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A 195, 747, 435 VERTEX GRAPH RELATED TO THE FISHER GROUP Fi_{23}, I

Peter Rowley and Louise Walker
Mathematics Department
University of Manchester
Manchester, UK

Abstract

The 195, 747, 435 vertex graph studied here is the point-line collinearity graph of a geometry for the second largest Fischer group Fi_{23}. In this paper and [7] a detailed description of this graph is obtained.

1. Introduction

It is the aim of this paper and [6] to lay bare the bones of $\mathcal{G}$, the point-line collinearity graph of Γ where Γ is a geometry associated with the second largest Fischer group Fi_{23}. The geometry Γ has rank 4 and is closely related to the transpositions of Fi_{23}. Diagrammatically we may describe Γ as follows:

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	$2^{11}M_{23}$	$2^{6+8}(A_7 \times S_3)$	$(2^2 \times 2_+^{1+8})3 \times U_4(2).2$	$2Fi_{22}$
	$\circ$	$\circ$	$\circ$	$\circ$
Type	0	1	2	3
(Fisher Name)	base	heptad	tri-transposition	transposition
(Geometric Name)	point	line	plane	hyperplane

The groups listed at the top are the stabilizers in Fi_{23} of the various objects in Γ – we have also given a description of each type of object in "transposition language" (see [1; p. 177] for more on this). The geometric names above are those we will use and are just meant to be names with no projective geometry connotations whatsoever.

Our anatomical description of $\mathcal{G}$ is given in terms of the geometry Γ. Consequently the residue of a point – intimately connected with the Steiner system $S(23, 7, 4)$ – is to the forefront in all that follows. Also very much in evidence is the residue of a hyperplane – we rely upon [5] for information about this geometry. A detailed discussion of Γ as it relates to $\mathcal{G}$ will be given in Section 2 though we remark here that Fi_{23} acts flag transitively on Γ and so, in particular, is a subgroup of Aut$\mathcal{G}$ acting transitively on the 195,747,435 vertices of $\mathcal{G}$. Also we note that $\mathcal{G}$ may be viewed as the graph where vertices are the bases (23 pair-wise commuting transpositions) with two vertices joined whenever they intersect in a heptad (of transpositions).

We now state our main results on the structure of $\mathcal{G}$. Our first theorem is a broad-brush description of $\mathcal{G}$. This also appears in [4; 2.21(iv)] and was obtained using extensive machine calculations. The results given in the present paper and [6] do not rely upon any machine calculations and moreover, Theorems 2-16 paint a much more detailed picture of the structure of $\mathcal{G}$. This detailed data on the point distribution of line orbits is deployed in the study [7] of the point-line collinearity graph of the maximal 2-local geometry for Fi_{24}', the largest simple Fischer group.

From now on we put $G = Fi_{23}$.

Theorem 1. *Let a be a fixed point of $\mathcal{G}$. Then G_a has 16 orbits $\Delta_j^i(a)$ upon the points of $\mathcal{G}$ whose sizes and collapsed adjacencies are given in Table 1 and Figure 1.*

$\Delta_j^i(a)$	$\|\Delta_j^i(a)\|$	$\Delta_j^i(a)$	$\|\Delta_j^i(a)\|$	$\Delta_j^i(a)$	$\|\Delta_j^i(a)\|$
$\Delta_1(a)$	$2.11.23$	$\Delta_3^3(a)$	$2^{10}.7.11.23$	$\Delta_4^2(a)$	$2^{12}.11.23$
$\Delta_2^1(a)$	$2^4.7.11.23$	$\Delta_3^4(a)$	$2^{10}.23$	$\Delta_4^3(a)$	$2^{12}.5.7.11.23$
$\Delta_2^2(a)$	$2^6.7.11.23$	$\Delta_3^5(a)$	$2^{12}.3.7.11.23$	$\Delta_4^4(a)$	$2^{16}.3.7.23$
$\Delta_3^1(a)$	$2^9.11.23$	$\Delta_3^6(a)$	$2^9.5.7.11.23$	$\Delta_4^5(a)$	$2^{15}.11.23$
$\Delta_3^2(a)$	$2^8.3.5.11.23$	$\Delta_4^1(a)$	$2^{13}.3.5.11.23$	$\Delta_4^6(a)$	$2^{15}.7.11.23$

Table 1.

The finer structure of $\mathcal{G}$, from which the information in Theorem 1 is derived, is the subject of Theorems 2-16. In each of these results a is a fixed point of $\mathcal{G}$ and for $x \in \Delta_j^i(a)$ we give the point distribution for each representative line l in a G_{ax}-orbit on $\Gamma_1(x)$. That is we state in which G_a-orbit each of the three points incident with l belong. So, for example, in Theorem 4 of the three points incident with $l \in \alpha_{1,1}(x, x+b, X(x,a))$ one is in $\Delta_2^2(a)$, one is in $\Delta_3^3(a)$ and one is in $\Delta_3^2(a)$ while for $l \in \alpha_{3,1}(x, x+b, X(x,a))$ one point is in $\Delta_2^2(a)$ and the other two in $\Delta_3^3(a)$.

The notation and conventions relating to the descriptions of G_{ax}-orbits on $\Gamma_1(x)$, as well as definitions of the $\Delta_j^i(a)$, are to be found in Section 2.

Theorem 2. *Let $x \in \Delta_1(a)$. Then $G_{ax} \sim 2^{10}2^4A_7$ (with $G_{ax}^{*x} \sim 2^4A_7$) has 3 orbits on $\Gamma_1(x)$ with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\{x+a\}$	*1*	$\{a\}2\Delta_1$
$\alpha_1(x, x+a)$	*112*	$\Delta_1 2\Delta_2^2$
$\alpha_3(x, x+a)$	*140*	$\Delta_1 2\Delta_2^1$

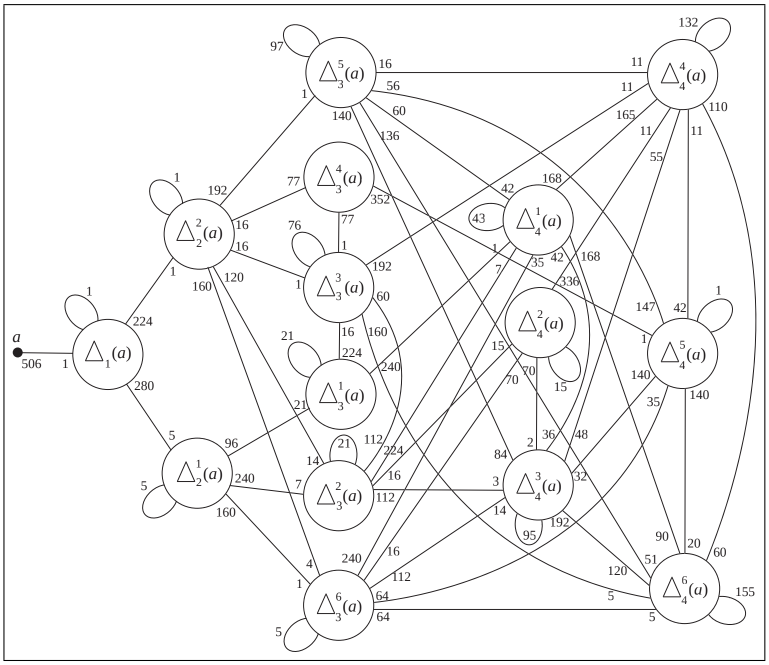


Figure 1.

Theorem 3. *Let* $x \in \Delta_2^1(a)$. *Then* $G_{ax} \sim 2^7 2^4 S_5 3$ *(with* $G_{ax}^{*x} \sim 2^4 S_5 3$*) has 4 orbits on* $\Gamma_1(x)$ *with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\alpha_3(x, T(x,a))$	*5*	$\Delta_1 2 \Delta_2^1$
$\alpha_2(x, T(x,a))$	*48*	$\Delta_1^2 2 \Delta_3^1$
$\alpha_0(x, T(x,a))$	*80*	$\Delta_2^1 2 \Delta_3^6$
$\alpha_1(x, T(x,a))$	*120*	$\Delta_2^1 2 \Delta_3^2$

For x in either of the G_a-orbits $\Delta_2^2(a)$, $\Delta_3^2(a)$, $\Delta_3^3(a)$, $\Delta_3^4(a)$ there is a unique hyperplane which is incident with both a and x. We denote this unique hyperplane by $X(a,x)$ (when viewed as being in Γ_a) and $X(x,a)$ (when viewed as being in Γ_x) – note that $X(a,x)$ and $X(x,a)$ both denote the same hyperplane of Γ.

Theorem 4. *Let* $x \in \Delta_2^2(a)$. *Then* $G_{ax} \sim 2^5 2^4 A_6$ *(with* $G_{ax}^{*x} \sim 2^4 A_6$*) has 5 orbits on* $\Gamma_1(x)$ *with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\{x+b\}$	*1*	$\Delta_1 2\Delta_2^2$
$\alpha_{1,1}(x, x+b, X(x,a))$	*16*	$\Delta_2^2\Delta_3^3\Delta_3^4$
$\alpha_{3,1}(x, x+b, X(x,a))$	*60*	$\Delta_2^2 2\Delta_3^2$
$\alpha_{3,0}(x, x+b, X(x,a))$	*80*	$\Delta_2^2 2\Delta_3^6$
$\alpha_{1,0}(x, x+b, X(x,a))$	*96*	$\Delta_2^2 2\Delta_3^5$

Theorem 5. *Let* $x \in \Delta_3^1(a)$. *Then* $G_{ax} \sim 2^2 L_3(4)2$ *(with* $G_{ax}^{*x} \sim L_3(4)2$*) has 3 orbits on* $\Gamma_1(x)$ *with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\alpha_2(x, D(x,a))$	*21*	$\Delta_2^1 2\Delta_3^1$
$\alpha_1(x, D(x,a))$	*112*	$\Delta_3^1 2\Delta_3^3$
$\alpha_0(x, D(x,a))$	*120*	$\Delta_3^1 2\Delta_4^1$

Theorem 6. *Let* $x \in \Delta_3^2(a)$. *Then* $G_{ax} \sim [2^7]L_3(2)$ *(with* $G_{ax}^{*x} \sim 2^3 L_3(2)$*) has 6 orbits on* $\Gamma_1(x)$ *with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\alpha_{0,1}(x, O(x,a), X(x,a))$	*7*	$\Delta_2^1 2\Delta_3^2$
$\alpha_{0,0}(x, O(x,a), X(x,a))$	*8*	$\Delta_3^2 2\Delta_4^2$
$\alpha_{4,1}(x, O(x,a), X(x,a))$	*14*	$\Delta_2^2 2\Delta_3^2$
$\alpha_{2,1}(x, O(x,a), X(x,a))$	*56*	$\Delta_3^2 2\Delta_3^3$
$\alpha_{4,0}(x, O(x,a), X(x,a))$	*56*	$\Delta_3^2 2\Delta_4^3$
$\alpha_{2,0}(x, O(x,a), X(x,a))$	*112*	$\Delta_3^2 2\Delta_4^1$

Theorem 7. *Let* $x \in \Delta_3^3(a)$. *Then* $G_{ax} \sim 22^4 A_6$ *(with* $G_{ax}^{*x} \sim 2^4 A_6$*) has 5 orbits on* $\Gamma_1(x)$ *with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\{x+b\}$	1	$\Delta_2^2\Delta_3^3\Delta_3^4$
$\alpha_{1,1}(x,x+b,X(x,a))$	16	$\Delta_3^1 2\Delta_3^3$
$\alpha_{3,1}(x,x+b,X(x,a))$	60	$\Delta_3^2 2\Delta_3^3$
$\alpha_{3,0}(x,x+b,X(x,a))$	80	$\Delta_3^3 2\Delta_4^6$
$\alpha_{1,0}(x,x+b,X(x,a))$	96	$\Delta_3^3 2\Delta_4^4$

Theorem 8. *Let $x \in \Delta_3^4(a)$. Then $G_{ax} \sim 2M_{22}$ (with $G_{ax}^{*x} \sim M_{22}$) has 2 orbits on $\Gamma_1(x)$ with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\alpha_1(x,X(x,a))$	77	$\Delta_2^2\Delta_3^3\Delta_3^4$
$\alpha_0(x,X(x,a))$	176	$\Delta_3^4 2\Delta_4^5$

Theorem 9. *Let $x \in \Delta_3^5(a)$. Then $G_{ax} \sim 2^4A_5$ (with $G_{ax}^{*x} \sim 2^4A_5$) has 6 orbits on $\Gamma_1(x)$ with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\{x+b\}$	1	$\Delta_2^2 2\Delta_3^5$
$\alpha_1(x,x+b,+)$	16	$\Delta_3^5\Delta_4^4\Delta_4^5$
$\alpha_3^{(1)}(x,x+b,-)$	40	$\Delta_3^5 2\Delta_4^3$
$\alpha_3^{(2)}(x,x+b,-)$	40	$\Delta_3^5\Delta_4^5\Delta_4^6$
$\alpha_3(x,x+b,+)$	60	$\Delta_3^5\Delta_4^1\Delta_4^3$
$\alpha_1(x,x+b,-)$	96	$2\Delta_3^5\Delta_4^6$

Theorem 10. *Let $x \in \Delta_3^6(a)$. Then $G_{ax} \sim [2^9]3^2$ (with $G_{ax}^{*x} \sim [2^7]3^2$) has 7 orbits on $\Gamma_1(x)$ with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\{x+b\}$	*1*	$\Delta_2^1 2\Delta_3^6$
$\alpha_{3,3}(x,x+b,TRI)$	*4*	$\Delta_2^2 2\Delta_3^6$
$\alpha_{3,0}(x,x+b,TRI)$	*16*	$\Delta_3^6\Delta_4^2\Delta_4^3$
$\alpha_{1,1}(x,x+b,TRI)$	*48*	$\Delta_3^6 2\Delta_4^3$
$\alpha_{3,2}(x,x+b,TRI)$	*48*	$\Delta_3^6 2\Delta_4^1$
$\alpha_{1,0}(x,x+b,TRI)$	*64*	$\Delta_3^6\Delta_4^5\Delta_4^6$
$\alpha_{3,1}(x,x+b,TRI)$	*72*	$\Delta_3^6 2\Delta_4^1$

Theorem 11. *Let $x \in \Delta_4^1(a)$. Then $G_{ax} \sim 2L_3(2)2$ (with $G_{ax}^{*x} \sim L_3(2)2$) has 8 orbits on $\Gamma_1(x)$ with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\{x+b\}$	*1*	$\Delta_3^1 2\Delta_4^1$
$\alpha_{3,2}(x,x+b,DUAD)$	*7*	$\Delta_3^2 2\Delta_4^1$
$\alpha_{1,2}(x,x+b,DUAD)$	*14*	$\Delta_3^6 2\Delta_4^1$
$\alpha_{3,0}^{\mathcal{L}}(x,x+b,DUAD)$	*21*	$\Delta_3^6 2\Delta_4^1$
$\alpha_{1,1}(x,x+b,DUAD)$	*42*	$\Delta_3^5\Delta_4^1\Delta_4^3$
$\alpha_{1,0}(x,x+b,DUAD)$	*56*	$\Delta_4^1 2\Delta_4^6$
$\alpha_{3,0}^{\mathcal{L}^c}(x,x+b,DUAD)$	*56*	$\Delta_4^1\Delta_4^4\Delta_4^6$
$\alpha_{3,1}(x,x+b,DUAD)$	*56*	$\Delta_4^1 2\Delta_4^4$

Theorem 12. *Let $x \in \Delta_4^2(a)$. Then $G_{ax} \cong A_8$ (with $G_{ax}^{*x} \cong A_8$) has 3 orbits on $\Gamma_1(x)$ with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\alpha_0(x,O(x,a))$	*15*	$\Delta_3^2 2\Delta_4^2$
$\alpha_4(x,O(x,a))$	*70*	$\Delta_3^6\Delta_4^2\Delta_4^3$
$\alpha_2(x,O(x,a))$	*168*	$\Delta_4^2 2\Delta_4^4$

Theorem 13. *Let $x \in \Delta_4^3(a)$. Then $G_{ax} \sim [2^6]3^2$ (with $G_{ax}^{*x} \sim [2^6]3^2$) has 8 orbits on $\Gamma_1(x)$ with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\alpha_{3,4\|0}(x, TRI, OCT)$	*2*	$\Delta_3^6\Delta_4^2\Delta_4^3$
$\alpha_{3,0}(x, TRI, OCT)$	*3*	$\Delta_3^2 2\Delta_4^3$
$\alpha_{1,0}(x, TRI, OCT)$	*12*	$\Delta_3^6 2\Delta_4^3$
$\alpha_{0,3\|1}(x, TRI, OCT)$	*32*	$2\Delta_4^3\Delta_4^5$
$\alpha_{1,2\|2}(x, TRI, OCT)$	*36*	$\Delta_3^5\Delta_4^1\Delta_4^3$
$\alpha_{0,1\|1}(x, TRI, OCT)$	*48*	$\Delta_3^5 2\Delta_4^3$
$\alpha_{2,1\|1}(x, TRI, OCT)$	*48*	$\Delta_4^3\Delta_4^4\Delta_4^6$
$\alpha_{1,2\|0}(x, TRI, OCT)$	*72*	$\Delta_4^3 2\Delta_4^6$

Theorem 14. *Let $x \in \Delta_4^4(a)$. Then $G_{ax} \cong L_2(11)$ (with $G_{ax}^{*x} \cong L_2(11)$) has 6 orbits on $\Gamma_1(x)$ with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\alpha_1(x, END, +)$	*11*	$\Delta_4^2 2\Delta_4^4$
$\alpha_1(x, END, -)$	*11*	$\Delta_3^3 2\Delta_4^4$
$\alpha_5(x, END, +)$	*11*	$\Delta_3^5\Delta_4^4\Delta_4^5$
$\alpha_3(x, END, +)$	*55*	$\Delta_4^1\Delta_4^4\Delta_4^6$
$\alpha_5(x, END, -)$	*55*	$\Delta_4^3\Delta_4^4\Delta_4^6$
$\alpha_3(x, END, -)$	*110*	$\Delta_4^1 2\Delta_4^4$

Theorem 15. *Let $x \in \Delta_4^5(a)$. Then $G_{ax} \cong A_7$ (with $G_{ax}^{*x} \cong A_7$) has 5 orbits on $\Gamma_1(x)$ with point distribution as follows:*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\{x + b\}$	*1*	$\Delta_3^4 2\Delta_4^5$
$\alpha_3(x, x + b, +)$	*35*	$\Delta_3^6\Delta_4^5\Delta_4^6$
$\alpha_1(x, x + b, +)$	*42*	$\Delta_3^5\Delta_4^4\Delta_4^5$
$\alpha_1(x, x + b, -)$	*70*	$2\Delta_4^3\Delta_4^5$
$\alpha_3(x, x + b, -)$	*105*	$\Delta_3^5\Delta_4^5\Delta_4^6$

Theorem 16. *Let $x \in \Delta_4^6(a)$. Then $G_{ax} \sim (3 \times A_5)2$ (with $G_{ax}^{*x} \sim (3 \times A_5)2$), G_{ax}^{*x} being the normalizer in G_x^{*x} of a group of order 3) has 8 orbits on $\Gamma_1(x)$ with point distribution as follows.*

ORBIT	*SIZE*	*POINT DISTRIBUTION*
$\alpha_{0,4}(x, TRI, FIX)$	*5*	$\Delta_3^3 2\Delta_4^6$
$\alpha_{3,1}(x, TRI, FIX)$	*5*	$\Delta_3^6\Delta_4^5\Delta_4^6$
$\alpha_{0,0}(x, TRI, FIX)$	*15*	$\Delta_3^5\Delta_4^5\Delta_4^6$
$\alpha_{2,0}(x, TRI, FIX)$	*18*	$2\Delta_3^5\Delta_4^6$
$\alpha_{1,3}(x, TRI, FIX)$	*30*	$\Delta_4^1\Delta_4^4\Delta_4^6$
$\alpha_{2,2}(x, TRI, FIX)$	*30*	$\Delta_4^3\Delta_4^4\Delta_4^6$
$\alpha_{0,2}(x, TRI, FIX)$	*60*	$\Delta_4^1 2\Delta_4^6$
$\alpha_{1,1}(x, TRI, FIX)$	*90*	$\Delta_4^3 2\Delta_4^6$

In the present paper we explore $\mathcal{G}$ as far as the third disc $\Delta_3(a)$, where a is a fixed point of $\mathcal{G}$. The analysis of $\Delta_3(a)$ is completed in [6] where we also carry out the dissection of $\Delta_4(a)$. We now discuss the contents of this paper and highlight some important features of the proofs of Theorems 1-16.

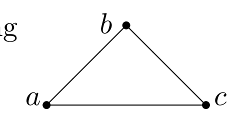
Section 2 begins with a quick reminder of some standard geometric notation before giving the promised further details on Γ. Then follows a long list of orbits on $\Gamma_1(x)$ ($x \in \Gamma_0$) for a variety of subgroups of $G_x^{*x} \cong M_{23}$. These orbits, and particularly their combinatorial description, lie at the heart of many of our later arguments.

The peeling back of the flesh of $\mathcal{G}$ gets underway in Section 3 where we examine the first two discs $\Delta_1(a)$ and $\Delta_2(a)$. We soon learn that $\Delta_2(a)$ is the union of two G_a-orbits, $\Delta_2^1(a)$ and $\Delta_2^2(a)$. The former of these G_a-orbits furnishes us with a useful configuration which we call a diamond. These are discussed after Lemma 3.9 with some of their properties stated in Lemmas 3.10 and 3.11. Diamonds are often used in the following way. We begin with a point, say x of $\mathcal{G}$ and two lines $x+y$

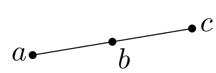
and $x+z$ (in $\Gamma_1(x)$) with $x+z \in \alpha_3(x, x+y)$. (For an explanation of $x+y$, $x+z$ and $\alpha_3(x, x+y)$, see Section 2.) Usually, from the choice of $x+y$ and $x+z$ we will know to which G_a-orbit x, y and z belong. Then shifting our view to other points of the diamond we seek to identify to which G_a-orbit they belong and as a consequence further increase our knowledge of $\mathcal{G}$. This type of strategy is frequently employed in [6] – for an inkling of what is in store see Lemma 4.5(iii). Also in Section 3 we meet $\tau(x)$. Lemma 3.2 gives a property of $\tau(x)$ that we use time and again.

In Section 4 we start to look at $\Delta_3(a)$ – this set breaks up into six G_a-orbits. For four of these orbits ($\Delta_3^i(a)$, $i = 1, 2, 3, 4$) we see that $\Gamma_3(a, x) \neq \emptyset$ (where $x \in \Delta_3^i(a)$, $i \in \{1, 2, 3, 4\}$). So, particularly in the light of Lemma 4.3, this is why hyperplane residues are important. For subsequent work in [6] on $\Delta_4(a)$ we single out for mention the summary results Lemmas 4.8 and 4.11 and Theorem 4.13.

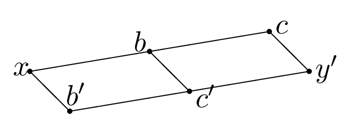
At certain points in this paper and [6] we will draw pictures depicting portions of $\mathcal{G}$. Rather than drawing



a, b, c points of $\mathcal{G}$; so $\{a, b, c\} = \Gamma_0(l)$) for some line $l$of Γ by Lemma 3.4) we usually draw



This is to simplify our pictures – usually we will have begun with collinear points a and c (adjacent points of $\mathcal{G}$) and later b comes in for attention. So, for example, the situation in Lemma 3.10 (using the notation there) is drawn thus



We will follow the ATLAS [1] in our description of group structures and our group theoretic notation is standard as given in either [3] or [8]. Also, if H and K are groups, $H \sim K$ means that H and K have the same shape. Finally, we recommend the reader to also look on Figure 1 as a useful navigational aid for keeping track of our whereabouts in the graph.

2. Notation and Line Orbits

First we review some standard geometric notation and begin by recalling the definition of a geometry. A geometry Γ is, strictly, a triple $(\Gamma, t, *)$ where Γ is a set, t is the type map $(t : \Gamma \to \{0, 1, \ldots, n-1\})$ and $*$ is a symmetric incidence relation on Γ with the property that whenever $x * y$ $(x, y \in \Gamma)$ then $t(x) \neq t(y)$. If t is onto, then Γ is said to have rank n.

Let $i \in \{0, 1, ..., n-1\}$, $x \in \Gamma$ and $\Sigma \subseteq \Gamma$. Then

$$\begin{aligned}
\Gamma_i &:= \{y \in \Gamma | \tau(y) = i\} \text{ (the objects of } \Gamma \text{ of type } i);\\
\Gamma_x &:= \{y \in \Gamma | x * y\} \text{ (the residue geometry of } x);\\
\Gamma(\Sigma) &:= \{y \in \Gamma | x * y \text{ for all } x \in \Sigma\}; \text{ and}\\
\Gamma_i(\Sigma) &:= \Gamma_i \cap \Gamma(\Sigma).
\end{aligned}$$

If $\Sigma = \{x_1, \ldots, x_k\}$, then we write $\Gamma(x_1, \ldots, x_k)$ and $\Gamma_i(x_1, \ldots, x_k)$ instead of $\Gamma(\{x_1, \ldots, x_k\})$ and $\Gamma_i(\{x_1, \ldots, x_k\})$. Note that $\Gamma_i(x) = \Gamma_x \cap \Gamma_i$. If G is a subgroup of $\mathrm{Aut}\,\Gamma$, then G_Σ or $G_{x_1 \ldots, x_k}$ denotes the subgroup of G fixing every object in $\Sigma = \{x_1, \ldots, x_k\}$. For $g \in G$ and $x \in \Gamma$, x^g is the image of x under g. Also we define

$$Q(x) := \{g \in G_x \mid g \text{ fixes every object in } \Gamma_x\}.$$

So $Q(x)$ is a normal subgroup of G_x. For $H \leq G_x$ we denote $HQ(x)/Q(x)$ by H^{*x}.

From now on Γ will be the rank 4 geometry introduced in Section 1 upon which $G = Fi_{23}$ acts flag transitively and $\mathcal{G}$ the point-line collinearity graph of Γ. The graph distance metric in $\mathcal{G}$ will be denoted by $d(,)$, and for $x \in \Gamma_0$

$$\Delta_i(x) = \{y \in \Gamma_0 \mid d(y,x) = i\} \text{ (the } i\text{th disc of } x).$$

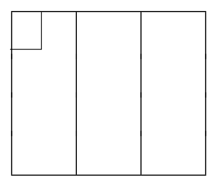
For $x, y \in \Gamma_0$, we put $\{x,y\}^\perp = \Delta_1(x) \cap \Delta_1(y)$.

Next we survey the properties of Γ that will be used in our analysis of $\mathcal{G}$. First we recall that Γ is a string geometry (meaning that for $0 \leq i < j < k \leq 3$ and $a_r \in \Gamma_r$, $r \in \{i,j,k\}$, $a_i * a_j$ and $a_j * a_k$ implies that $a_i * a_k$). Not surprisingly, in examining $\mathcal{G}$ the most important subgeometry of Γ is Γ_x, the residue geometry of a point x. Here we have $G_x/Q(x) \cong M_{23}$ with $Q(x)$ being the 11-dimensional irreducible $GF(2)M_{23}$ Todd-module. Γ_x and the induced action of $G_x/Q(x)$ is best viewed by taking a 23-element set, denoted by Ω_x endowed with the Steiner system $S(23,7,4)$. (Note the use of the word element so as to distinguish them from the points of Γ.) Then $\Gamma_x := \Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$ is a rank 3 geometry where Δ_0 consists of the heptads of the $S(23,7,4)$ on Ω_x, Δ_1 of all 3-element subsets of Ω_x and $\Delta_2 = \Omega_x$, with incidence given by (symmetrized) containment. The lines of Γ in Γ_x correspond to Δ_0 and the hyperplanes of Γ in Γ_x correspond to Δ_2.

For X a hyperplane of Γ, we have $G_X/Q(X) \cong Fi_{22}$ (the smallest Fischer group) with $|Q(X)| = 2$. We observe that if two points (bases) are both incident with the same hyperplane (transvection), and are collinear in Γ, then they are also collinear in Γ_X. The only other information about Γ_X pertinent here is the structure of the graph given in [5; Appendix 1] which is the induced subgraph $\Gamma_0 \cap \Gamma_X$ of $\mathcal{G}$. (Though [5] deals with the minimal parabolic geometries of Fi_{22}, note that Lemma 4.4 and discussion following in [5] show that it is the induced subgraph $\Gamma_0 \cap \Gamma_X$.)

Throughout this work, we adopt the following convention in order to avoid rampant notation. A line l and hyperplane X of Γ when viewed in the residue of some point x of Γ will metamorphose into (respectively) a heptad and an element of Ω_x. Equally, without further mention we shall regard heptads and elements of Ω_x as lines and hyperplanes of Γ_x.

Concerning the set Ω_x ($x \in \Gamma_0$), for concrete calculations we regard Ω_x as a subset of the MOG thus



(the top left-most element being removed). And of course we carry out these calculations in S(23,7,4) with the aid of Curtis's MOG [2].

As is observed in Lemma 3.3 two collinear points x and y of Γ_0 determine a unique line of Γ_1 – frequently we shall denote this line by $x+y$ (respectively $y+x$) to alert us to the fact that we are viewing the line in Γ_x (respectively Γ_y).

Fix $x \in \Gamma_0$ and let H be a subgroup of $L := G_x/Q(x)$ ($\cong M_{23}$). Before dealing with specific subgroups of H of M_{23} and their orbits on $\Gamma_1(x)$, we say a few words about their taxonomy. Frequently H may be specified as the subgroup of M_{23} stabilizing two particular subsets of Ω_x. Then, usually, the orbits of H upon $\Gamma_1(x)$ are determined by the intersections of the heptads (lines of Γ_x) with these subsets of Ω_x. Accordingly, the notation for an H-orbit on $\Gamma_1(x)$ is often of the form

$$\alpha_{i,j}(x, R, S).$$

The first entry x tells us we are working in the residue Γ_x, and R and S are subsets of Ω_x. So $l \in \alpha_{i,j}(x, R, S)$ means that l is a heptad of Ω_x with $|l \cap R| = i$ and $|l \cap S| = j$. In some instances the orbits may be described just using one subset of Ω_x, so the following is used

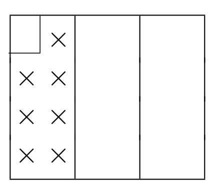
$$\alpha_i(x, R).$$

Frequently we have the case $|S| = 1$, say $S = \{X\}$. Then we write $\alpha_{i,j}(x, R, X)$ instead of $\alpha_{i,j}(x, R, \{X\})$ – for $l \in \alpha_{i,j}(x, R, X)$, $j = 0$ is, of course, equivalent to $X \notin l$ and $j = 1$ to $X \in l$. Still with the case when $|S| = 1$, we shall see instances where there is no obvious description of X. When this happens we use the following variant of the $\alpha_{i,j}(x, R, X)$ notation:

$$\alpha_i(x, R, +) \text{ or } \alpha_i(x, R, -).$$

Here $l \in \alpha_i(x, R, +)$ (respectively $\alpha_i(x, R, -)$) means $l \in \Gamma_1(x)$, $|R \cap l| = i$ and $X \in l$ (respectively $X \notin l$). There are some minor variations to the above scheme which we deal with as they arise.

In (2.1) - (2.14) we list data on the line orbits for various subgroups of M_{23} (to aid reference to these results, we indicate the G_a-orbit(s) where this information will be used). In the following statements, we first define the relevant subsets of Ω_x and then give the H-orbits, their sizes as well as a representative line (as a heptad in Ω_x) for each H-orbit. When mentioned, $x + b$ is some fixed line of $\Gamma_1(x)$ (so $b \in \Delta_1(x)$) and will be taken to be the standard heptad

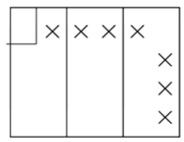


of Ω_x.

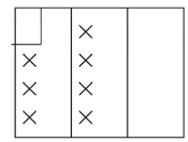
(2.1) ($\Delta_1(a)$ orbit) $H \sim 2^4 A_7$.

$\{x+b\}$ 1

$\alpha_1(x, x+b)$ 112

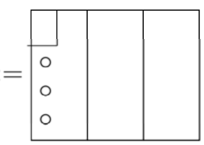


$\alpha_3(x, x+b)$ 140

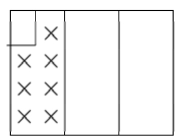


(2.2) ($\Delta_2^1(a)$ orbit) $H \sim 2^4 S_5 3$.

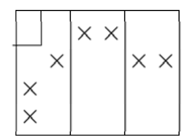
$T(x,a) :=$



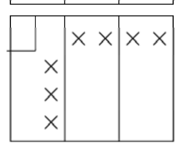
$\alpha_3(x, T(x,a))$ 5



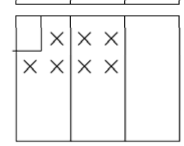
$\alpha_2(x, T(x,a))$ 48



$\alpha_0(x, T(x,a))$ 80

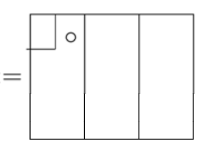


$\alpha_1(x, T(x,a))$ 120



(2.3) ($\Delta_2^2(a)$ and $\Delta_3^3(a)$ orbits) $H \sim 2^4 A_6$.

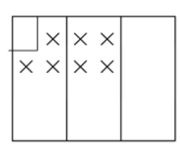
$X = X(x,a) :=$



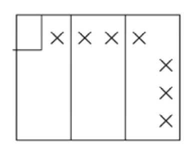
(so $X(x,a) \in x+b$)

$\{x+b\}$ 1

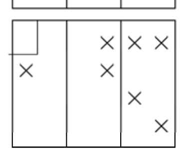
$\alpha_{3,1}(x, x+b, X)$ 60



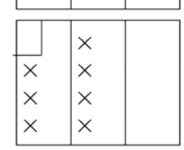
$\alpha_{1,1}(x, x+b, X)$ 16



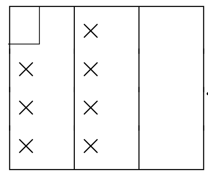
$\alpha_{1,0}(x, x+b, X)$ 96



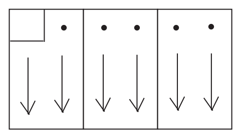
$\alpha_{3,0}(x, x+b, X)$ 80



Remark. Let $l \in \alpha_{3,0}(x, x+b, X(x,a))$ where

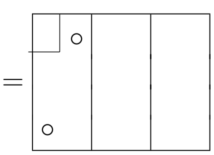
$l =$  .

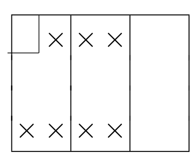
Then H_l, which has order $2^3.3^2$, has a normal subgroup of order 3 generated by ξ where

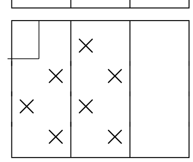
$\xi =$ 

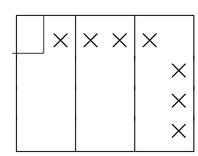
Further, ξ fixes 4 heptads in $\alpha_{1,1}(x, x+b, X(x,a))$, each of which form a diamond with l (see Section 3 for the definition of a diamond). Also H_l contains a subgroup isomorphic to $3 \times A_4$.

(2.4) ($\Delta_3^1(a)$ orbit) $H \sim L_3(4)2$.

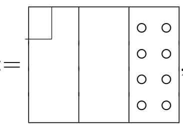
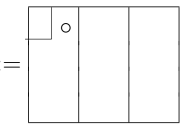
$D = D(x,a) :=$ 

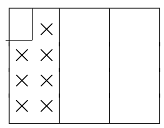
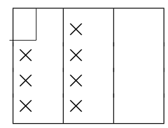
$\alpha_2(x,D)$ 21 

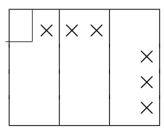
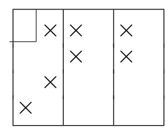
$\alpha_0(x,D)$ 120 

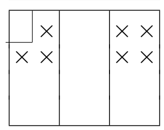
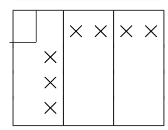
$\alpha_1(x,D)$ 112 

(2.5) ($\Delta_3^2(a)$ orbit) $H \sim 2^3L_3(2)$.

$O = O(x, a) :=$ , $X = X(x, a) :=$ 

$\alpha_{0,1}(x, O, X)$ 7  $\alpha_{0,0}(x, O, X)$ 8 

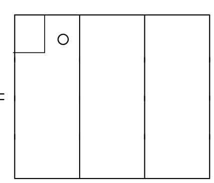
$\alpha_{4,1}(x, O, X)$ 14  $\alpha_{2,1}(x, O, X)$ 56 

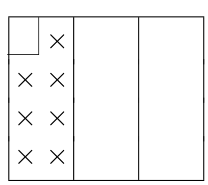
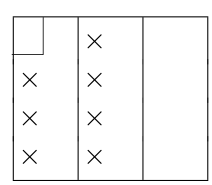
$\alpha_{4,0}(x, O, X)$ 56  $\alpha_{2,0}(x, O, X)$ 112 

Remark. (i) For $l \in \alpha_{0,0}(x, O(x, a), X(x, a))$, we have $H_l \cong L_3(2)$.

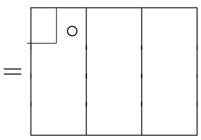
(ii) Assume that $l \in \alpha_{4,0}(x, O(x, a), X(x, a))$. Then $H_l \sim 2^3.3$, $H_l \cap O_2(H) = 1$ and $H_0 := \langle H_l, O_2(H)\rangle$ contains no normal subgroup of H_0 of order 2^2.

(2.6) ($\Delta_3^4(a)$ orbit) $H \cong M_{22}$.

$X = X(x, a) :=$ 

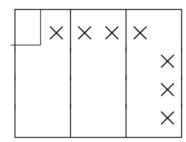
$\alpha_1(x, X)$ 77  $\alpha_0(x, X)$ 176 

(2.7) ($\Delta_3^5(a)$ orbit) $H \sim 2^4A_5 \leq 2^4A_7$ (the stabilizer of $x+b$) where the A_5 has orbits of sizes 1 and 6 on the elements of $x+b$, the standard heptad.

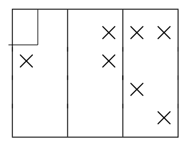
$X(x,a) :=$ 

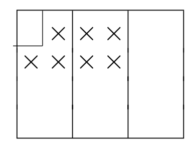
$\{x+b\}$ 1

$\alpha_3^{(1)}(x, x+b, -)$ 40 See remark below

$\alpha_1(x, x+b, +)$ 16 

$\alpha_3^{(2)}(x, x+b, -)$ 40 See remark below

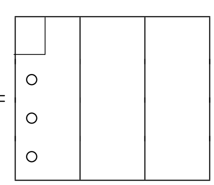
$\alpha_1(x, x+b, -)$ 96 

$\alpha_3(x, x+b, +)$ 60 

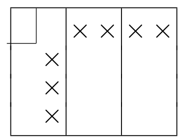
Remark. (i) For $l \in \alpha_1(x, x+b, +)$, $H_l \cong A_5$ and H_l has orbits of length 1 and 6 upon the elements of the heptad $x+b$.

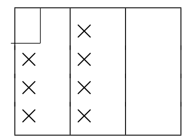
(ii) In (2.7) we have the exceptional degree 6 A_5 permutation representation making an appearance – since this not a 3-transitive permutation representation, $\alpha_3(x, x+b, -)$ splits into two orbits, called $\alpha_3^{(1)}(x, x+b, -)$ and $\alpha_3^{(2)}(x, x+b, -)$. In order to give representatives for each of these two orbits we need to specify $H = 2^4A_5$ "concretely". We will not do this since these two orbits will be distinguished via certain configurations in $\mathcal{G}$ (see [6; Section 5]).

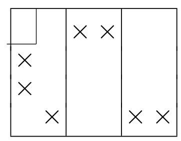
(2.8) ($\Delta_3^6(a)$ orbit) $H \sim [2^7]3^2 \leq 2^4A_7$ (the stabilizer of $x+b$); H is the stabilizer of a 3-set of $x+b$.

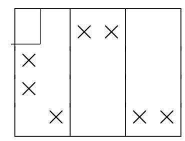
$T = TRI :=$ 

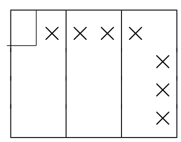
$\{x+b\}$ 1

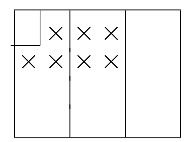
$\alpha_{3,0}(x, x+b, T)$ 16 

$\alpha_{3,3}(x, x+b, T)$ 4 

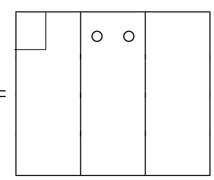
$\alpha_{3,2}(x, x+b, T)$ 48 

$\alpha_{3,2}(x, x+b, T)$ 48 

$\alpha_{1,0}(x, x+b, T)$ 64 

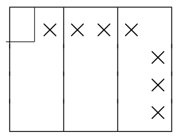
$\alpha_{3,1}(x, x+b, T)$ 72 

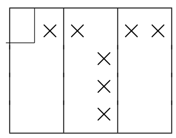
(2.9) $\Delta_4^1(a)$ orbit) $H \sim L_3(2)2(\leq L_3(4)2)$.

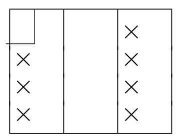
$D = \text{DUAD} :=$ 

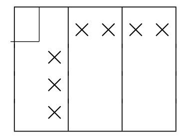
The 7 heptads in $\alpha_{3,2}(x, x+b, \text{DUAD})$ intersect the standard heptad in seven 3-element subsets and these 3-elements are the lines of a projective plane on the 7 elements of the standard heptad. Denote this collection of 3-elements of the standard heptad by $\mathfrak{L}$. Now $\alpha_{3,0}^{\mathfrak{L}}(x, x+b, \text{DUAD})$ consists of all heptads which, in addition to missing DUAD, intersect the standard heptad in a 3-element subset of $\mathfrak{L}$, and $\alpha_{3,0}^{\mathfrak{L}^c}(x, x+b, \text{DUAD}) = \alpha_{3,0}(x, x+b, \text{DUAD}) \backslash \alpha_{3,0}^{\mathfrak{L}}(x, x+b, \text{DUAD})$.

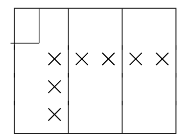
$\{x+b\}$ 1

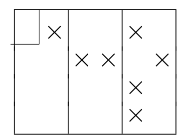
$\alpha_{1,2}(x, x+b, \mathrm{D})$ 14 

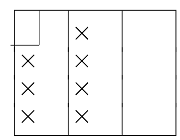
$\alpha_{1,1}(x, x+b, \mathrm{D})$ 42 

$\alpha_{3,0}^{\mathcal{L}^c}(x, x+b, \mathrm{D})$ 56 

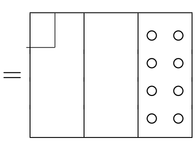
$\alpha_{3,2}(x, x+b, \mathrm{D})$ 7 

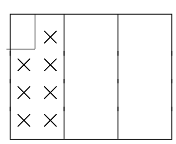
$\alpha_{3,0}^{\mathcal{L}}(x, x+b, \mathrm{D})$ 21 

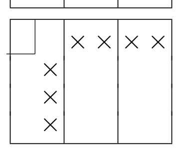
$\alpha_{1,0}(x, x+b, \mathrm{D})$ 56 

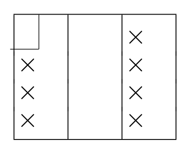
$\alpha_{3,1}(x, x+b, \mathrm{D})$ 56 

(2.10) ($\Delta_4^2(a)$ orbit) $H \cong A_8$.

$O = O(x,a) :=$ 

$\alpha_0(x, O)$ 15 

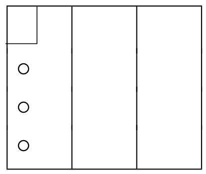
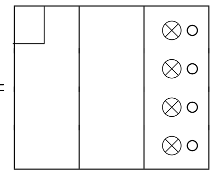
$\alpha_2(x, O)$ 168 

$\alpha_4(x, O)$ 70 

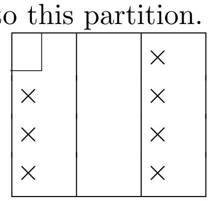
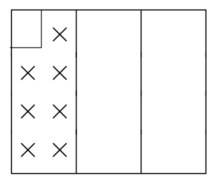
Remark. (i) For $l \in \alpha_4(x, O(x,a))$, $O_2(H_l) \cong 2^2$.

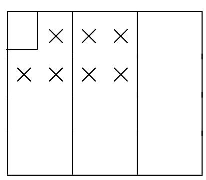
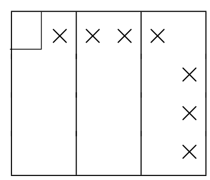
(ii) If $l \in \alpha_2(x, O(x,a))$, then $H_l \sim A_5 2$.

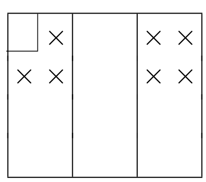
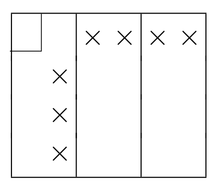
(2.11) ($\Delta_4^3(a)$ orbit) $H \sim [2^6]3^2$ (the stabilizer of OCT and the standard sextet).

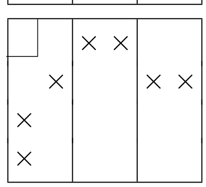
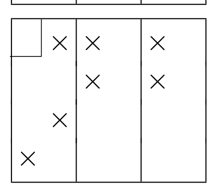
$T = TRI :=$  $O = OCT :=$ 

The orbits of H on $\Gamma_1(x)$ are parameterized by intersections with TRI and the partition of OCT into $4|4$ indicated by the $\otimes$'s and $\circ$'s. The subscript $j|k$ below describes how the intersection of a heptad with OCT splits with respect to this partition.

$\alpha_{3,4|0}(x, T, O)$ 2  $\alpha_{3,0}(x, T, O)$ 3 

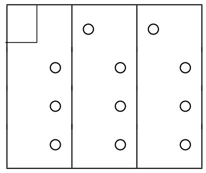
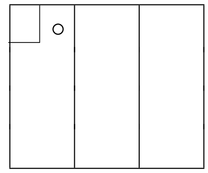
$\alpha_{1,0}(x, T, O)$ 12  $\alpha_{0,3|1}(x, T, O)$ 32 

$\alpha_{1,2|2}(x, T, O)$ 36  $\alpha_{0,1|1}(x, T, O)$ 48 

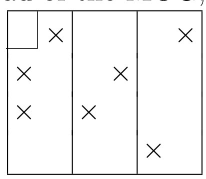
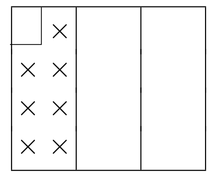
$\alpha_{2,1|1}(x, T, O)$ 48  $\alpha_{1,2|0}(x, T, O)$ 72 

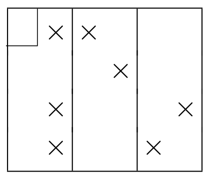
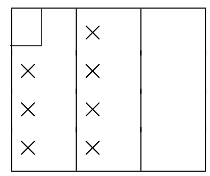
Remark. We note that TRI is the triad contained in all heptads in $\alpha_{3,4|0}(x, TRI, OCT)$ and $\alpha_{3,0}(x, TRI, OCT)$ and that the partition of the octad OCT is determined by the intersection with OCT of either of the heptads in $\alpha_{3,4|0}(x, TRI, OCT)$.

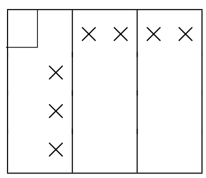
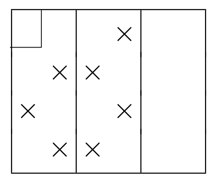
(2.12) ($\Delta_4^4(a)$ orbit) $H \cong L_2(11)$.

$END :=$  $X :=$ 

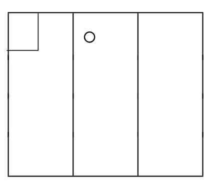
(So END is an endecad of the MOG; see [1].)

$\alpha_1(x, END, +)$ 11  $\alpha_3(x, END, +)$ 55 

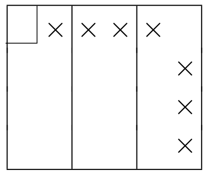
$\alpha_5(x, END, +)$ 11  $\alpha_1(x, END, -)$ 11 

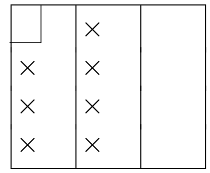
$\alpha_5(x, END, -)$ 55  $\alpha_3(x, END, -)$ 110 

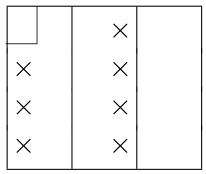
(2.13) ($\Delta_4^5(a)$ orbit) $H \cong A_7$.

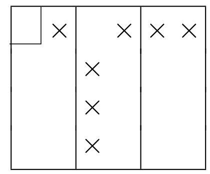
$X :=$ 

$\{x+b\}$ 1

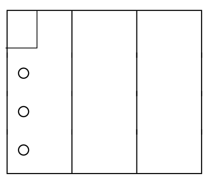
$\alpha_1(x, x+b, +)$ 42 

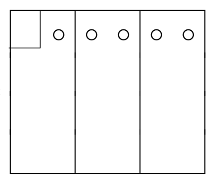
$\alpha_3(x, x+b, +)$ 35 

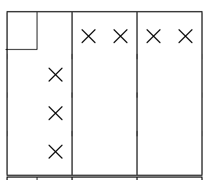
$\alpha_3(x, x+b, -)$ 105 

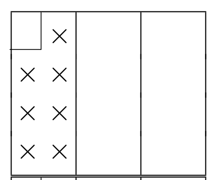
$\alpha_1(x, x+b, -)$ 70 

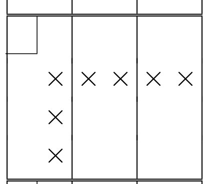
(2.14) ($\Delta_4^6(a)$ orbit) $H \sim (3 \times A_5)2$ ($= N_L(T)$ where $T \leq L$ has order 3). Also recall that H is a subgroup of a triad stabilizer and that T fixes exactly 5 elements of Ω_x.

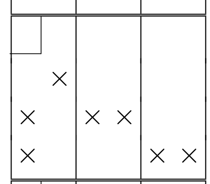
$T = TRI :=$ 

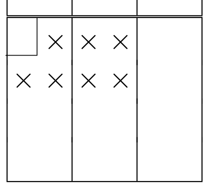
$F = FIX :=$ 

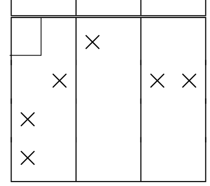
$\alpha_{0,4}(x, T, F)$ 5 

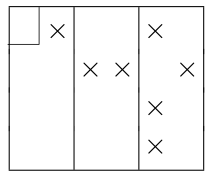
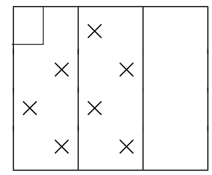
$\alpha_{3,1}(x, T, F)$ 5 

$\alpha_{0,0}(x, T, F)$ 15 

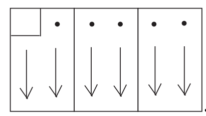
$\alpha_{2,0}(x, T, F)$ 18 

$\alpha_{1,3}(x, T, F)$ 30 

$\alpha_{2,2}(x, T, F)$ 30 

$\alpha_{0,2}(x,T,F)$ 60  $\alpha_{1,1}(x,T,F)$ 90 

Remark. The group T above is generated by

.

(2.15) Let a be a point of $\mathcal{G}$. (The notation, $T(c,a)$ and $X(c,a)$, for $c \in \Delta_2(a)$ is introduced in Section 3 while, for $d \in \Delta_3(a)$, $X(d,a)$ is given after Lemma 4.3 and $O(d,a)$ is given in Theorem 4.13(iv).)

(i) $\Delta_2^1(a) = \{x \in \Gamma_0 |$ there exists $b \in \{a,x\}^\perp$ such that $b+x \in \alpha_3(b, b+a)\}$.

(ii) $\Delta_2^2(a) = \{x \in \Gamma_0 |$ there exists $b \in \{a,x\}^\perp$ such that $b+x \in \alpha_1(b, b+a)\}$.

(iii) $\Delta_3^1(a) = \{x \in \Gamma_0 |$ there exists $c \in \Delta_2^1(a) \cap \Delta_1(x)$ such that $c+x \in \alpha_2(c, T(c,a))\}$.

(iv) $\Delta_3^2(a) = \{x \in \Gamma_0 |$ there exists $c \in \Delta_2^2(a) \cap \Delta_1(x)$ such that $c+x \in \alpha_{3,1}(c, c+b, X(c,a))$, where $\{b\} = \{a,c\}^\perp\}$.

(v) $\Delta_3^3(a) = \{x \in \Gamma_0 |$ there exists $c \in \Delta_2^2(a) \cap \Delta_1(x)$ such that $c+x \in \alpha_{1,1}(c, c+b, X(c,a))$ (where $\{b\} = \{a,c\}^\perp$) and c is the unique point in $\Gamma_0(X(a,c))$ lying in $\Delta_2^2(a) \cap \Delta_1(x)\}$.

(vi) $\Delta_3^4(a) = \{x \in \Gamma_0 |$ there exists $c \in \Delta_2^2(a) \cap \Delta_1(x)$ such that $c+x \in \alpha_{1,1}(c, c+b, X(c,a))$ (where $\{b\} = \{a,c\}^\perp$) and there are 77 points in $\Gamma_0(X(a,c))$ lying in $\Delta_2^2(a) \cap \Delta_1(x)\}$.

(vii) $\Delta_3^5(a) = \{x \in \Gamma_0 |$ there exists $c \in \Delta_2^2(a) \cap \Delta_1(x)$ such that $c+x \in \alpha_{1,0}(c, c+b, X(c,a))$, where $\{b\} = \{a,c\}^\perp\}$.

(viii) $\Delta_3^6(a) = \{x \in \Gamma_0|$ there exists $c \in \Delta_2^2(a) \cap \Delta_1(x)$ such that $c + x \in \alpha_{3,0}(c, c+b, X(c,a))$, where $\{b\} = \{a,c\}^{\perp}\}$.

(ix) $\Delta_4^1(a) = \{x \in \Gamma_0|$ there exists $d \in \Delta_3^1(a) \cap \Delta_1(x)$ such that $d + x \in \alpha_0(d, D(d,a))\}$.

(x) $\Delta_4^2(a) = \{x \in \Gamma_0|$ there exists $d \in \Delta_3^2(a) \cap \Delta_1(x)$ such that $d + x \in \alpha_{0,0}(d, O(d,a), X(d,a))\}$.

(xi) $\Delta_4^3(a) = \{x \in \Gamma_0|$ there exists $d \in \Delta_3^2(a) \cap \Delta_1(x)$ such that $d + x \in \alpha_{4,0}(d, O(d,a), X(d,a))\}$.

(xii) $\Delta_4^4(a) = \{x \in \Gamma_0|$ there exists $d \in \Delta_3^3(a) \cap \Delta_1(x)$ such that $d + x \in \alpha_{1,0}(d, d+b, X(d,a))$ where $\{b\} = \Delta_1(d) \cap \Delta_2^2(a)\}$.

(xiii) $\Delta_4^5(a) = \{x \in \Gamma_0|$ there exists $d \in \Delta_3^4(a) \cap \Delta_1(x)$ such that $d + x \in \alpha_0(d, X(d,a))\}$.

(xiv) $\Delta_4^6(a) = \{x \in \Gamma_0|$ there exists $d \in \Delta_3^3(a) \cap \Delta_1(x)$ such that $d + x \in \alpha_{3,0}(d, d+b, X(d,a))$ where $\{b\} = \Delta_1(d) \cap \Delta_2^2(a)\}$.

We remark that our notation has been chosen so as to mesh with that of [5] – so here our $\Delta_1(a), \Delta_2^1(a), \Delta_2^2(a), \Delta_3^1(a), \Delta_3^2(a), \Delta_3^3(a), \Delta_3^4(a)$ when intersected with Γ_X (for $X \in \Gamma_3(a)$) gives precisely the $\Delta_i^j(a)$ of [5].

3. The First Two Discs and Diamonds

Lemma 3.1 ("Three points on a line"). *For $l \in \Gamma_1$, $|\Gamma_0(l)| = 3$.*

Proof. Let $X \in \Gamma_3(l)$. Then Γ being a string geometry implies that $\Gamma_0(l) \subseteq \Gamma_0(X)$, whence Lemma 3.1 follows from [5; Lemma 4.4].

Since, for $X \in \Gamma_3$, $G_X \cong 2Fi_{22}$ and $|Z(G_X)| = 2$, let $\tau(X) \in G_X$ be such that $\langle \tau(X) \rangle = Z(G_X)$. (Of course $\tau(X)$ is just a transposition of $G = Fi_{23}$.) Note that, for $x \in \Gamma_0(X)$, we have $\tau(X) \in Q(x)$. Also, for $l \in \Gamma_1(X)$, Γ being a string geometry and $\tau(X) \in Q(X)$ means that $\tau(X)$ fixes each of the points of $\Gamma_0(l)$.

Lemma 3.2. *Suppose $x \in \Gamma_0$, $l \in \Gamma_1(x)$ and $X \in \Gamma_3(x)$. Then $\tau(X)$ interchanges $\Gamma_0(l)\backslash\{x\}$ if and only if $X \notin l$. (Recall our convention – in Γ_x, X is an element and l a heptad of Ω_x.)*

Proof. Since $G_{xX}^{*x} \cong M_{22}$, G_{xX} has two orbits upon the 253 heptads in Ω_x – those containing X (77) and those not containing X (176). If the result were false, then, since $\tau(X) \in Z(G_{xX})$, we infer that $\tau(X)$ fixes the points in $\Gamma_0(l)$ for all $l \in \Gamma_1(x)$. Because $Q(x)$ is an irreducible $GF(2)G_x$-module, $Q(x)$ then fixes the points in $\Gamma_0(l)$ for all $l \in \Gamma_1(x)$, contradicting [5; Lemma 4.2(iii)].

Lemma 3.3. *If x and y are collinear points of Γ, then $|\Gamma_1(x,y)| = 1$.*

Proof. Suppose we have $l, k \in \Gamma_1(x,y)$, and let $X \in \Gamma_3(l)$. Hence, as Γ is a string geometry, $X \in \Gamma_3(x) \cap \Gamma_3(y)$. So, in particular, $\tau(X)$ fixes y and thus, applying Lemma 3.2 in Γ_x, we get $X \in k$. That is $l, k \in \Gamma_1(X)$, whence $l = k$ by the structure of Γ_X (see [5; Lemma 4.4]).

Lemma 3.4. *If x, y and z form a triangle in $\mathcal{G}_0$ where $x, y, z \in \Gamma_0$, then $\{x, y, z\} = \Gamma_0(l)$ for some $l \in \Gamma_1$.*

Proof. Let $\{l\} = \Gamma_1(x,y)$, $\{m\} = \Gamma_1(y,z)$ and $\{k\} = \Gamma_1(z,x)$. Choose $X \in \Gamma_3(l)$. So $\tau(X) \in Q(x) \cap Q(y)$ and therefore $z^{\tau(X)} \in \Gamma_0(m) \cap \Gamma_0(k)$. If $k \notin \Gamma_X$, then $z^{\tau(X)} \neq z$ by Lemma 3.2. Therefore, using Lemma 3.3, $m = z + z^{\tau(X)} = k$, and then $k = l$, a contradiction. Thus $k \in \Gamma_X$, and so we get $l, k, x, y, z \in \Gamma_X$, whence $m \in \Gamma_X$ and then [5; Lemma 5.5] gives the lemma.

We now choose, and keep fixed, a point $a \in \Gamma_0$; our next result is about $\Delta_1(a)$, the first disc of a.

Lemma 3.5. (i) *$\Delta_1(a)$ is a G_a-orbit and, for $x \in \Delta_1(a)$, $G_{ax} \sim 2^{10}2^4A_7$ with $G^{*x}_{ax} \sim 2^4A_7$ (the stabilizer in G^{*x}_x of the line $x+a$); and*

(ii) $|\Delta_1(a)| = 2.11.23$.

Proof. Let $x \in \Delta_1(a)$, and select $X \in \Omega_a$ such that $X \notin \Gamma_3(a+x)$. Then $\tau(X)$ interchanges the two points in $\Gamma_0(a+x)\backslash\{a\}$ by Lemma 3.2 and therefore, as G_a is transitive on $\Gamma_1(a)$, we obtain (i). Since $\Gamma_1(a)$ has $253 = 11.23$ lines, (ii) follows using Lemma 3.1.

From Lemmas 3.4 and 3.5, (2.1) and the definition of $\Delta^1_2(a)$ and $\Delta^2_2(a)$ we see that Theorem 2 holds. We now proceed to examine $\Delta_2(a)$, the second disc of a.

Lemma 3.6. *Let $X \in \Gamma_3$. If $x, y \in \Gamma_0(X)$, then $\{x,y\}^\perp \subseteq \Gamma_0(X)$.*

Proof. Suppose we have $b \in \{x,y\}^\perp$ with $b \notin \Gamma_0(X)$. Then $b+x \neq b+y$. In Ω_x, $x+b$ is a heptad not containing X. So $b^{\tau(X)} \neq b$ by Lemma 3.2. Now $\tau(X) \in Q(x) \cap Q(y)$ and hence $b^{\tau(X)} \in \Gamma_0(x+b) \cap \Gamma_0(y+b)$. But then, by Lemma 3.3, $b+x = b+b^{\tau(X)} = b+y$, a contradiction.

Lemma 3.7. *Let $x \in \Delta^1_2(a)$. Then*

(i) *$|\{a,x\}^\perp| = 5$ and there are exactly 3 hyperplanes $\{X_1, X_2, X_3\}$ in $\Gamma_3(a,x)$, and $\{a,x\}^\perp \subseteq \Gamma_0(X_i)$ for $i = 1,2,3$;*

(ii) *$\Delta^1_2(a)$ is a G_a-orbit and $G_{ax} \sim 2^7 2^4 S_5 3$ (with $G^{*x}_{ax} \sim 2^4 S_5 3$); and*

(iii) $|\Delta^1_2(a)| = 2^4.7.11.23$.

Proof. Let $b \in \{a,x\}^\perp$ be such that $b+x \in \alpha_3(b, b+a)$. So (in Ω_b) $(b+a) \cap (b+x) = \{X_1, X_2, X_3\}$. Hence $a, x \in \Gamma_0(X_i)$ and consequently $\{a,x\}^\perp \subseteq \Gamma_0(X_i)$ for $i = 1,2,3$ by Lemma 3.6. From [5; Lemma 5.7(ii)] $|\{a,x\}^\perp| = 5$, and we have (i).

By Lemma 3.5(i) and (2.1) G_{ab} is transitive on the heptads of Ω_b intersecting $b+a$ in 3 elements. Also, by selecting a $Y \in \Gamma_3(b+a)\backslash\Gamma_3(b+x)$ we get that $\tau(Y)$ interchanges $\Gamma_0(b+x)\backslash\{b\}$ by Lemma 3.2. Hence, as G_a is transitive on $\Delta_1(a)$, $\Delta_2^1(a)$ is a G_a-orbit. Using $|\{a,x\}^\perp| = 5$, we may count thus

$$|\Delta_2^1(a)| = \frac{2.|\Delta_1(a)|.140}{5} = 2^4.7.11.23.$$

Now, by part (i), $[G_{ax} : G_{ax} \cap G_{X_1}] = 1$ or 3 and then combining $|\Delta_2^1(a)|$ with $\Delta_2^1(a)$ being a G_a-orbit and $|G_{ax} \cap G_{X_1}| = 2^{14}.3.5$ (see [5;Lemma 5.7(iii)]) yields $[G_{ax} : G_{ax} \cap G_{X_1}] = 3$. Appealing to [5; Lemma 5.7(iii)] again gives the shape of G_{ax}, and this completes the proof of Lemma 3.7.

Lemma 3.8. *Let $x \in \Delta_2^2(a)$. Then*

(i) $|\{a,x\}^\perp| = 1$ *and there is a unique hyperplane X in $\Gamma_3(a,x)$, and* $\{a,x\}^\perp \subseteq \Gamma_0(X)$*;*

(ii) $\Delta_2^2(a)$ *is a G_a-orbit and $G_{ax} \sim 2^5 2^4 A_6$ (with $G_{ax}^{*x} \sim 2^4 A_6$); and*

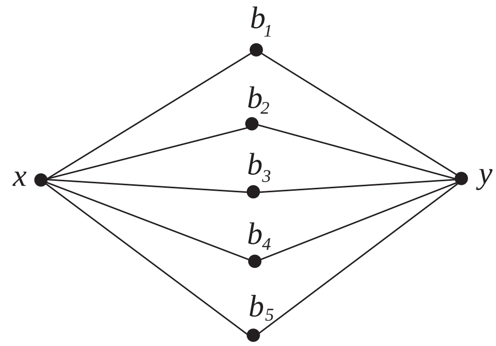
(iii) $|\Delta_2^2(a)| = 2^6.7.11.23$.

Proof. This result may be proved in the same manner as Lemma 3.7, using [5; Lemma 5.10] in place of [5; Lemma 5.7(iii)].

Lemma 3.9. $\Delta_2(a) = \Delta_2^1(a)\dot{\cup}\Delta_2^2(a)$.

Proof. Clearly, by Lemmas 3.3 and 3.4, we have $\Delta_2(a) = \Delta_2^1(a) \cup \Delta_2^2(a)$. So we only need to show that $\Delta_2^1(a) \cap \Delta_2^2(a) = \emptyset$. Suppose we have $x \in \Delta_2^1(a) \cap \Delta_2^2(a)$. Then there exist $b, b' \in \{a,x\}^\perp$ such that $|(b+a) \cap (b+x)| = 1$ and $|(b'+a) \cap (b'+x)| = 3$. Employing Lemma 3.7 gives $\{b, b'\} \subseteq \Gamma_0(X_i)$ for $i = 1,2,3$ where $\{X_1, X_2, X_3\} = \Gamma_3(a,x)$. But this contradicts $|(b+a) \cap (b+x)| = 1$.

Let $x, y \in \Gamma_0$ with $y \in \Delta_2^1(x)$ (so $x \in \Delta_2^1(y)$ also). By Lemma 3.7 $\{x, y\}^\perp = \{b_1, b_2, b_3, b_4, b_5\}$ and $\Gamma_3(x, y) = \{X_1, X_2, X_3\}$. So we have



and we will call such a configuration in $\mathcal{G}$ a **diamond**. Note that for $i \neq j, b_i \in \Delta_2^1(b_j)$. We will employ the same positional aid notation as for lines and denote $\{X_1, X_2, X_3\}$ by $T(x, y)$ so as to signal that we are viewing this as a subset of Ω_x.

A crucial observation, used repeatedly in later arguments, is that this set of hyperplanes $\{X_1, X_2, X_3\}$ manifests itself in the residue of any point in the diamond. That is, $T(b_i, b_j) = \{X_1, X_2, X_3\} = T(y, x)$; moreover $T(b_i, b_j) = (b_i + x) \cap (b_j + y)$, $T(x, y) = (x + b_i) \cap (x + b_j)$ and $T(y, x) = (y + b_i) \cap (y + b_j)$ $(1 \leq i < j \leq 5)$. Also $\{x + b_i | i = 1, \ldots, 5\}$ are precisely the 5 heptads which contain $T(x, y)$ – with a similar statement at other points in the diamond.

Lemma 3.10. *Let $y \in \Delta_2^1(x)$ and let $\{b, b'\} \subseteq \{x, y\}^\perp$ with $b \neq b'$ (so we are considering part of a diamond). If $\Gamma_0(x + b) = \{x, b, c\}$ and $\Gamma_0(y + b') = \{y, b', c'\}$, then $d(c, c') = 1$.*

Proof. Since we may choose $X \in \Gamma_3(x, b')$ with $X \notin x + b$ and $X \notin b' + y$, $b^{\tau(X)} = c$ and $y^{\tau(X)} = c'$ by Lemma 3.2. This proves the lemma as $d(b, y) = 1$.

In the last result of this section which follows directly from Lemma 3.10 and the discussion preceding it, we assume the notation given in the diamond above.

Lemma 3.11. (i) *Assume that* $\Gamma_0(x+b_1) = \{x, b_1, c\}$ *and, for* $i = 1, \ldots, 5$, $\Gamma_0(y+b_i) = \{y, b_i, c_i\}$. *Then* $\{c, y\}^{\perp} = \{b_1, c_2, c_3, c_4, c_5\}$.

(ii) *If* $X \in \Gamma_3(x)$, *then* $X \in \Gamma_3(x+b_i)$ *for some* $i \in \{1, \ldots, 5\}$.

Finally, in the situation of Lemma 3.8 we denote the unique hyperplane in $\Gamma_3(a, x)$ by $X(a, x)$, (respectively $X(x, a)$) if it is viewed as an element of Ω_a (respectively Ω_x).

4. Preliminaries on the Third Disc

In this section, in addition to establishing Theorems 3 and 4, we find the G_a-orbits of $\Delta_3(a)$, their sizes and determine the structure of G_{ax} for x in each of these orbits. This information, the salient points being summarized in Theorem 4.13, together with the data on line orbits in Section 2, serves as a launch pad for our investigations in [6].

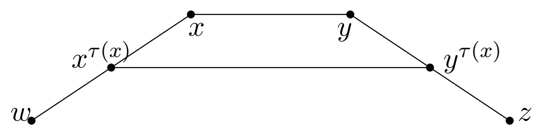
For $x \in \Delta_3^2(a) \cup \Delta_3^3(a) \cup \Delta_3^4(a)$ from the definitions (plus the fact that Γ is a string geometry), $X(a, c) \in \Gamma_3(a, x)$, where $c \in \Delta_2^2(a) \cap \Delta_1(x)$ is such that $c + x \in \alpha_{i,1}(c, c+b, X(c, a))$ ($i = 1$ or 3) and $\{b\} = \{a, c\}^{\perp}$. While for $x \in \Delta_3^1(a)$, two of the hyperplanes of $T(a, c)$ are in $\Gamma_3(a, x)$ (where $c \in \Delta_2^1(a) \cap \Delta_1(x)$ and $c + x \in \alpha_2(c, T(c, a))$). So it is not surprising that properties of these sets are very much tied up with the point-line collinearity graph of the Fi_{22}-geometry Γ_X ($X \in \Gamma_3$). Suppose $X \in \Gamma_3$ and $x, y \in \Gamma_0(X)$. If, say, x and y are distance 3 apart in the point-line collinearity graph of Γ_X, then, by Lemma 3.6, we also have $d(x, y) = 3$ (distance in $\mathcal{G}$).

Diamonds make their debut in our next argument.

Lemma 4.1. *Let* $X \in \Gamma_3$ *and let* $w, z \in \Gamma_0(X)$ *with* $w \neq z$. *If* $\{w, x, y, z\}$ *is a path in* $\mathcal{G}$ *of length 3, then either* $x, y \in \Gamma_0(X)$ *or* $d(w, z) = 1$.

Proof. Suppose the result is false. If either $x \in \Gamma_0(X)$ or $y \in$

$\Gamma_0(X)$, then Lemma 3.6 would force $x, y \in \Gamma_0(X)$. Thus we have $x, y \notin \Gamma_0(X)$. Hence $x^{\tau(X)} \neq x$ and $y^{\tau(X)} \neq y$ by Lemma 3.2. Also $x^{\tau(X)} \in \Gamma_0(w+x)$ and $y^{\tau(X)} \in \Gamma_0(z+y)$. So we have



If $d(x^{\tau(X)}, y) = 1$, then Lemmas 3.3 and 3.4 force $d(w, y) = 1$, whence $y \in \Gamma_0(X)$ by Lemma 3.6. So $d(x^{\tau(X)}, y) = 2$. If $x = y^{\tau(X)}$, then using Lemma 3.6 again gives $x \in \Gamma_0(X)$. Hence $x \neq y^{\tau(X)}$ and therefore $y \in \Delta_2^1(x^{\tau(X)})$ by Lemmas 3.8(i) and 3.9. (So we see part of a diamond.) Now using Lemma 3.10 yields $d(w, z) = 1$, contrary to our supposition. This completes the proof of the lemma.

Lemma 4.2. $\bigcup_{i=1}^{6} \Delta_3^i(a) \subseteq \Delta_3(a)$.

Proof. For $x \in \Delta_3^i(a)$ $(i = 1, 2, 3, 4)$ we have $a, x \in \Gamma_0(X)$ for some $X \in \Gamma_3$ with a and x distance 3 apart in the point-line collinearity graph of Γ_X. Therefore $x \in \Delta_3(a)$.

Suppose that $x \in \Delta_3^5(a)$. Then by (2.15) there exists $c \in \Delta_2^2(a)$ such that $c + x \in \alpha_{1,0}(c, c+b, X(c, a))$ (where $\{b\} = \{a, c\}^\perp$). If $d(a, x) = 1$, then $x \in \{a, c\}^\perp$ and so $x = b$, whereas $c + x \neq c + b$. So $d(a, x) \geq 2$. If $d(a, x) = 2$, then, by Lemmas 3.7(i) and 3.8(i), $a, x \in \Gamma_0(X)$ for some $X \in \Gamma_3$. Since $\{a, b, c, x\}$ is a path of length 3 in $\mathcal{G}$, Lemma 4.1 forces $b, c \in \Gamma_0(X)$. But then $X = X(a, c)$ and $X(c, a) \in c + x$, contradicting $c + x \in \alpha_{1,0}(c, c + b, X(c, a))$. Consequently $x \in \Delta_3(a)$. A similar argument yields that $\Delta_3^6(a) \subseteq \Delta_3(a)$, so proving the lemma.

Lemma 4.3. (i) *Let $x \in \bigcup_{i=1}^{4} \Delta_3^i(a)$, and suppose $X \in \Gamma_3(a, x)$. Then the vertices of every length 3 path from a to x are in $\Gamma_0(X)$.*

(ii) *Let $y \in \Delta_3^1(a)$ and $x \in \bigcup_{i=2}^{4} \Delta_3^i(a)$. Then for every $c \in \Delta_2^2(a) \cap \Delta_1(x)$, $\{X(a, c)\} = \Gamma_3(a, x)$ and for every $d \in \Delta_2^1(a) \cap \Delta_1(y)$,*

$T(a,d) \cap (d+y) = \Gamma_3(a,y)$. In particular, $|\Gamma_3(a,x)| = 1$ and $|\Gamma_3(a,y)| = 2$.

(iii) *For $i = 1,2,3,4$ and $X \in \Gamma_3(a)$, $\Gamma_0(X) \cap \Delta_3^i(a)$ is equal to the correspondingly named set in [5; Appendix 1].*

Proof. Since $d(a,x) = 3$, (i) follows immediately from Lemma 4.1. Note that one consequence of (i), using [5], is that for $y \in \Delta_3^1(a)$ and $d \in \Delta_2^1(a) \cap \Delta_1(y)$, we must have $d + y \in \alpha_2(d, T(d,a))$. Now, parts (ii) and (iii) are consequences of part (i).

Notation. For $x \in \bigcup_{i=2}^4 \Delta_3^i(a)$ we use $X(a,x)$ (or $X(x,a)$) to denote the unique hyperplane in $\Gamma_3(a,x)$ – by Lemma 4.3(ii) $X(a,x) = X(a,c)$ for any $c \in \Delta_2^1(a) \cap \Delta_1(x)$. While, for $y \in \Delta_3^1(a)$ we use $D(a,y)$ (or $D(y,a)$) to denote $\Gamma_3(a,y)$. Again we are using our "positional convention" – note that $D(a,y)$ (respectively $D(y,a)$) is a duad of Ω_a (respectively Ω_y).

Lemma 4.4. *For $i = 1,2,3,4,5,6$, $\Delta_3^i(a)$ is a G_a-orbit.*

Proof. By Lemma 4.3(iii) and [5] $\Gamma_0(X) \cap \Delta_3^i(a)$ ($i = 1,2,3,4$, and $X \in \Gamma_3(a)$) are G_{aX}-orbits. So, since G_a is transitive on $\Gamma_3(a)$, the lemma holds for $i = 1,2,3,4$. For x in $\Delta_3^5(a)$ or $\Delta_3^6(a)$ we have $c \in \Delta_2^2(a) \cap \Delta_1(x)$ for which $X(c,a) \notin c + x$. Hence $\Gamma_0(c+x)\backslash\{c\}$ is contained in a G_a-orbit by Lemma 3.2. Because $\Delta_2^2(a)$ is a G_a-orbit with $G_{ac}^{*c} \sim 2^4 A_6$, appealing to (2.3) we deduce that $\Delta_3^5(a)$ and $\Delta_3^6(a)$ are G_a-orbits too.

Lemma 4.5. *Let $x \in \Delta_3^6(a)$ and let $c_1 \in \Delta_2^2(a) \cap \Delta_1(x)$ be such that $c_1 + x \in \alpha_{3,0}(c_1, c_1 + b, X(c_1,a))$, where $\{b\} = \{a, c_1\}^\perp$. Then*

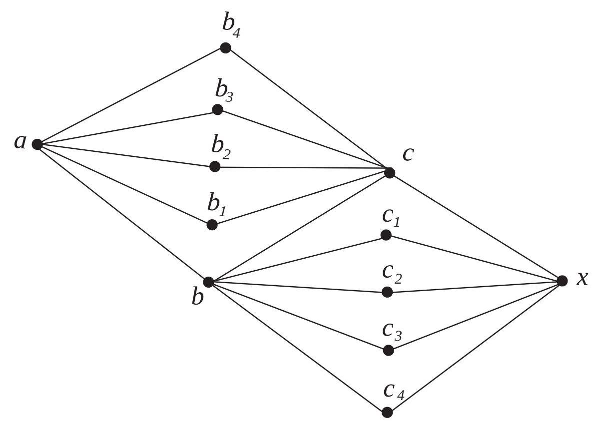
(i) *$|\{b,x\}^\perp \cap \Delta_2^2(a)| = 4$ and $|\{b,x\}^\perp \cap \Delta_2^1(a)| = 1$;*

(ii) *suppose $\{c_1,c_2,c_3,c_4\} = \{b,x\}^\perp \cap \Delta_2^2(a)$ and $\{c\} = \{b,x\}^\perp \cap \Delta_2^1(a)$. Then $X(a,c_i) \neq X(a,c_j)$ for $1 \leq i < j \leq 4$, and $X(a,c_i) \notin$*

$T(a,c)$ for $i = 1,2,3,4$. Further, $b + a = \{X(a,c_i)|i = 1,2,3,4\} \cup T(a,c)$; and

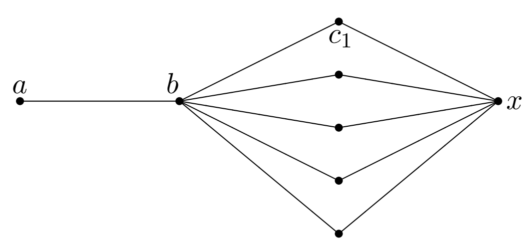
(iii) *set $\{a,c\}^{\perp} = \{b, b_1, b_2, b_3, b_4\}$.*

Then we have



with $c \in \Delta_2^1(a)$, $c_i \in \Delta_2^2(a)$, $b \in \Delta_2^1(x)$ and $b_i \in \Delta_2^2(x)$ $(i = 1,2,3,4)$.

Proof. First we observe that, because $c_1+x \in \alpha_{3,0}(c_1, c_1+b, X(c_1,a))$ and $d(a,x) = 3$, $\Gamma_3(a,x) = \emptyset$ by Lemma 4.1. Now $c_1 + x \in \alpha_3(c_1, c_1+b)$ implies that $|\{b,x\}^{\perp}| = 5$. Thus we have



Since $(b+a)\cap T(b,x) \subseteq \Gamma_3(a,x)$, $(b+a)\cap T(b,x) = \emptyset$ (in Ω_b). Therefore, as $\{b+y|y \in \{b,x\}^{\perp}\}$ intersects in $T(b,x)$ and their union is the whole of Ω_b, $b+a$ must intersect one of these heptads in 3 elements and four of these heptads in one element. This yields part (i).

If $X(a,c_i) = X(a,c_j)$ for $i \neq j$, then calling upon Lemma 3.6 gives $x \in \Gamma_0(X(a,c_i)) = \Gamma_0(X(a,c_j))$, contrary to $\Gamma_3(a,x) = \emptyset$. Similar considerations yield that $X(a,c_i) \notin T(a,c)$, and so (ii) holds. Noting that $a \in \Delta_3^6(x)$ we readily obtain (iii).

Our next result shows that looking out from a $\Delta_2^1(a)$ point, say c, "along" lines from $\alpha_1(c,T(c,a))$ or $\alpha_0(c,T(c,a))$ yields G_a-orbits already known to us.

Lemma 4.6. (i) $\Delta_3^2(a) = \{x \in \Gamma_0 |$ *there exists* $c \in \Delta_2^1(a)$ *such that* $c + x \in \alpha_1(c,T(c,a))\}$.

(ii) $\Delta_3^6(a) = \{x \in \Gamma_0 |$ *there exists* $c \in \Delta_2^1(a)$ *such that* $c + x \in \alpha_0(c,T(c,a))\}$.

Proof. Part (i) follows from Lemma 4.3(iii) and [5; Appendix 1]. Turning to part (ii), we claim that $T := \{x \in \Gamma_0 |$ there exists $c \in \Delta_2^1(a)$ such that $c + x \in \alpha_0(c,T(c,a))\}$ is a G_a-orbit. By Lemma 3.7(ii), for $c \in \Delta_2^1(a)$, $G_{ac}^{*c} \sim 2^4 S_5 3$. Now using the fact that $\Delta_2^1(a)$ is a G_a-orbit, Lemma 3.2 and (2.2) gives the claim. Let $x \in \Delta_3^6(a)$. From Lemma 4.5 there exists $c \in \Delta_2^1(a) \cap \Delta_1(x)$. Because $\Gamma_3(a,x) = \emptyset$ we must have $(c+x) \cap T(c,a) = \emptyset$ and hence $x \in T$. Since $\Delta_3^6(a)$ is also a G_a-orbit we infer that $T = \Delta_3^6(a)$, and (ii) holds.

We pause to remark that Lemma 4.6, together with (2.2), (2.3) and Lemmas 3.7 and 3.8, establish Theorems 3 and 4.

Lemma 4.7. $\bigcup_{i=1}^{6} \Delta_3^i(a) = \Delta_3(a)$.

Proof. Combining Lemmas 3.7(ii), 3.8(ii), 4.2, 4.6 together with (2.2), (2.3) and (2.15) yields the lemma.

Lemma 4.8. (i) $|\Delta_3^1(a)| = 2^9.11.23$ *and, for* $x \in \Delta_3^1(a)$, $G_{ax} \sim 2^2 L_3(4)2$ *with* $G_{ax}^{*x} \sim L_3(4)2$.

(ii) $\Delta_3^2(a)| = 2^8.3.5.11.23$ *and, for* $x \in \Delta_3^2(a)$, $G_{ax} \sim 2^7L_3(2)$ *with* $G_{ax}^{*x} \sim 2^3L_3(2)$. *Moreover,* $Q(a) \cap Q(x) =< \tau(X(a,x)) >$ *with* $Q(a)^{*x} \sim 2^3 \sim Q(x)^{*a}$.

(iii) $|\Delta_3^3(a)| = 2^{10}.7.11.23$ *and, for* $x \in \Delta_3^3(a)$, $G_{ax} \sim 2^5A_6$ *with* $G_{ax}^{*x} \sim 2^4A_6$.

(iv) $|\Delta_3^4(a)| = 2^{10}.23$ *and, for* $x \in \Delta_3^3(a)$, $G_{ax} \sim 2M_{22}$ *with* $G_{ax}^{*x} \cong M_{22}$.

Proof. For parts (ii)-(iv), putting $X = X(a,c)$, we have $G_{ax} = G_{axX}$ and, using Lemma 4.3(ii), $|\Delta_3^i(a)| = 23|\Delta_3^i(a) \cap \Gamma_0(X)|$ $(i = 2,3,4)$. Consulting [5; Appendix 1 and Lemma 6.17(iii)] then gives parts (ii)-(iv).

Let $x \in \Delta_3^1(a)$. Again employing Lemma 4.3(ii), (iii) and [5; Appendix 1], we have that

$$|\Delta_3^1(a)| = \frac{23|\Delta_3^1(a) \cap \Gamma_0(X)|}{2} = 2^9.11.23$$

(where $X \in D(a,x) = \Gamma_3(a,x)$). Moreover, by Lemma 4.3 (ii), for $X \in \Gamma_3(a,x)$, $[G_{ax} : G_{ax} \cap G_X] \leq 2$. Using $|\Delta_3^1(a)|$, $|G_{ax} \cap G_X|$ and the fact that $\Delta_3^1(a)$ is a G_a-orbit we find that $[G_{ax} : G_{ax} \cap G_X] = 2$, whence by [5; Appendix 1] we obtain part (i). The proof of the lemma is complete.

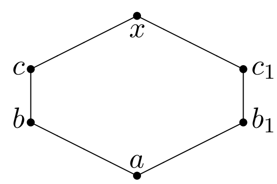
In Lemma 4.8 we have amassed a good deal of information about $\Delta_3^i(a)$ for $i = 1,2,3,4$. The next three results answer the corresponding questions for $\Delta_3^5(a)$ and $\Delta_3^6(a)$.

Lemma 4.9. *Let* $x \in \Delta_3^5(a)$ *and let* $c \in \Delta_2^2(a) \cap \Delta_1(x)$ *be such that* $c + x \in \alpha_{1,0}(c, c+b, X(c,a))$, *where* $\{b\} = \{a,c\}^\perp$. *Then*

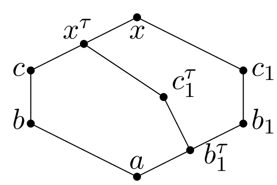
(i) $\Delta_2^2(a) \cap \Delta_1(x) = \{c\}$; *and*

(ii) $a \in \Delta_3^5(x)$.

Proof. We first prove part (i). Assume there exists $c_1 \in \Delta_2^2(a) \cap \Delta_1(x)$ with $c_1 \neq c$. Let $\{a, c_1\}^\perp = \{b_1\}$, and put $X = X(a, c)$ and $Y = X(a, c_1)$. Note that $b = b_1$ would imply, as $c \neq c_1$, that $x \in \Delta_2^1(b)$ and hence $c + x \in \alpha_3(c, c + b)$, which is not the case. So we have



Suppose that $b_1 \in \Gamma_0(X)$. By Lemma 4.1 we then get that either $x, c_1 \in \Gamma_0(X)$ or $d(c, b_1) = 1$. The former possibility cannot occur as $\Gamma_3(a, x) = \emptyset$, while $d(c, b_1) = 1$ implies (as $b \neq b_1$) that $c \in \Delta_2^1(a)$, whereas $c \in \Delta_2^2(a)$. Thus we conclude that $b_1 \notin \Gamma_0(X)$. So, by Lemma 3.2, $x^{\tau(X)} \neq x$ and $b_1^{\tau(X)} \neq b_1$ and hence we have (setting $\tau = \tau(X)$)



Since $\tau \in Q(a)$, c_1^τ, $b_1^\tau \in \Gamma_0(Y)$. Now, because $c_1 \neq c_1^\tau$ (else by Lemmas 3.3 and 3.4 $c_1 = a$), we may deploy Lemma 4.1 again to obtain either x^τ, $x \in \Gamma_0(Y)$ or $d(c_1, c_1^\tau) = 1$. Hence, as $\Gamma_3(a, x) = \emptyset$, $d(c_1, c_1^\tau) = 1$. However, as $c_1^\tau \neq b_1$ (because $c_1^\tau \in \Delta_2(a)$ and $b_1 \in \Delta_1(a)$), $c_1 \in \Delta_2^1(b_1^\tau)$, which gives $c_1 \in \Delta_2^1(a)$. From this contradiction we infer that $\Delta_2^2(a) \cap \Delta_1(x) = \{c\}$.

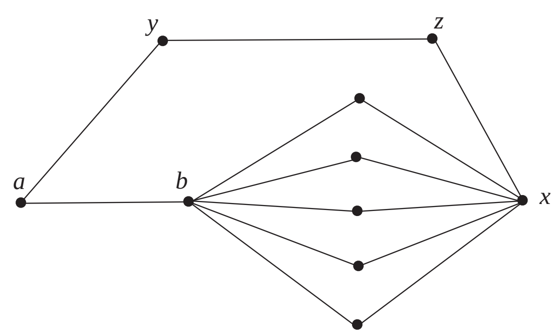
Note that part (i) and Lemma 4.5(i) together show that $\Delta_3^5(a) \neq \Delta_3^6(a)$.

Moving on to part (ii), part (i), Theorem 3 and Lemma 3.7(i) imply that $b \in \Delta_2^2(x)$. From the definition of $\Delta_3^5(a)$, $\Gamma_3(a, x) = \emptyset$ and so

$a \in \Delta_3^5(x) \cup \Delta_3^6(x)$ by Lemma 4.7. Since $b + c \in \alpha_1(b, b + a)$, the definition of $\Delta_3^5(x)$ gives $a \in \Delta_3^5(x)$.

Lemma 4.10. *For $x \in \Delta_3^6(a)$, $|\Delta_2^1(a) \cap \Delta_1(x)| = 1$ and $|\Delta_2^2(a) \cap \Delta_1(x)| = 4$. In particular, the configuration in Lemma 4.5(iii) shows $\Delta_1(x) \cap \Delta_2(a)$ and $\Delta_1(a) \cap \Delta_2(x)$.*

Proof. Suppose the result is false. Thus, appealing to Lemma 4.5(i), we have



where $b \in \Delta_2^1(x) \cap \Delta_1(a)$, $z \in (\Delta_2(a) \cap \Delta_1(x)) \backslash \{b, x\}^\perp$ and $y \in \{a, z\}^\perp$. Evidently $y \neq b$. By Lemma 4.5(ii) there exists $c \in \{b, x\}^\perp$ such that $\Gamma_3(a, c, y) \neq \emptyset$ and hence, using Lemma 4.1 and $\Gamma_3(a, x) = \emptyset$, we deduce that $d(y, c) = 1$. So $y \in \{a, c\}^\perp$ and therefore, as $y \neq b$, $c \in \Delta_2^1(a)$. But, by Lemma 4.5(iii), $y \in \Delta_2^2(x)$ which forces $z = c$, a contradiction.

Lemma 4.11. (i) $|\Delta_3^5(a)| = 2^{12}.3.7.11.23$ *and, for $x \in \Delta_3^5(a)$, $2^4 A_5 \sim G_{ax} \leq G_{acb}$ where $\{c\} = \Delta_2^2(a) \cap \Delta_1(x)$ and $\{b\} = \{a, c\}^\perp$. Furthermore $G_{ax}^{*x} \sim 2^4 A_5$ with the A_5 having orbits of length 1 and 6 upon $\Gamma_3(x + c)$, the orbit of length 1 being $\{X(x, b)\}$.*

(ii) $|\Delta_3^6(a)| = 2^9.5.7.11.23$ *and, for $x \in \Delta_3^6(a)$, $[2^9]3^2 \sim G_{ax} \leq G_{acx}$ where $\{c\} = \Delta_2^1(a) \cap \Delta_1(x)$.*

Proof. (i) Combining (2.3), Lemmas 3.8(iii) and 4.9(i) with the definition of $\Delta_3^5(a)$ ((2.15)) gives

$$|\Delta_3^5(a)| = 2^6.7.11.23.2.96 = 2^{12}.3.7.11.23.$$

Therefore, by Lemma 4.4, $|G_{ax}| = 2^7.3.5$. Since $\{a, b, c, x\}$ is the unique length 3 path from x to a, clearly $G_{ax} \leq G_{acb}$. Now, from (2.3), $G^{*c}_{cc+x} \cong A_5$ and so we see that $G_{ax} \sim 2^4 A_5$ with $2^4 \cong O_2(G_{ax}) = G_{ax} \cap Q(c)$. Also from (2.3) we have that G^{*c}_{cc+x} has orbits of length 1 and 6 upon the elements of the heptad $c+x$ (note that $(c+b) \cap (c+x)$ is the orbit of length 1). Since elements of Ω_c correspond to hyperplanes of Γ, this information translates to Γ_x, so it remains to show that $G^{*x}_{ax} \sim 2^4 A_5$. If this is not the case, then, as $O_2(G_{ax})$ is a G_{ax}-chief factor, $O_2(G_{ax}) \leq Q(x)$. Because $a \in \Delta_3^5(x)$ by Lemma 4.9(ii) we also get $O_2(G_{ax}) \leq Q(a)$. But then $2^4 \cong O_2(G_{ax}) \leq Q(a) \cap Q(c)$, which is untenable by [5; Lemma 5.10].

(ii) Putting together (2.3) and Lemmas 3.8(iii) and 4.10 yields

$$|\Delta_3^6(a)| = \frac{2^6.7.11.23.2.80}{4} = 2^9.5.7.11.23.$$

Hence $|G_{ax}| = 2^9 3^2$ by Lemma 4.4,so proving (ii).

From our knowledge of the sizes of $\Delta_3^i(a)$ ($i \in \{1, \ldots, 6\}$) in Lemmas 4.8 and 4.11, together with Lemma 4.7, we deduce the following useful fact.

Lemma 4.12. *For $i \in \{1, \ldots, 6\}$, if $x \in \Delta_3^i(a)$, then $a \in \Delta_3^i(x)$.*

Theorem 17. (i) *Let $x \in \Delta_2^1(a)$. Then G^{*x}_{xa} ($\sim 2^4 S_5 3$) is the stabilizer in G^{*x}_x of the triad $T(x, a)$ in Ω and for all five points $b \in \{a, x\}^\perp$, $x + b$ contains $T(x, a)$.*

(ii) *Let* $x \in \Delta_2^2(a)$*. Then* G_{xa}^{*x} ($\sim 2^4 A_6$) *is the stabilizer in* G_x^{*x} *of the heptad* $x+b$ *and the element* $X(x,a)$ *of* Ω_x *where* $\{b\} = \{a,x\}^{\perp}$.

(iii) *Let* $x \in \Delta_3^1(a)$*. Then* G_{xa}^{*x} ($\sim L_3(4)2$) *is the stabilizer in* G_x^{*x} *of the duad* $D(x,a)$*. Furthermore, for all 21 points* $c \in \Delta_2^1(a) \cap \Delta_1(x)$*, the heptad* $x+c$ *contains* $D(x,a)$ *in* Ω_x *and* $D(a,x) \subseteq T(a,c)$ *in* Ω_a.

(iv) *Let* $x \in \Delta_3^2(a)$*. Then* G_{xa}^{*x} ($\sim 2^3 L_3(2)$) *is the stabilizer in* G_x^{*x} *of an octad* $O(x,a)$ *and the element* $X(x,a)$ *of* Ω_x *where* $O(x,a)$ *has the following properties:–*

- *(a)* $(x+c) \cap O(x,a) = \emptyset$ *and* $X(x,a) \in x+c$ *in* Ω_x *for each of the 7 points* $c \in \Delta_2^1(a) \cap \Delta_1(x)$;
- *(b)* $|(x+c) \cap O(x,a)| = 4$ *and* $X(x,a) \in x+c$ *in* Ω_x *for each of the 14 points* $c \in \Delta_2^2(a) \cap \Delta_1(x)$;
- *(c)* $X(x,a) \notin O(x,a)$ *in* Ω_x*; and*
- *(d)* $T(x,b) \cap O(x,a) = \emptyset$ *in* Ω_x *for each* $b \in \Delta_1(a) \cap \Delta_2^1(x)$.

item Let $x \in \Delta_3^3(a)$*. Then* G_{xa}^{*x} ($\sim 2^4 A_6$) *is the stabilizer in* G_x^{*x} *of the heptad* $x+c$ *and the element* $X(x,a)$ *of* Ω_x *where* $\{c\} = \Delta_2^2(a) \cap \Delta_1(x)$*. Furthermore,* $\Gamma_0(x+c)\backslash\{x,c\} \subseteq \Delta_3^4(a)$.

(v) *Let* $x \in \Delta_3^4(a)$*. Then* G_{xa}^{*x} ($\cong M_{22}$) *is the stabilizer in* G_x^{*x} *of the element* $X(x,a)$ *of* Ω_x *and* $x+c$ *contains* $X(x,a)$ *for all the 77 points* $c \in \Delta_2^2(a) \cap \Delta_1(x)$.

(vi) *Let* $x \in \Delta_3^5(a)$*. Then* G_{xa}^{*x} ($\sim 2^4 A_5$) *stabilizes the heptad* $x+c$ *and the element* $X(x,b)$ *of* $x+c$ *in* Ω_x *where* $\{c\} = \Delta_2^2(a) \cap \Delta_1(x)$ *and* $\{b\} = \Delta_1(a) \cap \Delta_2^2(x)$.

(vii) *Let* $x \in \Delta_3^6(a)$*. Then* $G_{xa} \sim [2^9]3^2$ *stabilizes the triad* $T(x,b)$ *and the heptad* $x+c$ *which contains* $T(x,b)$*, where* $\{b\} = \Delta_1(a) \cap \Delta_2^1(x)$ *and* $\{c\} = \Delta_2^1(a) \cap \Delta_1(x)$.

Proof. For parts (i) and (ii) see Lemmas 3.7(i),(ii) and 3.8(i),(ii) respectively. Parts (iii)-(vi) follow from Lemma 4.8 and [5; Lemmas 6.7, 6.16, 7.8 and Proposition 7.13]. Finally parts (vii) and (viii) can be deduced from Lemmas 4.5, 4.9 and 4.11(i),(ii).

Lemma 4.13. *Let $x \in \Delta_3(a)$, with $c \in \Delta_2^1(a) \cap \Delta_1(x)$ and $b \in \{a, c\}^{\perp} \cap \Delta_2^1(x)$. Then,*

(i) *$x \in \Delta_3^1(a)$ if and only if $|T(b, x) \cap T(c, a)| = 2$ in Ω_c; and*

(ii) *$x \in \Delta_3^2(a)$ if and only if $|T(b, x) \cap T(c, a)| = 1$ in Ω_c; and*

(iii) *$x \in \Delta_3^6(a)$ if and only if $|T(b, x) \cap T(c, a)| = 0$ in Ω_c.*

Proof. If $x \in \Delta_3^6(a)$, then $\Gamma_3(a, x) = \emptyset$ implies that $T(b, x) \cap T(c, a) = \emptyset$ in Ω_c. Conversely, if $T(b, x) \cap T(c, a) = \emptyset$, then $\Gamma_3(a, x) = \emptyset$ and hence $x \in \Delta_3^6(a)$, which proves (iii).

Since $\Delta_2^2(a) \cap \Delta_1(x) = \emptyset$ for all $x \in \Delta_3^1(a)$, we must have $x \in \Delta_3^1(a)$ if and only if $|T(b, x) \cap T(c, a)| = 2$ in Ω_c. So (i) is proved. Part (ii) now follows using parts (i) and (iii) and (2.15).

Between them, the last two results of this section settle the question of adjacency within $\Delta_3(a)$, with the exception of edges between two points which are either both in $\Delta_3^5(a)$ or both in $\Delta_3^6(a)$.

Lemma 4.14. *Let $1 \leq i < j \leq 6$ and suppose that $x \in \Delta_3^i(a)$ and $y \in \Delta_3^j(a)$ with $d(x, y) = 1$. Then $x, y \in \Delta_3^1(a) \cup \Delta_3^2(a) \cup \Delta_3^3(a) \cup \Delta_3^4(a)$. Further, $X(a, y) \in D(a, x)$ (when $i = 1$) and $X(a, x) = X(a, y)$ (when $i \neq 1$), and exactly one of the following three possibilities hold.*

(i) *$i = 1$, $j = 3$ with $x + y \in \alpha_1(x, D(x, a))$ and $y + x \in \alpha_{1,1}(y, y + b, X(y, a))$ ($\{b\} = \Delta_1(y) \cap \Delta_2^2(a)$). Furthermore $\Gamma_0(y + x) \backslash \{x, y\} \subseteq \Delta_3^3(a)$.*

(ii) *$i = 2, j = 3$ with $x + y \in \alpha_{2,1}(x, O(x, a), X(x, a))$ and $y + x \in \alpha_{3,1}(y, y + b, X(y, a))$ ($\{b\} = \Delta_1(y) \cap \Delta_2^2(a)$). Furthermore $\Gamma_0(y + x) \backslash \{x, y\} \subseteq \Delta_3^3(a)$.*

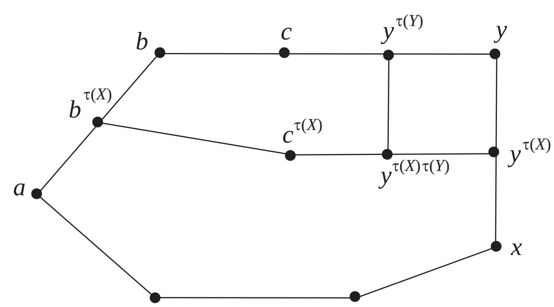
(iii) $i = 3, j = 4$ *with* $x + y = x + b$ *(where* $\{b\} = \Delta_1(x) \cap \Delta_2^2(a)$*) and* $y + x \in \alpha_1(y, X(y, a))$.

Proof. We begin by establishing that

(4.14.1) $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6\}$ cannot hold.

Suppose (4.15.1) is false. Then we have $x \in \bigcup_{l=1}^{4} \Delta_3^l(a)$ and $y \in \Delta_3^5(a) \cup \Delta_3^6(a)$ with $d(x, y) = 1$. By (2.15)(vii), (viii) we may choose $c \in \Delta_1(y) \cap \Delta_2^2(a)$. Let $\{b\} = \{a, c\}^\perp$. From Lemma 3.8(i) we have $\Gamma_3(a, c) = \{X(a, c)\}$ and, by Lemma 4.3(ii), $\Gamma_3(a, x) \neq \emptyset$. Let $X \in \Gamma_3(a, x)$, and set $X(a, c) = Y$.

Since $\Gamma_3(a, y) = \emptyset$, $y \notin \Gamma_0(X)$ and consequently $y^{\tau(X)} \neq y$. Also we have $b \notin \Gamma_0(X)$, for otherwise an appeal to Lemma 4.1 gives either $y \in \Gamma_0(X)$ or $d(b, x) = 1$, which cannot hold as $d(a, x) = 3$. Thus $b^{\tau(X)} \neq b$. Note that, as $\tau(X) \in Q(a)$, $c^{\tau(X)} \in \Gamma_0(Y)$. Because $y \notin \Gamma_0(Y)$ we have that $y^{\tau(X)} \notin \Gamma_0(Y)$. Therefore $y^{\tau(X)} \neq y$ and $y^{\tau(X)\tau(Y)} \neq y$. So the state of play is as follows.



Observe that $y + c \neq y + x \neq y^{\tau(X)} + c^{\tau(X)}$ (as $\Gamma_0(y + c) \cap \Delta_2(a) \neq \emptyset \neq \Gamma_0(y^{\tau(X)} + c^{\tau(X)}) \cap \Delta_2(a)$). Also, $d(y^{\tau(X)}, y^{\tau(Y)}) = 1$ yields, by Lemma 3.4, the impossible $c \in \Gamma_0(y+x)$. Hence we deduce that $y^{\tau(Y)} \in \Delta_2^1(y^{\tau(X)})$ with $y, y^{\tau(X)\tau(Y)} \in \{y^{\tau(Y)}, y^{\tau(X)}\}^\perp$ and $y \neq y^{\tau(X)\tau(Y)}$. Calling upon Lemma 3.10 gives $d(c, c^{\tau(X)}) = 1$. This in turn implies that $c \in \Delta_2^1(b^{\tau(X)})$. But then $b + c \in \alpha_3(b, b + a)$ which contradicts Lemma 3.8(i) and the fact that $c \in \Delta_2^2(a)$. This completes the proof of (4.15.1).

(4.14.2) $i = 5$ and $j = 6$ cannot hold.

Again we suppose this statement is false. So we have $x \in \Delta_3^5(a)$, $y \in \Delta_3^6(a)$ with $d(x, y) = 1$. Putting $m = |\Delta_1(x) \cap \Delta_3^6(a)|$ and $n = |\Delta_1(y) \cap \Delta_3^5(a)|$ we obtain $m|\Delta_3^5(a)| = n|\Delta_3^6(a)|$. Hence $24m = 5n$, using Lemma 4.11. Therefore $5|m$. Now $\Delta_1(x) \cap \Delta_3^6(a)$ must be a union of G_{ax}-orbits, and so, by (2.7) and Lemma 4.11(i), we deduce that $m \geq 40$. Hence $24.40 \leq 5n$, which gives $n \geq 192$. Let $b \in \Delta_2^1(y) \cap \Delta_1(a)$ – by Lemma 4.5(iii) such a point b exists. From Lemma 4.12 $a \in \Delta_3^5(x)$. In view of (4.15.1) (with x playing the role of a, a the role of y and b the role of x) $b \notin \Delta_3^1(x) \cup \Delta_3^2(x)$. Using Lemma 4.12 again gives $x \notin \Delta_3^1(b) \cup \Delta_3^2(b)$. Hence $(y + x) \cap T(y, b) = \emptyset$ and therefore $y + x \in \alpha_0(y, T(y, b))$. This shows that $n \leq 80.2 = 160$, contrary to the earlier prediction of $n \geq 192$. Thus we have verified (4.15.2).

By (4.15.1) and (4.15.2) we have that $x, y \in \bigcup_{l=1}^4 \Delta_3^l(a)$. Now we prove that

(4.14.3) $X(a, y) \in D(a, x)$ (when $i = 1$) and $X(a, x) = X(a, y)$ (when $i \neq 1$).

If (4.15.3) is false, then we may find $X \in \Gamma_3(a, x)$ and $Y \in \Gamma_3(a, y)$ for which $y \notin \Gamma_0(X)$ and $x \notin \Gamma_0(Y)$. Then employing Lemma 3.2, $y^{\tau(X)} = x^{\tau(Y)}$, which, by Lemma 4.4, forces $\Delta_3^i(a) = \Delta_3^j(a)$. This, by Lemma 4.8, is clearly impossible.

From (4.15.3) we get that $x + y \in \Gamma_1(X)$ for some $X \in \Gamma_3(a, y, x)$ and then, consulting [5], we obtain one of the three listed possibilities, so proving the lemma.

Lemma 4.15. *Suppose $x, y \in \Delta_3^i(a)$ where $i \in \{1, 2, 3, 4\}$. If $d(x, y) = 1$, then $D(a, x) = D(a, y)$ (for $i = 1$) and $X(a, x) = X(a, y)$ (for $i \neq 1$).*

Proof. Suppose the lemma is false. Then we must have $X \in \Gamma_0(a, x)$ and $Y \in \Gamma_0(a, y)$ such that $X \notin D(a, y)$ and $Y \notin D(a, x)$

(if $i = 1$), $X \neq X(a, y)$ and $Y \neq X(a, x)$ (if $i \neq 1$). So $y \notin \Gamma_0(X)$ and $x \notin \Gamma_0(Y)$. Therefore, by Lemma 3.2, $y \neq y^{\tau(X)} \in \Gamma_0(x+y)$, in addition to $y^{\tau(X)} \in \Gamma_0(Y)$. Hence $x + y \in \Gamma_1(Y)$ and then $x \in \Gamma_0(Y)$, a contradiction.

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