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A NEW FORMALISM FOR THE STUDY OF NATURAL TENSOR FIELDS OF TYPE $(0, 2)$ ON MANIFOLDS AND FIBRATIONS

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Abstract

In order to study tensor fields of type $(0, 2)$ on manifolds and fibrations we introduce a new formalism that we called s-space. The s-spaces induced a one to one correspondence between the $(0, 2)$ tensor fields and some differential matricial applications. Using this relationship, we generalized the concept of natural tensor without making use of the theory of natural operators and differential invariants.

1. Introduction

In [9], Kowalski and Sekizawa defined and characterized the *natural tensor fields* on the tangent bundle *TM* of a manifold *M*. They called a tensor $\tilde{g}$ of type $(0, 2)$ on *TM natural* if it cames from a second order natural operator of a metric *g* on *M*. They showed that there exist *natural F-metrics* ξ_1, ξ_2 and ξ_3 (i.e. a bundle morphism of the form $\xi : TM \oplus TM \oplus TM \to M \times \mathbb{R}$ linear in the second and in

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the third argument) derived from *g*, such that $\tilde{g} = \xi_1^{s,g} + \xi_2^{h,g} + \xi_3^{v,g}$ with ξ_1 and ξ_3 symmetric, where $\xi_1^{s,g}$, $\xi_2^{h,g}$ and $\xi_3^{v,g}$ are the classical Sasaki, horizontal and vertical lift of ξ_1, ξ_2 and ξ_3, respectively. Also, Kowalski and Sekizawa [10] study the *natural tensor fields* on the linear frame bundles of a manifold endowed with a linear connection.

In [2], Calvo and Keilhauer showed that any $(0, 2)$ tensor field on *TM* over a Riemannian manifold (M, g) admits a global matrix representation. Using this one to one relationship, they defined and characterized what they called *natural tensor*. In the symmetric case, this concept coincides with the one of Kowalski and Sekizawa. Keilhauer [7] defined and characterized the tensor fields of type $(0, 2)$ on the linear frame bundle of a Riemannian manifold *LM* endowed with a linear connection. The *natural tensors* on the tangent and cotangent bundle T^*M of a semi Riemannian manifold was characterized by Araujo and Keilhauer in [1]. The main idea of [1], [2] and [7] is to use a suitable fiber bundle *P* in order to see the tensor fields on *TM*, T^*M and *LM* as matricial functions from *P* to $\mathbb{R}^{m\times m}$. The principal difference between [9] and [10], [1, 2] and [7] is that these last works do not make use of the theory of differential invariant developed by Krupka [11] (see [8, 12]).

The aim of this work is generalized the notion of natural tensor fields in the sense of [1], [2] and [7] to manifolds and fibrations. With this purpose we introduce the concept of *s-space*. In Section 2, we define and give some examples of *s-spaces*. We see general properties of *s-spaces*, for example, that there exists a one to one relationship between the tensor fields of type $(0, 2)$ and some types of matricial maps. This relationship allows us to study the tensor fields in the sense of [2]. We characterize the *s-spaces* which its group acts without fixed point. We study some general statement of *morphisms* of *s-spaces* and tensor fields on manifolds in Section 3. In Section 4, we define *connections* on *s-spaces*, that coincides with the well known notion of connection when the *s-space* is also a principal fiber bundle. We give a condition that a *s-space* endowed with a connection has to be satisfied in order to has a parallelizable space manifold. Also, using a *connection* we show a useful way of lift metrics to the space manifold of the *s-space*. The concept of *s-space* gives several notions of naturality. The λ-*natural* and λ-*natural tensors with* *respect to a fibration* are defined in Section 5. We give examples and see that these notions extend the idea of naturality of [1], [2] and [7]. In Section 7, we define the notion of *atlas of s-spaces* and use them to generalize the λ-*naturality*. In Section 8, we consider some *s-spaces* over a Lie group and characterized the *natural tensors fields* on it. Finally, we study the bundle metrics on a principal fiber bundle endowed with a linear connection.

2. s-spaces

Definition 2.1. Let M be a manifold of dimension n. Then a collection $\lambda = (N, \psi, O, R, \{e_i\})$ is called a *s-space over M* if:

(a) N is a manifold.

(b) $\psi : N \to M$ is a submersion.

(c) O is a Lie group and R is a right action of the group O over N which is transitive in each fibers. The action also must satisfied that $\psi \circ R_a = \psi$ for all $a \in O$.

(d) $e_i : N \to TM$, with $1 \le i \le n$, are differential functions such that $\{e_1(z), ..., e_n(z)\}$ is a basis of $M_{\psi(z)}$ for all $z \in N$.

If $\psi(z) = p$, then $\{e_1(z), ..., e_n(z)\}$ and $\{e_1(z.a), ..., e_n(z.a)\}$ are bases of M_p. Therefore there exists an invertible matrix $L(z, a)$ such that $\{e_i(z.a)\} = \{e_i(z)\}$. $L(z, a)$, (i.e. $e_i(z.a) = \sum_{j=1}^{n} e^l(z) L_i^l(z, a)$ for $1 \le i \le n$). If the matrix L only depends of the parameter of the Lie group O, we have a differentiable map $L : O \to GL(n)$ such that

$$\{e_i\} \circ R_a = \{e_i\} \cdot L(a).$$

We called this map the *base change morphism* of the s-space λ. It easy to see that L is a group morphism. In this case, we define that λ have a *rigid base change*. From now on, we will consider only this class of s-spaces.

In the sequel, unless otherwise stated, $\dim M = n$, $\dim O = k$ and we will denote the Lie algebra of O by $\mathfrak{o}$. Also, we assume that all tensor are of type $(0, 2)$.

Example 2.2. The linear frame bundle of M induces a s-space $\lambda = (LM, \pi, GL(n), (\cdot), \{\pi_i\})$ over M, where π is the projection of the bundle, $(\cdot)$ is the canonical action of the general linear group over LM and $\pi_i(p, u) = u_i$. The base change morphism is given by $L(a) = a$ for all $a \in GL(n)$. This example shows that every manifolds admits a s-space. For simplicity of notation, let us denote this s-space with LM. If we consider a Riemannian metric on M or an orientation, then the bundle of orthonormal frames and the bundle of orientated bases induced similar s-spaces over M.

Example 2.3. Let $\alpha = (P, \pi, G, \cdot)$ be a principal fiber bundle over M, and ω be a connection on α. Let $\lambda = (N, \psi, O, R, \{e_i\})$, where

(a) $N = \{(p, u, w) : p \in P, u$ and w are a bases of $M_{\pi(p)}$ and $\mathfrak{g}$, respectively$\}$.

(b) $\psi(p, u, w) = p$.

(c) $O = GL(n) \times GL(k)$ and $R_{(a,b)}(p, u, w) = (p, u \cdot a, w \cdot b)$.

(d) For $1 \leq i \leq n$ and $1 \leq j \leq k$, $e_i(p, u, w)$ is the horizontal lift with respect to ω of u_i at p and $e_{n+j}(p, u, w)$ is the only vertical vector on P_p such that $\omega(p)(e_{n+j}(p, u, w)) = w_j$.

λ is a s-space over P and its base change morphism is given by $L(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

Example 2.4. This example can be found in [7]. Let M be *a* manifold and ∇ be a linear connection on it. Let $K : TTM \to TM$ be the connection function induced by ∇ (i.e., K is the unique function that for $v \in M_p$ satisfies that $K|_{TM_v} : TM_v \to M_p$ is a surjective linear map and for any vector field Y on M such that $Y(p) = v$, $K(Y_{*_p}(w)) = \nabla_w Y)$. For $1 \leq i, j \leq n$, consider the 1-forms θ^i and ω^i_j defined by $\pi_{*_{(p,u)}}(b) = \sum_{i=1}^{n} \theta^i(p, u)(b)u_i$ and $K((\pi_j)_{*_{(p,u)}}(b)) = \sum_{i=1}^{n} \omega^i_j(p, u)(b)u_i$. Let $\lambda = (LM \times GL(n), \psi, GL(n), R, \{H_i, V^i_j\})$, where $\psi(p, u, b) = (p, u \cdot b)$, the action is

$R_a(p, u, b) = (p, u \cdot a, a^{-1}b)$ and $\{H_i, V_j^i\}$ is dual to $\{\theta^i, \omega_j^i\}$. λ is a s-space over the frame bundle of M with base change morphism $L(a) \equiv Id_{n\times n}$.

The importance of the s-spaces for the study of the tensor fields is given by the following proposition.

Proposition 2.5. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ *be a s-space over M with base change morphism L. There is a one to one correspondence between tensor fields of type* $(0, 2)$ *on M and the differentiable maps* ${}^{\lambda}T : N \to \mathbb{R}^{n\times n}$ *that satisfy the invariance property*

$$ {}^{\lambda}T \circ R_a = (L(a))^t \cdot {}^{\lambda}T \cdot L(a). $$

Proof. Let T be a tensor on M. Consider the matrix function ${}^{\lambda}T : N \to \mathbb{R}^{n\times n}$ defined by $[{}^{\lambda}T(z)]_j^i = T(\psi(z))(e_i(z), e_j(z))$. For $a \in O$, we have that the (i, j) entry of the matrix ${}^{\lambda}T(z \cdot a)$ is $[{}^{\lambda}T(z \cdot a)]_j^i = T(\psi(z \cdot a))(e_i(z \cdot a), e_j(z \cdot a)) =$ $T(\psi(z))\left(\sum_{r=1}^{n} e_r(z)L(a)_i^r, \sum_{s=1}^{n} e_s(z)L(a)_j^s\right) = \sum_{r,s=1}^{n} L(a)_i^r \cdot {}^{\lambda}T(z)_s^r \cdot L(a)_j^s$, hence ${}^{\lambda}T$ satisfies the invariance property. Let $F : N \to \mathbb{R}^{n\times n}$ be a differentiable function that satisfies the invariance property. If X is a vector field on M, then it induces a map ${}^{\lambda}X = (x_1, ..., x_n) : N \to \mathbb{R}^n$ such that $X(\psi(z)) = \sum_{i=1}^{n} x_i(z)e_i(z)$. It is easy to check that ${}^{\lambda}X \circ R_a = {}^{\lambda}X \cdot [L(a)^t]^{-1}$. Then, we define $T(p)(X, Y) = {}^{\lambda}X(z) \cdot F(z) \cdot ({}^{\lambda}Y(z))^t$, where $\psi(z) = p$. Consider z and $\bar{z}$ such that $\psi(z) = \psi(\bar{z}) = p$. Since O acts transitively on the fibers of N, there exists $a \in O$ that satisfies $\bar{z} = z \cdot a$. Therefore, ${}^{\lambda}X(\bar{z}) \cdot F(\bar{z}) \cdot ({}^{\lambda}Y(\bar{z}))^t = {}^{\lambda}X(z) \cdot (L(a)^t)^{-1} \cdot L(a)^t \cdot {}^{\lambda}F(z) \cdot L(a) \cdot (L(a))^{-1}({}^{\lambda}Y(z))^t$ $= {}^{\lambda}X(z) \cdot F(z) \cdot ({}^{\lambda}Y(z))^t$, what it proves that T it is well defined. Given X and Y vector fields on M, $T(X, Y) : M \to \mathbb{R}$ is a differentiable function because $T(X, Y) \circ \psi$ is differentiable and ψ is a submersion. Since T is $\mathcal{F}(M)$-bilinear, we conclude that T is a tensor of type $(0, 2)$ on M. Finally, it is clear that ${}^{\lambda}T = F$. □

Theorem 2.6. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ *be a s-space over M, such that O acts without fixed point. Then* (N, ψ, O, R) *is a principal fiber bundle over M.*

Let us denote by $z \sim z'$ the equivalence relation induced by the action of the group O on the manifold N. To prove the previous Theorem we will need the following next two lemmas.

Lemma 2.7. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ *be a s-space over M. Then* N/O *has differentiable manifold structure and* $\pi : N \to N/O$ *is a submersion.*

Proof. Consider the map $\rho : N \times N \to M \times M$ defined by $\rho(z, z') = (\psi(z), \psi(z')) \cdot \rho$ is a submersion since ψ it is. Let the set $\overline{\Delta} = \{(z, z') : z \sim z'\}$ and Δ be the diagonal submanifold of $N \times N$. Since $z \sim z'$ if and only if $\psi(z) = \psi(z')$, we have that $\overline{\Delta} = \rho^{-1}(\Delta)$. Therefore $\overline{\Delta}$ is a closed submanifold of $N \times N$. It is well known that if a group O acts on a manifold N, N/O has a structure of differentiable manifold such that the canonical projection π is a submersion if and only if $\overline{\Delta}$ is a closed submanifold of $N \times N$. In this case, the differentiable structure of N/O is unique. □

Lemma 2.8. *Under the hypotheses of the previous lemma*:

(i) N/O *is diffeomorphic to M.*

(ii) $\ker \pi_* = \ker \psi_*$.

Proof. Let $f : N/O \to M$ defined by $f([z]) = \psi(z)$. By definition $f \circ \pi = \psi$, then f is differentiable function and $\ker \pi_* \subseteq \ker \psi_*$. On the other hand, let $g : M \to N/O$ be the function defined by $g(p) = \pi(z)$ where $z \in N$ satisfies that $\psi(z) = p$. Since O acts transitively on the fibers of N, g is well defined. As $\pi = g \circ \psi$ we have that g is a differentiable function and that $\ker \psi_* \subseteq \ker \pi_*$. An easy verification shows that $g \circ f = Id_{N/O}$ and $f \circ g = Id_M$. □

Remark 2.9. If $\lambda = (N, \psi, O, R, \{e_i\})$ is a s-space over M, then (N, ψ, O, R) is a principal fiber bundle over N/O.

Proof of Theorem 2.6. It remains to prove that (N, ψ, O, R) satisfies the local triviality property, (i.e. all $p \in M$ has an open neighbourhood U on M, and a diffeomorphism $\tau : \psi^{-1}(U) \to U \times O$ such that $\tau = (\psi, \phi)$, where $\phi(z \cdot a) = \phi(z) \cdot a$

for all $a \in O$). Let $p \in M$, take $[z_0] \in N/O$ such that $f([z_0]) = p$. As (N, ψ, O, R) is a principal fiber bundle over N/O. Then there exist an open neighbourhood V of $[z_0]$ and a diffeomorphism $\overline{\tau} = (\pi(z), \overline{\phi}(z))$ such that satisfy the local triviality property. $U = f(V)$ is an open neighbourhood of p on M, since f is a diffeomorphism, and it satisfies that $\psi^{-1}(U) = \pi^{-1}(V)$. Finally, if we define $\tau : \psi^{-1}(U) \to U \times O$ by $\tau(z) = (\psi(z), \overline{\phi}(z))$, U and τ satisfy the local triviality property on p. □

Remark 2.10. Note that there exist s-spaces that are not principal fiber bundles. For example, let $\lambda = (\mathbb{R}^n \times (\mathbb{R}^n - \{0\}), pr_1, GL(n), R, \{e_i\})$ over R^n, where $pr_1(p, q) = p$, $R_a(p, q) = (p, q \cdot a)$ and $e_i(p, q) = \left.\frac{\partial}{\partial u_i}\right|_p$ is the basis of $\mathbb{R}^n_p$ induced by the canonical coordinate system of $\mathbb{R}^n$.

We say that a s-space $\lambda = (N, \psi, O, R, \{e_i\})$ over M is a *principal fiber bundle* if (N, ψ, O, R) is a principal fiber bundle over M.

We denote by $S_z = \{a \in O : z \cdot a = z\}$ the stabilizer's group of the action R at z. It is well known that if for a point $z \in N$ the orbit $z \cdot O$ is locally closed (i.e. if $w \in z \cdot O$, there exists an open neighbourhood V of w on N, such that $V \cap z \cdot O$ is a closed set of V), then $z \cdot O$ is a submanifold of N and $f_z([a]) = z \cdot a$ is a diffeomorphism between O/S_z and $z.O$.

Proposition 2.11. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ *be a s-space over M. Then*:

(i) *There exists* $s \in \mathbb{N}_0$ *such that* $\dim S_z = s$ *for all* $z \in N$.

(ii) $\dim N = \dim M + \dim O - s$.

Proof. Let $z \in N$ and $\psi(z) = p$. Since f is a submersion we have that $\dim N = \dim \ker \psi_{*_z} + \dim M$ and $\dim \ker \psi_{*_z} = \dim \psi^{-1}(p)$. Note that $z \cdot O = \psi^{-1}(p)$, since O acts transitively on the fibers. As $\psi^{-1}(p)$ is locally closed, it follows that $\dim O/S_z = \dim \psi^{-1}(p)$. Therefore, $\dim N = \dim M + \dim O - \dim S_z$ for all z, so S_z is of constant dimension. □

Given a s-space λ over M, it will be important to know which are the tensors on M that satisfy that ${}^{\lambda}T$ is a constant matrix. It is clear that not for every matrix $A \in \mathbb{R}^{n\times n}$ there exists a tensor T on M such that ${}^{\lambda}T = A$. A necessary and sufficient conditions for this holds is that $L(a)^t \cdot A \cdot L(a) = A$ for all $a \in O$. In that case, we said that λ *admits matrix representations* of type A. To finish the section we will state some conditions in order to guarantee that a s-space admits matrix representations of certain class of diagonal matrices.

For $\nu = 0, 1, ..., n-1$, we denote by I_ν the following matrix of $\mathbb{R}^{n\times n}$

$$I_\nu = \left(\begin{array}{cccccc} -1 & & & & & \\ & \ddots^{\,\nu} & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots^{\,n-\nu} & \\ & & & & & 1 \end{array}\right) \text{ if } \nu \geq 1 \text{ and } I_0 = Id_{n\times n}.$$

With O_ν we denote the orthonormal group of index ν. If $\nu = 0$, then $O_0 = O(n)$.

Proposition 2.12. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ *be a s-space over M with base change morphism L. If* $0 \leq \nu \leq n-1$, *the following conditions are equivalent*:

(i) $Img(L) \subseteq O_\nu$.

(ii) λ *admits matrix representations of type* I_ν.

(iii) *There is a semi-Riemannian metric on M of signature* ν *such that* $\{e_1(z), ..., e_n(z)\}$ *is an orthonormal basis of* $M_{\psi(z)}$ *for all* $z \in N$.

(iv) *There exists a tensor T on M that satisfies* ${}^{\lambda}T(z) = I_\nu$ *for all* $z \in \psi^{-1}(p_0)$ *and for a* $p_0 \in M$.

Proof. (i) $\Rightarrow$ (ii) Consider the constant map $F \equiv I_\nu$. Since F satisfies the invariance property, it follows from the Proposition 2.5 the existence of a tensor such that ${}^{\lambda}T = I_\nu$. (ii) $\Rightarrow$ (iii) If ${}^{\lambda}T = I_\nu$, then T is a semi-Riemannian metric of index ν and $T(\psi(z))(e_i(z), e_j(z)) = [I_\nu]^i_j$. (iii) $\Rightarrow$ (iv) Is immediately. (iv) $\Rightarrow$ (i) Let $a \in O$ and z_0 such that $\psi(z_0) = p_0$. Then $I_\nu = I_\nu(z_0.a) = L(a)^t \cdot I_\nu \cdot L(a)$ for all $a \in O$. □

The next Proposition is a consequence of the fact that $O(m) \cap O_\nu = \{D \in O(m) : D = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with $A \in O(\nu)$ and $B \in O(m - \nu)\}$.

Proposition 2.13. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ *be a s-space over M with base change morphism L and* $1 \leq \nu \leq n - 1$. λ *admits matrix representations of type* I_0 *and* I_ν *if and only if there exist differentiable functions* $L_1 : O \to O(\nu)$ *and* $L_2 : O \to (n - \nu)$ *such that*

$$L(a) = \begin{pmatrix} L_1(a) & 0 \\ 0 & L_2(a) \end{pmatrix}.$$

Proposition 2.14. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ *be a s-space over M with O connected.* λ *admits matrix representations of type* I_ν *for all* $0 \leq \nu \leq n - 1$ *if and only if* λ *admits matrix representations of type A for all constant matrix* $A \in \mathbb{R}^{n \times n}$.

Proof. If λ admits matrix representations of type $I_0, I_1, \ldots, I_\nu$, from the proposition above we have that $L(a) = \begin{pmatrix} \pm 1 & & & \\ & \ddots^{\,\nu} & & \\ & & \pm 1 & \\ & & & l(a) \end{pmatrix}$ with $l(a) \in O(n - \nu)$. Since L is differentiable and $L(ab) = L(a).L(b)$, we see that $L(a) = \begin{pmatrix} Id_{\nu \times \nu} & 0 \\ 0 & f(a) \end{pmatrix}$. If $\nu = n$, then $L \equiv I_{n \times n}$ and the proposition it follows. □

3. Morphisms of s-spaces

Definition 3.1. Let $\lambda = (N, \psi, O, R, \{e_i\})$ and $\lambda' = (N', \psi', O', R', \{e_i'\})$ be s-spaces over M. We call a pair (f, τ) a morphism of s-spaces between λ and λ' if

(a) $f : N \to N'$ is a differentiable function.

(b) $\tau : O \to O'$ is a morphism of Lie groups.

(c) $\psi' \circ f = \psi$.

(d) $f(z \cdot a) = f(z) \cdot \tau(a)$ for all $z \in N$ and $a \in O$.

Note that if λ and λ' are principal fiber bundles, (f, τ) is a principal bundle morphism.

Example 3.2. Let $\lambda = (N, \psi, O, R, \{e_i\})$ be a s-space over M and let LM be the s-space induced by the linear frame bundle of M. Consider the pair $(\Gamma, L) : \lambda \to LM$, where $\Gamma(z) = (\psi(z), e_1(z), ..., e_n(z))$ and L is the base change morphism of λ, then (Γ, L) is a morphism of s-spaces.

Remark 3.3. Let λ and λ' be s-spaces over M and let $(f, \tau) : \lambda \to \lambda'$ be a morphism between them. If λ' is a principal fiber bundle and τ is injective, then λ is a principal fiber bundle.

Remark 3.4 It is easy to see that if τ is surjective then f is also surjective. If O' acts without fixed point, then we have that τ is surjective if and only if f is surjective; the injectivity of τ implies that of f; and if τ is bijective then so is f. If O and O' act without fixed point, then f is injective if and only if τ is it.

Let $(f, \tau) : \lambda \to \lambda'$ be a morphism of s-spaces. As $\psi'(f, (z)) = \psi(z)$ we have that $\{e_i'(f(z))\}$ and $\{e_i(z)\}$ are bases of $M_{\psi(z)}$. Therefore, there exists $C(z) \in GL(n)$ that satisfies $\{e_i'(f(z))\} = \{e_i(z)\} \cdot C(z)$. We called the function $C : N \to GL(n)$ the *linking map* of (f, τ). For example, the linking map of the morphism given in Example 3.2 is $C(z) = Id_{n\times n}$. Let λ be a s-space over M with base change morphism L and let $a_0 \in O$. Consider $(f, \tau) : \lambda \to \lambda$ defined by $f(z) = R_{a_0}$ and $\tau(b) = Ad(a_0^{-1})(b)$, then $C(z) = L(a_0)$.

The linking map of a morphism (f, τ) satisfies that $C(z.a) = (L(a))^{-1} \cdot C(z)$. $L'(\tau(a))$, where L and L' are the base change morphism of λ and λ', respectively. The relationship between two linking maps is given by $C_{(g,\gamma)}(z) = C_{(f,\tau)}(z)$ $\cdot L'(a(z))$, where $a : N \to O$ is a differentiable function.

Let $\lambda = (N, \psi, O, R, \{e_i\})$ be a s-space over M and consider a function $F : N \to \mathbb{R}^{n\times n}$. We say that F comes from a tensor if there exists a tensor T on M such that ${}^{\lambda}T = F$. In this case, we say that F is the matrix representation (or the induced matrix function by) of T with respect to λ.

Proposition 3.5. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ *and* $\lambda' = (N', \psi', O', R', \{e'_i\})$ *be two s-spaces over M with base change morphism L and* L' *respectively, and let* $(f, \tau) : \lambda \to \lambda'$ *be a morphism. If* ${}^{\lambda'}T$ *is the matrix representation of T with respect to* λ', *then* ${}^{\lambda'}T \circ f$ *comes from a tensor if and only if*

$$(L(a))^t \cdot ({}^{\lambda'}T \circ f)(z).L(a) = (L'(\tau(a)))^t \cdot ({}^{\lambda'}T \circ f)(z) \cdot L'(\tau(a))$$

for all $z \in N$ *and* $a \in O$.

Proof. If ${}^{\lambda'}T \circ f$ comes from a tensor, then it satisfies $({}^{\lambda'}T \circ f)(z \cdot a) = (L(a))^t \cdot ({}^{\lambda'}T \circ f)(z) \cdot L(a)$. Therefore, ${}^{\lambda'}T(f(z \cdot a)) = L'(\tau(a))^t \cdot {}^{\lambda'}T(f(z)) \cdot L'(\tau(a))$. The other implication follows by a verification of the invariance property. □

Remark 3.6. Let T be a tensor on M. From the above Proposition it follows that until the kth iteration of T by (f, τ) comes from a tensor on M if and only if $L^t \cdot (C^t)^j \cdot {}^{\lambda}T \cdot C^j \cdot L = (L' \circ \tau)^t \cdot (C^t)^j \cdot {}^{\lambda}T \cdot C^j \cdot (L' \circ \tau)$ for all $1 \leq j \leq k$.

Corollary 3.7. *The following sentences are equivalent*:

(i) *For all tensor T on* M, ${}^{\lambda'}(T \circ f)$ *comes from a tensor on M.*

(ii) $L' \circ \tau = \pm L$.

Proposition 3.8. *Let* $(f, \tau) : \lambda \to \lambda'$ *be a morphism of s-spaces and let T be a tensor on M. Then*

$$({}^{\lambda'}T \circ f)(z) = (C(z))^t \cdot {}^{\lambda}T(z) \cdot C(z),$$

where C is the linking map of (f, τ).

Proof.

$$[({}^{\lambda'}T \circ f)(z)]^i_j = T(\psi'((f(z))))(e'_i(f(z)), e'_j(f(z)))$$

$$= T(\psi(z))\left(\sum_{r=1}^{m}(C(z))_i^r e_r(z),\ \sum_{s=1}^{m}(C(z))_j^s e_s(z)\right)$$

$$= \sum_{r,s=1}^{m}(C(z))[{}^{\lambda}T(z)]_s^r \cdot (C(z))_j^s. \qquad \square$$

Definition 3.9. Let $(f, \tau) : \lambda \to \lambda'$ be a morphism of s-spaces and T be a tensor on M. We say that T is *invariant* by (f, τ) if ${}^{\lambda'}T \circ f = {}^{\lambda}T$. Let us denote with $I_{(f, \tau)}$ the subspace of the invariant tensors of (f, τ).

Let λ be a s-space over M. If $(f, \tau) : \lambda \to LM$ is the morphism given in the Example 3.2, then all the tensors are invariant. Given a s-space $\lambda = (N, \psi, O, R, \{e_i\})$ and $T \neq 0$, then there exists $a \in GL(n)$ and $z \in N$ such that $a^t \cdot T(z) \cdot a \neq T(z)$. Therefore, if we consider the s-space $\lambda' = (N, \psi, O, R, \{e_i'\})$, where $\{e_i'\} = \{e_i\} \cdot a$, T is not an invariant tensor by the morphism (Id_N, Id_O).

Proposition 3.10. *Let $(f, \tau) : \lambda \to \lambda'$ be a morphism and T be a tensor on M. If there exists $k \in \mathbb{N}$ such that the kth iteration by (f, τ) of T is an invariant tensor, then T is an invariant tensor.*

Proof. Let us denoted by ${}^{\lambda}T^j$ and ${}^{\lambda'}T^j$ the matrix representation of the jth iteration of T with respect to λ and λ', respectively. ${}^{\lambda}T^k = {}^{\lambda'}T^k \circ f = C^t \cdot {}^{\lambda}T^k \cdot C$, since the kth iteration is an invariant tensor. On the other hand, ${}^{\lambda}T^k = ({}^{\lambda'}T^{k-1} \circ f)$ $= C^t \,{}^{\lambda}T^{k-1} C = C^t \cdot ({}^{\lambda'}T^{k-2} \circ f) \cdot C = (C^t)^2 \cdot {}^{\lambda} T^{k-2} \cdot C^2 = (C^t)^{k-1} \cdot {}^{\lambda} T \cdot C^{k-1}$, hence ${}^{\lambda}T = C^t \cdot {}^{\lambda}T \cdot C$. $\square$

Let T be a tensor on M and $\lambda = (N, \psi, O, R, \{e_i\})$ be a s-space over M. For each $z \in N$, consider the Lie subgroup of $GL(n)$ defined by $G_T^{\lambda}(z) = \{D \in GL(n) : D^t \cdot {}^{\lambda}T(z) \cdot D = {}^{\lambda}T(z)\}$. We call it the *group of invariance* of T at z. For simplicity of notation we write $G_T(z)$ instead of $G_T^{\lambda}(z)$. A tensor T is invariant by (f, τ) if and only if $C(z) = G_T(z)$ for all $z \in N$.

If $\psi(z) = \psi(z')$, then we have that $G_T(z) \simeq G_T(z')$. This is because $\varphi_a : G_T(z') \to G_T(z)$, defined by $\varphi_a(D) = L(a) \cdot D \cdot L(a^{-1}) = Ad(L(a))(D)$ for $a \in O$ such that $z' = z \cdot a$, is a homomorphism of Lie groups. We called the subset $F_T = \{(z, g) : z \in N$ and $g \in G_T(z)\}$ of $N \times GL(n)$ the invariance set of T. If there is a tensor T on M that admits a matrix representation of the type $\alpha \cdot Id_{n\times n}$ with $a \neq 0$, then $F_T = N \times O(n)$. Let λ be the s-space of Example 2.4. If T is the tensor on LM such that ${}^{\lambda}T = \begin{pmatrix} 0 & Id_{m\times m} \\ -Id_{m\times m} & 0 \end{pmatrix}$ with $m = \dfrac{n + n^2}{2}$. Then $F_T = LM \times GL(n) \times \mathcal{S}_m$, where $\mathcal{S}_m$ denotes the symplectic group of $\mathbb{R}^{2m\times 2m}$. In general, F_T does not has a manifold structure. The invariant tensor by a morphism $(f, \tau) : \lambda \to \lambda'$ are those that satisfy that $(z, C(z)) \in F_T$ for any z in N.

Remark 3.11. Let $(f, \tau) : \lambda \to \lambda'$ be a morphism with linking map C. If $T \in I_{(f,\tau)}$ and T is non-degenerated, then $\det(C(z)) = \pm 1$ for any z in N.

4. Connections on s-spaces

Given $\lambda = (N, O, \psi, \mathbb{R}, \{e_i\})$ a s-space over M, for $z \in N$ let us denote by V_z the vertical subspace at z induced by the projection ψ (i.e. $V_z = \ker \psi_{*_z}$). Note that $\dim V_z = k - s$, where s is the dimension of the stabilizer S_z and $k = \dim O$. We adapt the concept of connection in fibrations (see [13]) to s-spaces as follows:

Definition 4.1. A connection on a s-space λ over M is $(1, 1)$ tensor ϕ on N that satisfies:

(1) $\phi_z : N_z \to V_z$ is a linear map.

(2) $\phi^2 = \phi$, ϕ is a projection to the vertical subspace.

(3) $\phi_{z.a}((R_a)_{*_z}(b)) = (R_a)_{*_z}(\phi(b))$.

Note that (3) has sense because $(R_a)_{*_z}(V_z) = V_{z\cdot a}$.

We called to $H_z = \ker \phi_z$ the *horizontal subspace* at z. It is clear that

$N_z = H_z \oplus V_z$. Since $\phi_{za}((R_a)_{*_z}(\phi(z)(b))) = (R_a)_{*_z}(\phi(z)(b)) = (R_a)_{*_z}(0) = 0$, $(R_a)_{*_z}(H_z) = H_{z\cdot a}$. As in the case of connections in principal fiber bundles we have that: There is a connection ϕ on λ if and only if there exists a differentiable distribution on $N(z \to H_z)$ such that $N_z = H_z \oplus V_z$ and $H_{z\cdot a} = (R_a)_{*_z}(H_z)$. If we have a distribution with these properties, we define $\phi(z)(b) = b^v$, where $b = b^h + b^v$.

Definition 4.2. Let $\lambda = (N, \psi, O, R, \{e_i\})$ be a s-space over M endowed with a connection ϕ. Let $v \in M_p$ and $z \in \psi^{-1}(p)$. We called *horizontal lift* of v at z to the unique vector $v_z^h \in N_z$ such that $\psi_{*_z}(v_z^h) = v$ and $v_z^h \in H_z$.

Given a vector field X on N, let $H(X)$ and $V(X)$ the vector fields that satisfy that $H(X)(z) \in H_z, V(X)(z) \in V_z$ and $X(z) = H(X)(z) + V(X)(z)$ for all $z \in N$. We called $H(X)$ and $V(X)$ the *horizontal* and the *vertical projections* of X. Is easy to see that $H(X)$ and $V(X)$ are smooth vector fields if X is a smooth vector field.

Proposition 4.3. *Let X be a vector field on M. Then there exists a unique vector field* X^h *on N such that* $X^h(z) \in H_z$ *and* $\psi_{*_z}(X^h(z)) = X(\psi(z))$ *for all* $z \in N$.

Proof. Let $p_0 \in M$ and $z_0 \in N$ such that $\psi(z_0) = p_0$. As ψ is a submersion, there exist two charts (U, x) and (V, y) centered at p_0 and z_0 respectively that satisfy $\psi(U) \subseteq V$ and $y \circ \psi \circ x^{-1}(a_1, ..., a_n, a_{n+1}, ..., a_m) = (a_1, ..., a_n)$. If $X(p) = \sum_{i=1}^n \rho^i(p)\frac{\partial}{\partial y_i}\Big|_p$ for $p \in U$, let the vector field on V defined by $\tilde{X}_U(z) = \sum_{i=1}^n (\rho^i \circ \psi)(z)\frac{\partial}{\partial x_i}\Big|_z$, then we have that $\psi_*(\tilde{X}) = X \circ \psi$. For this reason, we can take an open covering $\{U_i\}_{i\in I}$ of N such that for each U_i we have a field $\tilde{X}_i \in \chi(U_i)$ that satisfies the previous property. Let $\{\zeta_i\}_{i\in I}$ be a unit partition subordinate to the covering $\{U_i\}_{i\in I}$. Consider the vector field $\tilde{X} \in \chi(N)$ given by $\tilde{X} = \sum_{i\in I} \zeta_i \cdot \tilde{X}_i$. $\tilde{X}$ satisfies that $\psi_{*_z}(\tilde{X}(z)) = X(\psi(z))$ for all $z \in N$. Finally,

$H(\tilde{X})$ is the vector fields that we looked for. The uniqueness follows from the fact that $\psi_{*_z}|_{H_z}: H_z \to M_{\psi(z)}$ is an isomorphism. □

Remark 4.4. The horizontal distribution $z \to H_z$ is trivial since $\{e_i^h(z) = (e_i(z))_z^h\}_{i=1}^n$ is a base of H_z for all $z \in N$ and $\{e_i^h\}_{i=1}^n$ are smooth vector fields.

For any z in N let the function $\sigma_z : O \to N$ given by $\sigma_z(a) = z \cdot a$. If $X \in \mathfrak{o}$, let $V(X)(z) = (\sigma_z)_{*_e}(X) \in V_z$, where e is the unit element of O. If the group O acts effectively and $X \neq 0$ is easy to see that V is not the null vector field. If O acts without fixed point, then $V(X)(z) \neq 0$ for all $z \in N$ and $X \neq 0$. Anyway if $\{X_1, ..., X_k\}$ is a base of $\mathfrak{o}$, then $\{V(X_1)(z), ..., V(X_k)(z)\}$ spanned V_z. It is not difficult to see that $\ker(\sigma_z)_{*_e} = T_e S_z$. The 1-forms θ_i on N defined by $\psi_{*_z}(b) = \sum_{i=1}^n \theta^i(z)(b) e_i(z)$ are a basis of the null space of the vertical subspace. Straightforward calculations show that the 1-forms θ_i satisfy that

$$L(a).\begin{pmatrix} \theta^1(z.a)((R_a)_{*_z}(b)) \\ \vdots \\ \theta^n(z.a)((R_a)_{*_z}(b)) \end{pmatrix} = \begin{pmatrix} \theta^1(z)(b) \\ \vdots \\ \theta^n(z)(b) \end{pmatrix} \text{ for all } z \in N \text{ and } a \in O.$$

Proposition 4.5. *Let λ be a s-space over M such that exists a subspace $\tilde{V}$ of $\mathfrak{o}$ that satisfies $\dim \tilde{V} = k - s$ $(s = \dim S_z)$ and $\tilde{V} \cap T_e S_z = \{0\}$ for all $z \in N$. If λ admits a connection, then the tangent bundle of N is trivial.*

Proof. Let $\{X_1, ..., X_{k-s}\}$ be a base of $\tilde{V}$, then the vertical vector fields $V_i(z) = (\sigma_z)_{*_e}(X_i)$ with $i = 1, ..., k - s$ are a base of V_z for all $z \in N$. Therefore the frame $\{e_1^h, ..., e_n^h, V_1, ..., V_{k-s}\}$ trivializes the tangent bundle of N. □

Remark 4.6. With the same hypothesis of the Proposition above is easy to see that $\{\theta^1(z), ..., \theta^n(z), W^1(z), ..., W^{k-s}(z)\}$, where W_i are the 1-forms defined by $\phi_z(b) = \sum_{i=1}^{k-s} W^i(z)(b) V_i(z)$, is a basis of N_z^*. Note that it is the dual base of $\{e_1^h(z), ..., e_n^h(z), V_1(z), ..., V_{k-s}(z)\}$.

Remark 4.7. Let $\lambda = (N, \psi, O, R, \{e_i\})$ be a s-space over M that is also a principal fiber bundle. It is well known that every principal fiber bundle admits a smooth distribution that is transversal to the vertical distribution and is invariant by the action of the group O, see [4], so there exists a connection on λ. On the other hand, the group O acts on N without fixed point and the hypothesis of the Proposition 4.5 are satisfied. Therefore, the tangent bundle of N is trivial.

Remark 4.8. Let G be a metric on N such that the maps R_a are isometries for any a in O. If O is compact and N is a closed manifold, then N admits *a* metric with this property (see [4]). Let H_z be the subspace of N_z orthogonal to V_z. It is easy to see that $z \to H_z$ induces a connection on λ.

Remark 4.9. In the situation of Proposition 4.5, we can lift a metric G on M to *a* metric $\tilde{G}$ on N in a natural way as follows:

$$\tilde{G} = \psi^*(G) + \sum_{i=1}^{k-s} W^i \otimes W^i.$$

The projection $\psi : (N, \tilde{G}) \to (M, G)$ is a Riemannian submersion. The metric $\tilde{G}$ can be very useful because using the fundamental equations of a Riemannian submersion [16] we can relate the curvature tensors of both metrics. If we chose appropriately the s-space over M, the calculation of the curvature tensor of (M, G) can be simplified. For example, In [6] (see also [5]), the curvature tensor of the tangent bundle of a Riemannian manifold endowed with certain class of metrics is computed using this technique.

Remark 4.10. Let λ be a s-space over M and let ∇ be a linear connection on M with connection function K. Consider $K^i : TN \to TM$ defined by

$$K_z^i(b) = K((e_i)_{*_z}(b))$$

and let $H_z = \{b \in N_z : K_z^i(b) = 0 \text{ for } i = 1, ..., n\}$. This smooth distribution is invariant by the group action but it is not necessary complementary to V_z. If $F_z : N_z$ $\to M_{\psi(z)} \times \overbrace{M_{\psi(z)} \times \cdots \times M_{\psi(z)}}^{n \text{ times}}$ is given by $F_z(b) = (\psi_{*_z}(b), K_z^1(b), ..., K_z^n(b))$, it is not difficult to see that the following facts are equivalent:

(i) F_z is injective and $(M_{\psi(z)} \times 0 \times \cdots \times 0) \in Img\ F_z$.

(ii) $N_z = H_z \oplus V_z$.

So if λ satisfies (i) and (ii) we have that the distribution $z \to H_z$ induces a connection on λ. If G is a metric on M let the $(0, 2)$ symmetric tensor on N given by

$$\tilde{G}(A, B) = c(z)G(\psi_{*_z}(A), \psi_{*_z}(B)) + \sum_{i=1}^{n} l_i(z)G(K^i(A), K^i(B)),$$

where c, l_i are positive differentiable functions. If F is injective, $\tilde{G}$ is a Riemannian metric. If λ is the s-space LM, $c = 1$ and $l_i = 1$ for $i = 1, ..., n$, then $\tilde{G}$ is the well know Sasaki-Mok metric (see [3] and [15]).

5. Natural Tensor Fields

5.1. Natural tensor fields on fibrations

In this section, we will study certain class of tensors on a manifolds and fibrations. With a tensor T on a fibration we want to mean that T is a tensor on the space manifold of the fibration. If $\alpha = (P, \pi, \mathbb{F})$ is a fibration we will consider a particular class of s-spaces over P in order to take into account the structure of the fibration for the study of the tensors on it.

Definition 5.1. Let $\alpha = (P, \pi, \mathbb{F})$ be a fibration on M and $\lambda = (N, \psi, O, R, \{e_i\})$ be a s-space over P. Then we say that λ is a *trivial s-space over* α if $N = N' \times \mathbb{F}$.

Example 5.2. The s-space $\lambda = (LM \times GL(n), \psi, GL(n), R, \{H_i, V_j^i\})$ given in the Example 2.4 is a trivial s-space over the linear frame bundle of M.

Definition 5.3. Let $\alpha = (P, \pi, \mathbb{F})$ be a fibration and $\lambda = (N \times \mathbb{F}, \psi, O, R, \{e_i\})$ be a trivial s-space over α. Then we say that a tensor T on P is λ-*natural* with respect to α if ${}^{\lambda}T(z, w) = {}^{\lambda}T(w)$ (i.e. its matrix representation depends only of the parameter w of the fiber $\mathbb{F}$).

Remark 5.4. Let M be *a* manifold endowed with *a* linear connection ∇ and a Riemannian metric g. If we consider the s-spaces $\lambda = (LM \times GL(n), \psi, GL(n), R,$

$\{H_i, V^i_j\})$ (Example 2.4) and $\lambda' = (O(M) \times GL(n), \psi, O(n), R, \{H_i, V^i_j\})$, where $O(M)$ is the manifold of orthonormal bases of (M, g), the action of the orthonormal group and the projection are similar to that ones of λ. The λ-naturality and λ'-naturality with respect to $(LM, \pi, GL(n))$ agree with the concept of natural tensor with respect to the connection ∇ and with respect to the metric g given in [7].

Remark 5.5. There exist s-spaces such that the concept of λ-natural with respect to the fibration agree with the known cases of naturality. So, our definition also generalizes the notion of natural tensor on the tangent and the cotangent bundle of a Riemannian (see [2] and Example 6.2) and semi-Riemannian manifold (see [1]).

5.2. Natural tensor fields on manifolds

In view of the definition of λ-natural with respect to a fibration, it seems interesting to ask what it means to be λ-natural with respect to a manifold? A manifold M can be viewed as a trivial fibration $\alpha_M = (M \times \{a\}, pr_1, \{a\})$. Therefore, there is a one to one correspondence between the s-spaces over λ and the trivial s-spaces over α. A s-space $\lambda = (N, \psi, O, R, \{e_i\})$ over M induced the $\lambda' =$ $(N \times \{a\}, \psi, O, R, \{e_i\})$ over α. A tensor T on M induce a tensor T' on $M \times \{a\}$. Then T' is λ'-natural with respect to a α if and only if ${}^{\lambda'}T'(z, a) = {}^{\lambda'}T'(a)$, hence T' is λ'-natural with respect to a α if and only if ${}^{\lambda}T$ is a constant map. This suggests the following definition:

Definition 5.6. Let λ be a s-space over M and T *a* tensor on M. Then we say that T is λ-*natural* if ${}^{\lambda}T$ is a constant map.

Example 5.7. Let (M, g) be a Riemannian manifold and let $\lambda = (O(M), \pi,$ $O(n), \cdot, \{\pi_i\})$ be the s-space over M induced by the orthonormal frame bundles of M. Since $L(a) = a$ for all $a \in O(n)$, T is λ-natural if and only if ${}^{\lambda}T = k.Id_{n\times n}$ (T is a scalar multiple of the metric g).

Example 5.8. Suppose that the map F of the Remark 4.10 is bijective. Let $\beta =$ $(N, id_N, \{1\}, (\cdot), \{(e_i(z))^h, (e_j(z))^{v(i)}_z\})$ be the s-space over the space manifold of λ, where $\{1\}$ is the trivial group, $(\cdot)$ is the trivial action, $(e_i(z))^h$ is the horizontal lift of $e_i(z)$ at z and $(e_j(z))_z^{v(i)}$ satisfies that $K^i((e_j(z))_z^{v(i)}) = e_j(z)$. If G is a metric on M and $\tilde{G}$ is the generalizes Sasaki-Mok metric on N then

$$^{\beta}\tilde{G}(z) = \begin{pmatrix} [^{\lambda}G] & 0 & \cdots & 0 \\ 0 & [^{\lambda}G] & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & [^{\lambda}G] \end{pmatrix},$$

so $\tilde{G}$ is β natural if and only if G is λ-natural.

Remark 5.9. Let $\alpha = (p, \pi, \mathbb{F})$ be a fibration on M and λ a trivial s-space over α. λ is also a s-space over P. If a tensor T on P is λ-natural then T is λ-natural with respect to α. The converse implication not necessarily holds. Let $\lambda = (O(M) \times GL(n), \psi, O(n), R, \{H_i, V_j^i\})$ over LM. There are more λ-natural tensors with respect to LM than constant maps, see [7].

Remark 5.10. Consider the s-space LM and let T be a LM-natural tensor on M. Let $A \in R^{n\times n}$ such that $^{LM}T \equiv A$. Since the base change morphism of LM is the identity of $GL(n)$, $A = a^t \cdot A \cdot a$ for all $a \in GL(n)$, hence T must be the null tensor. Therefore, for a manifold M the null tensor is the only one that is λ-natural for all the s-spaces over M.

Remark 5.11. If T is λ-natural, we have that $N \times \mathrm{Im}(L) \subseteq F_T$, where $F_T = N \times G$ with G a subgroup of $GL(n)$.

Let $\lambda = (N, O, \psi, R, \{e_i\})$ be a s-space over M. Note that if T is λ-natural and $(f, \tau): \lambda \to \lambda$ is a morphism of s-spaces then $T \in I_{(f,\tau)}$. On the other hand, if $T \in I_{(f,\tau)}$ for all (f, τ) automorphism of λ, then $^{\lambda}T$ is constant in each fiber of N. A necessary and sufficient condition for a tensor T to have a constant matrix representation in each fiber is that $T \in I_{(f_a, \tau_a)}$ for all $a \in O$, where (f_a, τ_a) is the morphism defined by $f_a(z) = R_a(z)$ and $\tau_a(b) = a^{-1}b \cdot a$.

Let us see some facts about the relationship between the natural tensors and the morphisms of s-spaces. The next two proposition follow from Proposition 3.8.

Proposition 5.12. *Let* λ *and* λ' *be two s-spaces over M and* $(f, \tau) : \lambda \to \lambda'$ *be a morphism with linking map C. If T is a* λ'*-natural tensor with* ${}^{\lambda'}T = A \in \mathbb{R}^{n\times n}$, *then T is* λ*-natural if and only if* $(C(z)^{-1})^t \cdot A \cdot C(z)^{-1}$ *is a constant map.*

Proposition 5.13. *Let* $(f, \tau) : \lambda \to \lambda'$ *be a morphism of s-spaces with linking map C and T a tensor on M that is* λ *and* λ' *natural. Let A and* $B \in R^{n\times n}$ *such that* ${}^{\lambda}T = A$ *and* ${}^{\lambda'}T = B$, *then* $C(z)^t \cdot A \cdot C(z) = B$ *for all* $z \in N$.

In particular, if $\lambda = \lambda'$, the image of the linking map of any automorphism has to be included in the group of invariance of all the λ-natural tensors. For example, if $\lambda = (LM \times GL(n), \psi, GL(n), R, \{H_i, V_j^i\})$ and (f, τ) is an automorphism of λ with linking map *C*, then $C(z) = Id_{(n+n^2)\times(n+n^2)}$ for all $z \in LM \times GL(n)$.

Proposition 5.14. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ *and* $\lambda' = (N', \psi', O', R', \{e_i'\})$ *be two s-spaces over* M, $(f, \tau) : \lambda \to \lambda'$ *be a morphism of s-space, T* a λ'-natural *tensor and let* $A \in R^{n\times n}$ *such that* ${}^{\lambda'}T = A$. *Then* ${}^{\lambda'}T \circ f$ *comes from a tensor on M if and only if* $(L(a))^t \cdot A \cdot L(a) = A$ *for any a in O.*

Proof. Since T is λ'-natural, $(L'(a'))^t \cdot A \cdot L'(a') = A$ for all $a' \in O'$, then the Proposition follows from Proposition 3.5. □

Remark 5.15. There are tensors on *M* that are not λ-natural for any s-space over *M*. Let *T* be a not null tensor on *M*, then there exists $p \in M$ such that $T(p) : M_p \times M_p \to \mathbb{R}$ is not the null bilinear form. Let *f* be a differentiable function on *M* that satisfies $f(p) = 1$ and $f(q) = 0$ for $q \neq p$. Consider the tensor $\tilde{T}$ defined by $\tilde{T}(\xi) = f(\xi) \cdot T(\xi)$. If $\tilde{T}$ is λ-natural, then ${}^{\lambda}\tilde{T} \equiv A$ and since $\tilde{T}(q) = 0$, A must be the zero matrix. For $z' \in \psi^{-1}(p)$, we have that ${}^{\lambda}\tilde{T}(z') = [\tilde{T}(q)(e_i(z'), e_j(z'))] =$ $f(p)[T(p)(e_i(z') \cdot e_j(z'))] \neq 0$, hence *T* is not λ-natural.

Proposition 5.16. *Let T be a symmetric tensor on M with index and constant rank. Then there is a s-space* λ *over M such that T is* λ*-natural.*

Proof. If $rank(T) = 0$, then T is the null tensor and T is λ-natural for all λ.

Suppose that $rank(T) = r \geq 1$ and $index(T) = r - s$. For every $p \in M$ there is a basis $\{v_1, ..., v_s, v_{s+1}, ..., v_r, v_{r+1}, ..., v_n\}$ of M_p that diagonalizes the matrix of $T(p)$, i.e.

$$[T(p)(v_i, v_j)] = \begin{pmatrix} Id_{s\times s} & 0 & 0 \\ 0 & -Id_{(r-s)\times(r-s)} & 0 \\ 0 & 0 & 0 \end{pmatrix} = I_{sr}.$$

Let $\lambda = (N, \pi, O, \cdot, \{\pi_i\})$, where $N = \{(q, v) \in LM : [T(p)(v_i, v_j)] = I_{sr}\}$,

$$O = \begin{pmatrix} O(s) & 0 & 0 \\ 0 & O(r-s) & 0 \\ 0 & 0 & GL(n-r) \end{pmatrix};$$ the action, the projection and the maps π_i are similar to those of LM. Then ${}^{\lambda}T = I_{sr}$. □

6. Subs-spaces

Let $\lambda = (N, \psi, O, R, \{e_i\})$ and $\lambda' = (N', \psi', O', R', \{e_i'\})$ be s-spaces over M and N respectively and $h : M \to M'$ be a differentiable function. Let $f : N \to N'$ be a differentiable function and $\tau : O \to O'$ a group morphism.

Definition 6.1. We said that (f, τ) is a *morphism* of s-spaces over *h* if $f(z \cdot a) = f(z) \cdot \tau(a)$ for all $z \in N$ and $a \in O$ and $\psi' \circ f = h \circ \psi$.

This definition generalizes the concept of morphism of s-spaces. If λ and λ' are s-spaces over M and $(f, \tau) : \lambda \to \lambda'$ is a morphism of s-spaces, then (f, τ) is a morphism over Id_M.

Example 6.2. Let (M, g) be a Riemannian manifold and let $\lambda = (O(M) \times \mathbb{R}^n, \psi, O(n), R, \{e_i\})$ be the s-space over TM where the projection is defined by $\psi(p, u, \xi) = \left(p, \sum_{i=1}^n u_i \xi^i\right)$, $O(n)$ acts on $O(M) \times \mathbb{R}^n$ by $R_a(p, u) = (p, u \cdot a, \xi \cdot a)$.

For $1 \leq i \leq n$, let $e_i(p, u, \xi) = (\pi_{*_{\psi(p,u,\xi)}} \times K_{\psi(p,u,\xi)})^{-1}(u_i, 0)$ and $e_{n+i}(p, u, \xi) = (\pi_{*_{\psi(p,u,\xi)}} \times K_{\psi(p,u,\xi)})^{-1}(0, u_i)$, where K is the connection map induced by the

Levi-Civita connection of g. Before we see an example of subs-space, let us make a brief comment. The tensors on TM that are λ naturals with respect to TM agree with the ones of Calvo-Keilhauer [2]. The Sasaki metric G_S and the Cheeger-Gromoll metric G_{cg} are λ-naturals with respect to TM. The matrix representation of the Sasaki metric and the Cheeger-Gromoll metric are ${}^{\lambda}G_S(p, u, \xi) = \begin{pmatrix} Id_{n\times n} & 0 \\ 0 & Id_{n\times n} \end{pmatrix}$ and ${}^{\lambda}G_{cg}(p, u, \xi) = \begin{pmatrix} Id_{n\times n} & 0 \\ 0 & \dfrac{1}{1+|\xi|^2}(Id_{n\times n} + (\xi)^t \cdot \xi) \end{pmatrix}$, respectively.

Consider the s-space $\lambda' = (O(M), \psi', O(n-1), R', \{e_i'\})$ over the unitary tangent bundle T_1M of M, where $\psi'(p, u) = (p, u_n)$ and the action of $O(n-1)$ on $O(M)$ is given by $R_a'(p, u) = \left(p, \sum_{i=1}^{n-1} u_i a_1^i, ..., \sum_{i=1}^{n-1} u_i a_{n-1}^i, u_n\right)$. The maps $\{e_i'\}$ are defined by $e_i'(p, u) = (\pi_{*_{\psi(p,u)}} \times K_{\psi(p,u)})^{-1}(u_i, 0)$ if $1 \le i \le n$ and by $e_{n+i}'(p, u) = (\pi_{*_{\psi(p,u)}} \times K_{\psi(p,u)})^{-1}(0, u_i)$ if $1 \le i \le n-1$. Let $f : O(M) \to O(M) \times \mathbb{R}^n$ and $\tau : O(n-1) \to O(n)$ defined by $f(p, u) = (p, u, v)$, where v is the nth vector of the canonic base of $\mathbb{R}^n$, and $\tau(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. Then $(f, \tau) : \lambda \to \lambda'$ is a morphism of s-spaces over the inclusion map of T_1M in TM.

Let M and M' be two manifolds of dimension n and n' respectively. Let $\lambda = (N, \psi, O, R, \{e_i\})$ and $\lambda' = (N', \psi', O', R', \{e_i'\})$ be two s-spaces over M and M' and $(f, \tau) : \lambda \to \lambda'$ a morphism of s-space over an inmersion $h : M \to M'$. For every $z \in N$, $h_{*_{\psi(z)}}(M_{\psi(z)})$ is a subspace of dimension n of $M'_{\psi(f(z))}$ and it is generated by $\{h_{*_{\psi(z)}}(e_1(z)), ..., h_{*_{\psi(z)}}(e_n(z))\}$. As $\{e_i'(f(z))\}$ is a base of $M'_{\psi'(f(z))}$, for every $z \in N$ there exists a matrix $A(z) \in \mathbb{R}^{n'\times n'}$ with $\operatorname{rank}(A(z)) = n$ that satisfies

$$\{h_{*_{\psi(z)}}(e_1(z)), ..., h_{*_{\psi(z)}}(e_n(z)), \overbrace{0, ..., 0}^{n'-n}\} = \{e_1'(f(z)), ..., e_{n'}'(f(z))\} \cdot A(z).$$

In the previous example, $A(p, u) = \begin{pmatrix} Id_{(2n-1)\times(2n-1)} & 0 \\ 0 & 0 \end{pmatrix}$. If $M = M'$ and h is the identity map then (f, τ) is a morphism of s-spaces and $A(z) = C^{-1}(z)$ is C is the linking map of (f, τ). In this situation, we have the following definition:

Definition 6.3. λ is a subs-space of λ' if there exists a morphism of s-spaces (f, τ) over an injective inmersion $h : M \to M'$ such that f is an inmersion and the map A induced by (f, τ) is constant. In this case, we said that λ is a *subs-space* of λ' with morphism (f, τ) over h. A s-space $\lambda = (N, \psi, O, R, \{e_i\})$ is included in $\lambda' = (N', \psi', O', R', \{e_i'\})$ if $N \subseteq N'$.

Example 6.4. Let M be a parallelizable manifold, V a vectorial space and V' a subspace of V. Let $GL(V)$ be the group of linear isomorphisms of V and let $GL(V, V')$ be the subgroup of linear isomorphisms of V with the property that $T(V') = V'$. Consider the s-space $\lambda = (M \times V,\ pr_1,\ GL(V),\ R_f,\ \{e_i\})$ over M, where the action is defined by $R_f(p, z) = (p, f(z))$ for $(p, z) \in M \times V$ and $f \in GL(V)$, and $e_i = \overline{e}_i \circ pr_1$, where $\{\overline{e}_1, \ldots, \overline{e}_n\}$ are the vector fields that trivialized the tangent bundle of M. If $\lambda' = (M \times V',\ pr_1,\ GL(V, V'),\ R_f,\ \{e_i\})$, then λ' is a subs-space of λ.

Proposition 6.5. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ *and* $\lambda' = (N', \psi', O', R', \{e_i'\})$ *be two s-spaces over M such that* λ *is a subs-space of* λ' *with morphism* (f, τ) *over the identity map of M. If a tensor T on M is* λ'*-natural then T is* λ*-natural*.

Proof.

$$\begin{aligned}
[{}^{\lambda}T(z)]_{ij} &= T(\psi(z))(e_i(z), e_j(z)) \\
&= T(\psi'(f(z)))\left(\sum_{l=1}^{n} e_l'(z)A_i^l,\ \sum_{s=1}^{n} e_s'(z)A_j^s\right) \\
&= \sum_{ls}^{n} A_i^l . A_j^s [{}^{\lambda'}T]_{ij},
\end{aligned}$$

then ${}^{\lambda}T$ is a constant map. □

Remark 6.6. The converse statement does not holds in general. Let (M, g) be a Riemannian manifold and $O(M)$ be the s-space induced by the principal bundle of orthonormal frames. If $i_{O(M)} : O(M) \to LM$ and $i_{O(n)} : O(n) \to GL(n)$ are the respective inclusion functions, then $O(M)$ is a subs-space of *LM* with morphism $(i_{O(M)}, i_{O(n)})$ over the identity map of *M*. We known that there are $O(M)$-natural tensors that are not *LM*-natural.

Let *T* be a tensor on *M* and let ${}^{LM}T : LM \to R^{n\times n}$ be the matrix map induced by the s-space *LM*. Given a s-space $\lambda = (N, \psi, O, R, \{e_i\})$ over *M* we have a morphism $(\Gamma, L): \lambda \to LM$ (see Example 3.2). It is clear that ${}^{\lambda}T = {}^{LM}T \circ \Gamma$, thus if *T* is λ-natural then there exists a matrix $A \in \mathbb{R}^{n\times n}$ such that $Img\,\Gamma \subseteq ({}^{LM}T)^{-1}(A)$.

Proposition 6.7. Let *T* be a tensor on *M*. There exists λ a s-space over *M* such that *T* is λ-natural if and only if there exist *a* matrix $A \in R^{n\times n}$ and a subs-space of *LM* included in $({}^{LM}T)^{-1}(A)$.

Proof. Suppose that *T* is λ-natural $(\lambda = (N, \psi, O, R, \{e_i\}))$ and let $A \in R^{n\times n}$ such that ${}^{\lambda}T = A$. Let $\lambda' = (\Gamma(N), \pi, L(O), R', \{\pi_i\})$, where π, R' and $\{\pi_i\}$ are induced by *LM*. The map $\pi : \Gamma(N) \to M$ is a submersion. Since $\pi(\Gamma(N)) = \psi(N) = M$, π is surjective. Let $p \in M$ and $z \in \psi^{-1}(p)$. Then $\pi(\Gamma(z)) = p$. Given $v \in M_p$ there exists $w \in N_z$ such that $\psi_{*_z}(w) = v$. Let α be a curve on *N* that satisfies $\alpha(0) = z$ and $\dot{\alpha}(0) = w$. Then for $\beta(t) = \Gamma(\alpha(t))$ we have that $\beta(0) = \Gamma(z)$ and $\pi_{*_{\Gamma(z)}}(\dot{\beta}(0)) = D|_0\ (\pi(\beta(t))) = \psi_{*_z}(w) = v$, so $\pi_{*_{\Gamma(z)}} : N_{\Gamma(z)} \to M_p$ is surjective. On the other hand, it is clear that $L(O)$ acts transitively on $\Gamma(N)$, so λ' is a s-space and it is a subs-space of *LM* with morphism $(i_{\Gamma(N)}, i_{L(O)})$ over the identity map of *M*.

Conversely, suppose that there exists $\lambda = (N, \psi, O, R\{e_i\})$ a s-space over *M* that is also a subs-space of *LM* with morphism (f, τ) over the identity map, and suppose that $f(N) \subseteq ({}^{LM}T)^{-1}(A)$ for a matrix $A \in R^{n\times n}$. Since $\{e_i(z)\} = \{\pi_i(f(z))\} \cdot B$ for $B \in GL(n)$, this implies that $[{}^{\lambda}T(z)] = [T(\psi(z))(e_i(z), e_j(z))] = B^t \cdot [T(\psi(z))(\pi_i(f(z)), \pi_j(f(z)))] \cdot B = B^t \cdot A \cdot B$. □

7. Atlas of s-spaces

Definition 7.1. Let M be a manifold and let $\mathcal{A} : \{\lambda_i = (N_i, \psi_i, O_i, R_i, \{e_l\})\}_{i\in I}$ be a collection of s-spaces over M. Then the collection $\mathcal{A}$ is called an *Atlas* of *s-spaces* if for each pair $(i, j) \in I \times I$ there is a morphism of s-spaces $(f_{ij}, \tau_{ij}) : \lambda_i \to \lambda_j$ such that $f_{ij} : N_i \to N_j$ is a diffeomorphism.

We said that the s-spaces λ and β are *compatible* if there exists a morphism $(f_{\lambda\beta}, \tau_{\lambda,\beta}) : \lambda \to \beta$ and $(f_{\beta\lambda}, \tau_{\beta,\lambda}) : \beta \to \lambda$ such that $f_{\lambda\beta}$ and $f_{\beta\lambda}$ are diffeomorphisms. Hence, an atlas is a set of compatible s-spaces over M. We say that $\mathcal{A}$ is a maximal atlas if $\mathcal{A} \subseteq \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. In other words, if $\mathcal{A}$ is a s-space compatible with the s-spaces of $\mathcal{A}$ then $\lambda \in \mathcal{A}$. If λ is a s-space over M let us notate with $\mathcal{A} = \langle\lambda\rangle$ the maximal atlas generated by λ. Let $\mathcal{A}$ be a maximal atlas. Then it follows from the definition that $\mathcal{A} = \langle\lambda\rangle$ for every $\lambda \in \mathcal{A}$. Note that there are different maximal atlases over a manifold. Consider a metric on M, then $\langle LM\rangle$ and $\langle O(M)\rangle$ are maximal s-spaces but they are different because LM and $O(M)$ are not compatible.

Let λ be a s-space over M, then $\mathcal{A} = \{\lambda\}$ is an atlas. Therefore, the concept of atlas is a generalization of the notion of s-space.

Example 7.2. *Let* $\lambda = (N, \psi, O, R, \{e_i\})$ be a s-space over M and let $A : N \to GL(n)$ be a differentiable function. Consider $\lambda_A = (N, \psi, O, R, \{e_l^A\})$, where $e_l^A(z) = \sum_{i=1}^n e_i(z) A_l^i(z)$. The collection $\mathcal{A} = \{\lambda_A\}_{A\in\mathcal{F}(M)}$ is an atlas of s-spaces.

Example 7.3. Let M be a parallelizable manifold and $\{H_i\}_{i=1}^n$ the vector fields that trivialize the tangent bundle of M. Let (N, g) be a Riemannian manifold such that its isometry group $I_{(N,g)}$ acts transitively on N. Let $\lambda_{(N,g)} = (M \times N, pr_1, I_{(N,g)}, R_f, \{H_i \circ pr_1\})$, where the action of $I_{(N,g)}$ on $M \times N$ is given by $R_f(z, p) = (z, f(p))$. If (N', g') is isometric to (N, g), then $\lambda_{(N,g)}$ is compatible with $\lambda_{(N',g')}$. If N' is not diffeomorphic to N, then $\langle\lambda_{(N,g)}\rangle$ and $\langle\lambda_{(N',g')}\rangle$ are different atlases.

Definition 7.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two atlases of s-spaces over M and F a collection of morphisms of s-spaces from a s-space of $\mathcal{A}$ to a s-space of $\mathcal{B}$. Then F will be called a *morphism* between the atlas $\mathcal{A}$ and $\mathcal{B}$ if for every $\lambda \in \mathcal{A}$ and $\beta \in \mathcal{B}$ there exists $(f, \tau) \in F$ such that $(f, \tau) : \lambda \to \beta$.

Remark 7.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two atlas over M, $\lambda_0 \in \mathcal{A}$, $\beta_0 \in \mathcal{B}$ and $(f_0, \tau_0) : \lambda_0 \to \beta_0$. Consider $F = \{f_{\beta_0\beta} \circ f_0 \circ f_{\lambda\lambda_0}, \tau_{\beta_0\beta} \circ \tau_0 \circ \tau_{\lambda\lambda_0}\}_{\lambda\in\mathcal{A},\, \beta\in\mathcal{B}}$, where $(f_{\beta_0\beta}, \tau_{\beta_0\beta}) : \beta_0 \to \beta$ and $(f_{\lambda\lambda_0}, \tau_{\lambda\lambda_0}) : \lambda \to \lambda_0$ are the morphisms that show the compatibility between β and β_0 and between λ and λ_0 respectively. Then F is morphism of atlases between $\mathcal{A}$ and $\mathcal{B}$.

Remark 7.6. If λ is a s-space over M we have a canonical morphism $(\Gamma_\lambda, L_\lambda) : \lambda \to LM$ (see Example 3.2), hence for every s-space λ we have a morphism between the atlases $\langle\lambda\rangle$ and $\langle LM\rangle$. But this property do not characterize $\langle LM\rangle$. In other words, if a s-space β satisfies that for every λ there exists a morphism $(f_\lambda, \tau_\lambda) : \lambda \to \beta$, β is not necessarily compatible with LM.

$$\begin{tikzcd}
 & \lambda \arrow[dl, "(\Gamma_\lambda, L_\lambda)"'] \arrow[dr, "(f_\lambda, \tau_\lambda)"] & \\
LM \arrow[rr, bend right, "(f_{LM}, \tau_{LM})"'] & & \beta \arrow[ll, bend right, "(\Gamma_\beta, L_\beta)"']
\end{tikzcd}$$

Consider a parallelizable Riemannian manifold (M, g). Let $\{H_i\}_{i=1}^n$ be orthonormal fields that trivialized the tangent bundle of M. If $\lambda = (N, \psi, O, R, \{e_i\})$ is a s-space over M, let $(f_\lambda, \tau_\lambda) : \lambda \to O(M)$ be a morphism defined by $f(z) = (\psi(z), H_1(\psi(z)), \ldots, H_n(\psi(z)))$ and $\tau(a) = Id_{n\times n}$. Therefore, for every maximal atlas $\mathcal{A}$ there is a morphism between it and $O(M)$, but $O(M)$ is not compatible with LM.

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, "F_{\mathcal{A},LM}"'] \arrow[dr, "F_{\mathcal{A},O(M)}"] & \\
< LM > \arrow[rr, bend right, "F_{LM,O(M)}"'] & & < O(M) > \arrow[ll, bend right, "F_{O(M),LM}"']
\end{tikzcd}$$

There are more atlases with this property. If (M, g) is an oriented manifold, the maximal atlas generated by the s-space induced by the principal fiber bundles of orthonormal oriented bases $SL(M)$ have this property. The atlas $\langle(M, Id_M, \{1\}, R_1, \{H_i\})\rangle$, where R_1 is the trivial action, is another example.

Definition 7.7. Let $\mathcal{A}$ be an atlas of s-spaces over *M*. Then a tensor *T* on *M* will be called $\mathcal{A}$ *-natural* if *T* is λ-natural for all $\lambda \in \mathcal{A}$.

Note that the concept of $\mathcal{A}$ -naturality generalized the notion of λ-naturality. If we consider the atlas $\mathcal{A} = \{\lambda\}$, then *T* is $\mathcal{A}$ -natural if and only if *T* is λ-natural.

Example 7.8. Let λ be a s-space over *M* and consider the subatlas of the atlas given in the Example 7.2 defined by $\mathcal{A} = \{\lambda_A\}_{A \in GL(n)}$. *T* is $\mathcal{A}$ -natural if and only if *T* is λ-natural. Let *T* be a λ-natural tensor on *M* and $\mathcal{A}' = \{\lambda_A\}_{A \in \mathcal{F}(N, G_T)}$. Then *T* is $\mathcal{A}'$ -natural and it has the same matrix representation in all the s-spaces of the atlas.

Remark 7.9. If $\mathcal{A}$ is a maximal atlas then the unique $\mathcal{A}$ -natural tensor is the null tensor. Let $\lambda = (N, \psi, O, R, \{e_i\}) \in \mathcal{A}$ and $f : N \to \mathbb{R}$ be a differentiable function such that $f(z) \neq 0$ for all $z \in N$ and f^2 is not constant. If $\lambda' = (N, \psi, O, R, \{f \cdot e_i\})$, then we have that $\lambda' \in \mathcal{A}$, but the null tensor is the only one that is λ-natural and λ' -natural at the same time.

Definition 7.10. Let $\mathcal{A}$ be an atlas of s-spaces over *M* and *T* a tensor on *M*. Then *T* is called $\mathcal{A}$ *-weak natural* if there exists $\lambda \in \mathcal{A}$ such that *T* is λ-natural.

If $\mathcal{A} = \{\lambda\}$ or $\mathcal{A}$ is the atlas of Example 7.8, the concept of $\mathcal{A}$ -natural and $\mathcal{A}$ -weak natural coincide.

For study the naturality of tensors on a fibration α it will be useful consider the atlases $\mathcal{A}$ such that all its s-spaces are trivial over α. An atlas with this property will be called a *trivial atlas over* α. The following definition is a generalization of the concept of naturality with respect to a fibration:

Definition 7.11. Let $\mathcal{A}$ be a trivial atlas over a fibration $\alpha = (P, \pi, \mathbb{F})$ and *T* a tensor on *P*. Then *T* is $\mathcal{A}$ -natural with respect to α if *T* is λ-natural with respect to α for all $\lambda \in \mathcal{A}$.

Example 7.12. Let $\alpha = (P, \pi, G, \cdot)$ be a principal fiber bundle on (M, g) endowed with a connection ω. For every $W = \{W_1, ..., W_k\}$ basis of $\mathfrak{g}$ let $\lambda_W = (M, \psi, O, R, \{e_i^W\})$, where $N = \{(p, u, b) : p \in P, u$ is an orthonormal base of

$M_{\pi(p)},\ b \in G\}$, $\psi(q, u, b) = q \cdot b$, $O = O(n) \times G$ and the action R is defined by $R_{(h, a)}(q, u, b) = (qa, uh, a^{-1}b)$. For $1 \leq i \leq n$, $e_i^W(p, u, g)$ is the horizontal lift of u_i with respect to ω at $p.g$ and for $1 \leq j \leq k$, $e_{n+j}(p, u, g)$ is the only one vertical vector on $P_{p.g}$ such that $\omega(p)(e_{n+j}(p, u, g)) = W_j$. $\mathcal{A} = \{\lambda_W\}_{W \in L_{\mathfrak{g}}}$ is a trivial atlas over α. An easy computation shows that the set of $\mathcal{A}$-natural tensors with respect to α is the set of tensors T whose matrix representation with respect to some λ_W is ${}^{\lambda_W}T(q, u, a) = \begin{pmatrix} f(a).Id_{n\times n} & 0 \\ 0 & B(a) \end{pmatrix}$, where $f : G \to \mathbb{R}$ and $B : G \to \mathbb{R}^{k\times k}$ are differentiable functions.

As above, if $\mathcal{A}$ is a maximal trivial atlas over α the only $\mathcal{A}$-natural tensor with respect to α is the null tensor. So we have a weak definition of naturality for this case too. We say that T is $\mathcal{A}$-weak natural with respect to α if T is λ-natural with respect to α for some $\lambda \in \mathcal{A}$.

8. Examples

We conclude showing some examples of s-spaces:

8.1. Lie groups

Let G be a Lie group of dimension k. We denote with e the unit element of G. If $v = \{v_1, ..., v_n\}$ is a base of $\mathfrak{g}$, let H_i^v be the unique left invariant vector field on G such that $H_i^v(e) = v_i$.

Example 8.1. Given v a basis of $\mathfrak{g}$, let $\lambda^v = (N, \psi, G, R, \{e_i^v\})$ be the s-space over G defined by $N = G \times G$, $\psi(g, h) = g.h$, $R_a(g, h) = (g.a, a^{-1} \cdot h)$ and $e_i^v(g, h) = H_i^v(g \cdot h)$ for $1 \leq i \leq k$. Since $e_i^v \circ R_a(g, h) = e_i^v(g, h)$, the base change morphism L^v is equal to the identity matrix of $R^{k\times k}$. Therefore, if T is a tensor on G, then it satisfies that

$$ {}^{\lambda^v}T \circ R_a = {}^{\lambda^v}T. $$

For this reason, all constant matricial maps come from a tensor and the λ^v-natural tensors are in a one to one relation with the matrices of $\mathbb{R}^{k\times k}$.

Suppose that ${}^{\lambda^v}T$ depends only of one parameter, for example ${}^{\lambda^v}T(g, h) = {}^{\lambda^v}T(h)$. Since $[{}^{\lambda^v}T(g', h')]_{ij} = [{}^{\lambda^v}T(g'hh'^{-1}, h')]_{ij} = T(g'h)(H_i^v(g'h), H_j^v(g'h)) = [{}^{\lambda^v}T(g', h)]_{ij} = [{}^{\lambda^v}T(g, h)]_{ij}$, T is λ^v-natural. Therefore, T is λ^v-natural if and only if T is ${}^{\lambda^v}T$ depends of one parameter. The left invariant metrics are tensors of this type.

Let v' be another basis of $\mathfrak{g}$ and consider $\lambda^{v'}$. If $a_{vv'} \in GL(k)$ is the matrix that satisfies $v' = a_{vv'}v$, then we have that $e_i^{v'}(g, h) = e_i^v(g, h) \cdot a_{vv'}$ and ${}^{\lambda^{v'}}T = (a_{vv'})^t \cdot {}^{\lambda^v}T \cdot a_{vv'}$ for a tensor T on M. Thus the set of λ^v-natural tensors is independent of the choice of the basis v. We can observe that $(Id_{G\times G}, Id_G)$ is a morphism of s-spaces with linking map equals $a_{vv'}$, so $T \in I_{(Id_{G\times G}, Id_G)}$ if and only if $a_{vv'} \in G_T(g, h)$.

Example 8.2. Let $\lambda = \{N, \psi, O, R, \{e_i\}\}$ be the s-space over G defined by $N = G \times L\mathfrak{g} = \{(g, v) : g \in G$ and v is a basis of $\mathfrak{g}\}$, $\psi(g, v_1, ..., v_n) = g$, $O = GL(n)$, $R_\xi(g, v) = (g, v \cdot a)$ and $e_i(g, v) = H_i^v(g)$. Since $\{e_i\} \circ R_\xi = \{e_i\} \cdot \xi$, ${}^{\lambda}T \circ R_\xi = \xi^t \cdot {}^{\lambda}T \cdot \xi$ for all $\xi \in GL(k)$. Therefore, the null vector is only one that is λ-natural.

The left invariant metrics on G are not λ-natural but for a metric T on G we have that T is a left invariant metric if and only if ${}^{\lambda}T(g, v) = {}^{\lambda}T(v)$. If T is a left invariant metric, then

$$\begin{aligned}[{}^{\lambda}T(g, v)]_{ij} &= T(g)((L_g)_{*_e}(v_i), (L_g)_{*_e}(v_j)) \\ &= T(e)((L_{g-1})_{*_g}((L_g)_{*_e}(v_i)), (L_{g-1})_{*_g}((L_g)_{*_e}(v_i))) \\ &= T(e)(v_i, v_j) = [{}^{\lambda}T(e, v)]_{ij}.\end{aligned}$$

Suppose that the matrix representation induced by T depends only of the parameter of $\mathfrak{g}$. Let $g, h \in G$ and $w, v \in T_gG$, we have to see that $T(g)(v, w)$

$= T(hg)((L_h)_{*_g}(v), (L_h)_{*_g}(w))$. Let $\{u_1, ..., u_n\}$ be a basis of $\mathfrak{g}$. If $v = \sum_{i=1}^{n} v_i (L_g)_{*_e}(u_i)$ and $w = \sum_{i=1}^{n} w_i (L_g)_{*_e}(u_i)$. Then $(L_h)_{*_g}(v) = \sum_{i=1}^{n} v_i (L_{hg})_{*_e}(u_i)$ and $(L_h)_{*_g}(w) = \sum_{i=1}^{n} w_i (L_{hg})_{*_e}(u_i)$. Hence,

$$T(hg)((L_h)_{*_g}(v), (L_h)_{*_g}(w)) = (v_1, ..., v_n) \cdot {}^{\lambda}T(hg, u) \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = T(g)(v, w).$$

Let T be a tensor such that ${}^{\lambda}T(g, v)$ depends only of v. We know that ${}^{\lambda}T(g, v \cdot \xi) = (\xi)^t \cdot {}^{\lambda}T(e, v) \cdot \xi$ for all $\xi \in GL(k)$. Fixed $v_0 \in L_{\mathfrak{g}}$ and let $F : L_{\mathfrak{g}} \to GL(k)$ defined by $v = v_0 \cdot F(v)$. Then ${}^{\lambda}T(g, v) = (F(v))^t \cdot {}^{\lambda}T(e, v_0) \cdot F(v)$ for all $(g, v) \in G \times L_{\mathfrak{g}}$. Therefore, ${}^{\lambda}T$ depends only of the parameter of $L_{\mathfrak{g}}$ if and only if there exists $A \in R^{k \times k}$ and a differentiable function $F : L_{\mathfrak{g}} \to GL(k)$, that satisfies $F(w \cdot \xi) = F(w) \cdot \xi$, such that ${}^{\lambda}T(g, w) = (F(w))^t \cdot A \cdot F(w)$.

Example 8.3. Fixed $v \in L_{\mathfrak{g}}$ and consider $\lambda^v = (G \times O(k), \psi, O(k), R, \{e_i^v\})$, where $\psi(g, \xi) = g$, $R_a(g, \xi) = (g, \xi a)$, $e_i^v(g, \xi) = H_i^{v \cdot \xi}(g) \cdot \lambda$ is a s-space over G with base change morphism $L = Id_{O(k)}$. If T is a tensor of M, then ${}^{\lambda}T \circ R_a = a^t \cdot {}^{\lambda}T \cdot a$. Therefore, T is λ-natural if and only if ${}^{\lambda}T(g, \xi) = f(g) \cdot Id_{k \times k}$ with $f : G \to \mathbb{R}$ a differentiable function. Is easy to see that ${}^{\lambda}T((g, \xi) \cdot a) = (\xi a)^t \cdot {}^{\lambda}T(g, Id) \cdot (\xi a)$, hence the matrix representation of T depends only of the parameter of $O(k)$ if and only if ${}^{\lambda}T(g, \xi) = \xi^t \cdot A \cdot \xi$ with $A \in \mathbb{R}^{n \times n}$.

8.2. Bundle metrics

Let $\alpha = (P, \pi, G, \cdot)$ be a principal fiber bundle over a Riemannian manifold (M, g) endowed with a connection ω. Let us denote with $\mathcal{M}_{\mathrm{ad}}(\mathfrak{g})$ the set of metrics on $\mathfrak{g}$ that are invariant by the adjoint map **ad**. Consider the metric on P defined by

$$h(p)(X, Y) = g(\pi(p))(\pi_{*_p}(X), \pi_{*_p}(Y)) + (l \circ \pi)(p)(\omega(X), \omega(Y)) \qquad (1)$$

where $l : M \to \mathcal{M}_{\mathrm{ad}}(\mathfrak{g})$. If G is compact, $\mathcal{M}_{\mathrm{ad}}(\mathfrak{g}) \neq \varnothing$, and if $\mathfrak{g}$ is also a simple algebra, then essentially there is only one conformal class of positive defined **ad**-invariant metric [14]. If l is a constant function, h is called a *bundle metric*. It is easy to see that $\pi : (P, h) \to (M, g)$ is a Riemannian submersion.

Let l_0 be an **ad**-invariant metric on $\mathfrak{g}$. In the following we are going to consider the s-space $\lambda = (N, \psi, O, R, \{e_i\})$ over P given by $N = \{(q, u, v, g) : q \in P, u$ is an orthonormal base of $M_{\pi(q)}, v$ is an orthonormal base of $\mathfrak{g}$ with respect to l_0 and $g \in G\}$, $\psi(q, u, v, g) = q \cdot g$, $O = O(n) \times O(k) \times G$ and the action is defined by $R_{(a,b,h)}(q, u, v, g) = (qh, ua, vb, h^{-1}g)$. For $1 \le i \le n$, $e_i(q, u, v, g)$ is the horizontal lift with respect to ω of u_i at $q \cdot g$ and, for $1 \le j \le k$, $e_{n+j}(q, u, v, g)$ is the unique vertical vector on $P_{p \cdot g}$ such that $\omega(q \cdot g)(e_{n+j}(q, u, v, g)) = v_j$. λ is a trivial s-space over α.

Let G be a compact Lie group with $\mathfrak{g}$ a simple algebra and h a metric on P of the type of (1). Then we have the following proposition:

Proposition 8.4. *h is λ-natural with respect to α if and only if h is a bundle metric*.

Proof. ${}^{\lambda}h(q, u, v, g)$ is the matrix of $h(q \cdot g)$ with respect to the base $\{e_i(q, u, v, g), e_{n+i}(q, u, v, g)\}$. For $1 \le i, j \le n$, we have that:

$$h(q.g)(e_i(q, u, v, g), e_j(q, u, v, g)) = g(u_i, u_j) + 0 = \delta_{ij}.$$

For $1 \le i \le n$ and $1 \le j \le k$:

$$\begin{aligned} &h(qg)(e_i(q, u, v, g), e_{n+j}(q, u, v, g)) \\ &= 0 = h(qg)(e_{n+j}(q, u, v, g), e_i(q, u, v, g)) \end{aligned}$$

and for $1 \le i, j \le k$:

$$h(q \cdot g)(e_{n+i}(q, u, v, g), e_{n+j}(q, u, v, g)) = l \circ \pi(qg)(v_i, v_j) = f(\pi(q)) \cdot \delta_{ij},$$

because $\mathfrak{g}$ has essentially one **ad**-invariant metric. Since

$${}^{\lambda}h(q, u, v, g) = \begin{pmatrix} Id_{n\times n} & 0 \\ 0 & f(\pi(q)).Id_{k\times k} \end{pmatrix},$$

h is λ-natural with respect to α if and only if f is a constant map, that is to say that h is a bundle metric. □

Remark 8.5. If $\mathfrak{g}$ has different **ad**-invariant metrics and h is a metric of the type of (1), then ${}^{\lambda}h : N \to \mathbb{R}^{(n+k)\times(n+k)}$ only depends of the parameter of G if $l = \delta \cdot l_0$ with δ a constant. In general, the metrics of type (1) that are λ-natural with respect to α are the bundle metrics induced by the **ad**-invariant metric l_0.

Remark 8.6. The s-space λ depends of l_0 and ω. Let ω' be another connection on α and consider the s-space λ' induced by it. The difference between λ^{ω} and $\lambda^{\omega'}$ are the maps $e_i : N \to TP$ and $e_i' : N \to TP$. Let

$$A(p, u, v, g) = \begin{pmatrix} a_1(p, u, v, g) & a_2(p, u, v, g) \\ a_4(p, u, v, g) & a_3(p, u, v, g) \end{pmatrix} \in GL(n+k)$$

be the matricial map that satisfies $\{e_i', e_{n+j}'\} = \{e_i, e_{n+j}\}.A$, where $a_1(p, u, v, g) \in \mathbb{R}^{n\times n}$, $a_2(p, u, v, g) \in \mathbb{R}^{n\times k}$, $a_3(p, u, v, g) \in \mathbb{R}^{k\times k}$ and $a_4(p, u, v, g) \in \mathbb{R}^{k\times n}$. Since $e_{n+j}(p, u, v, g) = e_{n+j}'(p, u, v, g)$, we have that $a_2 \equiv 0$ and $a_3 \equiv Id_{k\times k}$. If T is a tensor, then

$$^{\lambda^{\omega'}}T(p, u, v, g) = \begin{pmatrix} a_1^t(p, u, v, g) & a_4^t(p, u, v, g) \\ 0 & Id_{k\times k} \end{pmatrix}$$

$$.^{\lambda^{\omega}}T(p, u, v, g) \cdot \begin{pmatrix} a_1(p, u, v, g) & 0 \\ a_4(p, u, v, g) & Id_{k\times k} \end{pmatrix}.$$

Suppose as in the proposition above that there is essentially one **ad**-invariant metric. Then if h is a metric of type (1) we have that

$$^{\lambda^{\omega'}}h(p, u, v, g)$$

$$= \begin{pmatrix} a_1^t(p,u,v,g)a_1(p,u,v,g) & f(\pi(p))\cdot a_4^t(p,u,v,g) \\ \quad + f(\pi(p))a_4^t(p,u,v,g)\cdot a_4(p,u,v,g) & \\ f(\pi(p))a_4(p,u,v,g) & f(\pi(p))\cdot Id_{k\times k} \end{pmatrix}$$

Therefore, if the connections satisfy that $a_1 \in O(n)$ and a_4 is a constant map, then h is λ-natural with respect to α if and only if h is λ'-natural with respect to α. In this situation h is a bundle metric.

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References

[1] J. Araujo and G. R. Keilhauer, Natural tensor field of type (0, 2) on the tangent and cotangent bundle of a semi-Riemannian manifold, Acta Univ. Palacki. Olomuc., Fac. rer. nat. Mathematica 39 (2000), 7-16.

[2] M. C. Calvo and G. R. Keilhauer, Tensor field of type (0, 2) on the tangent bundle of a Riemannian manifold, Geometriae Dedicata 71 (1998), 209-219.

[3] L. A. Cordero and M. De León, On the curvature of the induced Riemannian metric on the frame bundle of a Riemannian manifold, J. Math. Pures Appl. 65(9) (1986), 81-91.

[4] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, Modern Geometry, Methods and Applications. Part II, The Geometry and Topology of Manifolds, Graduate Texts in Mathematics, 104, Springer-Verlag, New York, 1985, 430 pp.

[5] G. Henry, Tensores Naturales Sobre Variedades *y* Fibraciones, Doctoral Thesis. Universidad de Buenos Aires, 2009.
http://digital.bl.fcen.uba.ar/Download/Tesis/Tesis 4540 Henry.pdf

[6] G. Henry and G. R. Keilhauer, Some relationships between the geometry of the tangent bundle and the geometry of the Riemannian base manifold, preprint.

[7] G. R. Keilhauer, Tensor field of type (0, 2) on linear frame bundles and cotangent bundles, Rend. Sem. Mat. Univ. Padova 103 (2000), 51-64.

[8] I. Kolar, P. Michor and J. Slovak, Natural Operations in Differential Geometry, Springer-Verlag, Berlin, 1993, 434 pp.

[9] O. Kowalski and M. Sekisawa, Natural transformation of Riemannian metrics on manifolds to metrics on tangent bundles - a classification, Bull Tokyo Gakugei Univ. 4 (1988), 1-29.

[10] O. Kowalski and M. Sekisawa, Natural transformation of Riemannian metrics on manifolds to metrics on linear frame bundles - a classification, Differential Geometry and its Applications, Proceedings of the Conference, August 24-30, Brno, Czechoslovakia, 1986, pp. 149-178.

[11] D. Krupka, Elementary theory of differential invariants, Arch. Math. (Brno) 14(4) (1978), 207-214.

[12] D. Krupka and J. Janyska, Lectures on differential invariants, Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis, Mathematica, 1, University J. E. Purkyne, Brno, 1990, p. 195.

[13] P. Michor, Gauge Theory for Fibers Bundles, Extended version of a series of lectures held at the Institute of Physics of the University of Napoli, 1988.

[14] J. Milnor, Curvatures of Lef invariant metrics on Lie groups, Adv. Math. 21 (1976), 293-329.

[15] K. P. Mok, On the differential geometry of frame bundles of Riemannian manifolds, J. Reine Angew. Math. 302 (1978), 16-31.

[16] B. O’Neill, The fundamental equations of a submersion, Michigan Math. J. 13 (1966), 459-469.