26 Units; setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 25 Units, setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 27 Units. Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 4 at the rate

of 16 Units/day until its surplus level reaches 0; continue producing product type 4 at the demand rate of 2.8 Units/day until the surplus level of product type 5 drops to 19 Units; at this moment, increase the production rate of product type 4 to 14 Units/day until its surplus level reaches 27 Units.

Now, we have come to the end of one complete cyclic schedule. The machine is setup again to start with product type 2 and the processes continue on and on.

6. Conclusion

In conclusion, the goal of the Production and Setup Scheduling Problem (PSSP) is to minimize the total backlog, inventory and setup cost incurred over a finite horizon. The optimal solution provides the optimal production rate and setup switching epochs as a function of the state of the system (backlog and inventory levels).

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