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A NOTE ON THE COMPLEX ROOTS OF TWISTED q-EULER POLYNOMIALS

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Abstract

In this paper, observing an interesting phenomenon of 'scattering' of the zeros of twisted q-Euler polynomials $E_{n,q,w}(x)$, we investigate the complex roots of twisted q-Euler polynomials $E_{n,q,w}$.

1. Introduction

In this paper, we investigate the complex roots of twisted q-Euler polynomials $E_{n,q,w}$. The outline of this paper is as follows: In Section 2, we introduce twisted q-Euler polynomials $E_{n,q,w}(x)$. In Section 3, we display distribution and structure of the zeros of twisted q-Euler polynomials $E_{n,q,w}(x)$ by using computer. By using the results of our paper, the readers can observe the regular behavior of the roots of twisted q-Euler polynomials $E_{n,q,w}(x)$. Finally, we carry out computer experiments for demonstrating a remarkably regular structure of the complex roots of twisted q-Euler polynomials $E_{n,q,w}(x)$. Throughout this paper, we always make use of the following notations: $\mathbb{C}$ denotes the set of complex numbers and

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$$[x]_q = \frac{1-q^x}{1-q}.$$

First, we introduce the q-Euler numbers and Euler polynomials. The q-Euler numbers $E_{n,q}$ are defined by the generating function:

$$F(t) = \frac{[2]_q}{qe^t + 1} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \text{ cf. [2]}, \tag{1.1}$$

where we use the technique method notation by replacing $(E_q)^n$ by $E_{n,q} (n \geq 0)$ symbolically. We consider the Euler polynomials $E_n(x)$ as follows:

$$F(x, t) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{1.2}$$

2. Twisted q-Euler Numbers and Polynomials

Our primary aim in this section is to introduce the twisted q-Euler numbers $E_{n,q,w}$ and polynomials $E_{n,q,w}(x)$ and investigate their properties. Let q be a complex number with $|q| < 1$ and w be the p^N th root of unity. By the meaning of (1.1) and (1.2), let us define the twisted q-Euler numbers $E_{n,q,w}$ and polynomials $E_{n,q,w}(x)$ as follows:

$$F_{q,w}(t) = \frac{[2]_q}{wqe^t + 1} = \sum_{n=0}^{\infty} E_{n,q,w} \frac{t^n}{n!}, \tag{2.1}$$

$$F_{q,w}(x, t) = \frac{[2]_q}{wqe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}(x) \frac{t^n}{n!}. \tag{2.2}$$

The following elementary properties of twisted q-Euler numbers $E_{n,q,w}$ and polynomials $E_{n,q,w}(x)$ are readily derived from (2.1) and (2.2). We, therefore, choose to omit the details involved.

Proposition 1 (The several values)**.**

$$E_{0,q,w}(x) = \frac{[2]_q}{1+qw},$$

$$E_{1,q,w}(x) = \frac{[2]_q(-qw+x+qwx)}{(1+qw)^2},$$

$$E_{2,q,w}(x) = \frac{[2]_q(-qw+q^2w^2-2qwx-2q^2w^2x+x^2+2qwx^2+q^2w^2x^2)}{(1+qw)^3}.$$

Proposition 2. *For any positive integer n, we have*

$$E_{n,q,w}(x) = \sum_{k=0}^{n}\binom{n}{k}E_{k,q,w}x^{n-k}.$$

Proposition 3. *For* $n \geq 0$, *we have*

$$wq(E_{q,w}+1)^n + E_{n,q,w} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

with the usual convention about replacing $(E_{q,w})^n$ *by* $E_{n,q,w}$ *in the binomial expansion*.

Proposition 4 (Differential relation)**.**

$$\frac{\partial}{\partial x}E_{n,q,w}(x) = nE_{n-1,q,w}(x).$$

Proposition 5 (Integral formula)**.**

$$\int_a^b E_{n-1,q,w}(x)\,dx = \frac{1}{n}(E_{n,q,w}(b) - E_{n,q,w}(a)).$$

Theorem 6 (Addition theorem)**.**

$$E_{n,q,w}(x+y) = \sum_{k=0}^{n}\binom{n}{k}E_{k,q,w}(x)\,y^{n-k}.$$

Theorem 7 (Difference equation).

$$wqE_{n,q,w}(x+1) + E_{n,q,w}(x) = [2]_q x^n.$$

Theorem 8 (Theorem of complement).

$$E_{n,q,w}(1+x) = (-1)^n w^{-1} E_{n,q^{-1},w^{-1}}(-x),$$

$$E_{n,q,w}(1-x) = (-1)^n w^{-1} E_{n,q^{-1},w^{-1}}(x).$$

3. Distribution of Zeros of Twisted q-Euler Polynomials

This section aims to discover new interesting pattern of the zeros of twisted q-Euler polynomials $E_{n,q,w}(x)$ and to demonstrate the benefit of using numerical investigation to support theoretical prediction. First, we investigate the zeros of twisted q-Euler polynomials $E_{n,q,w}(x)$ by using computer. Let $w = e^{\frac{2\pi i}{N}}$ in $\mathbb{C}$. We plot the zeros of $E_{n,q,w}(x)$ for $N = 5, 7, 9, 11$ (Figure 1).

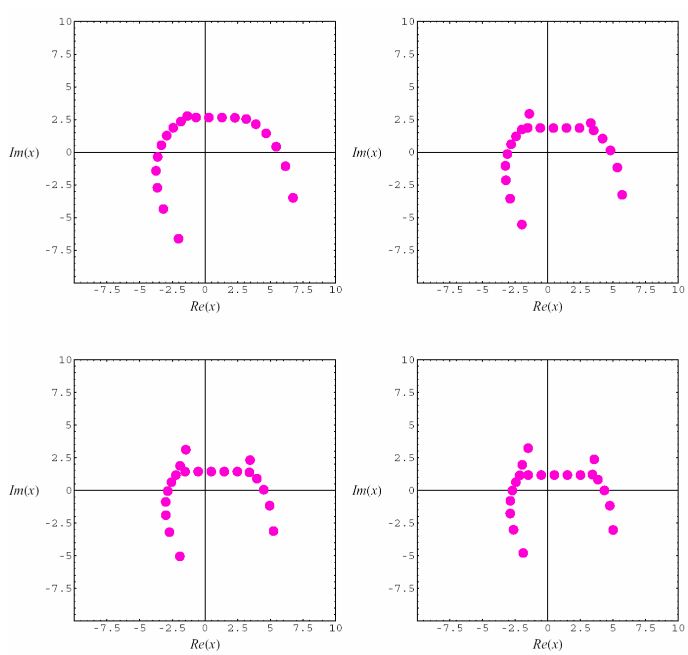

Figure 1. Zeros of $E_{n,q,w}(x)$.

In Figure 1(top-left), we choose $n = 20,\ q = 1/2$ and $w = e^{\frac{2\pi i}{5}}$. In Figure 1(top-right), we choose $n = 20,\ q = 1/2$ and $w = e^{\frac{2\pi i}{7}}$. In Figure 1(bottom-left), we choose $n = 20,\ q = 1/2$ and $w = e^{\frac{2\pi i}{9}}$. In Figure 1(bottom-right), we choose $n = 20,\ q = 1/2$ and $w = e^{\frac{2\pi i}{11}}$.

Stacks of zeros of $E_{n,q,w}(x)$ for $q = 1/2,\ 1 \le n \le 20$ from a 3-D structure are presented (Figure 2). In Figure 2, we choose $w = e^{\frac{2\pi i}{5}}$.

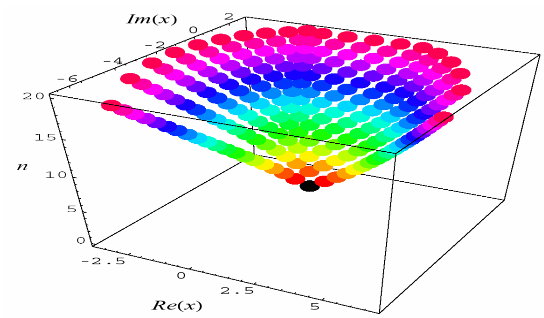

Figure 2. Stacks of zeros $E_{n,q,w}(x)$ for $1 \le n \le 20$.

Our numerical results for numbers of real and complex zeros of $E_{n,q,w}(x)$ are displayed in Table 1.

Table 1. Numbers of real and complex zeros of $E_{n,q,w}(x)$

	$w = e^{\pi i}$		$w = e^{\frac{2\pi i}{3}}$		$w = e^{\frac{2\pi i}{5}}$	
degree n	real zeros	complex zeros	real zeros	complex zeros	real zeros	complex zeros
1	1	0	0	1	0	1
2	0	2	0	2	0	2
3	1	2	0	3	0	3
4	0	4	0	4	0	4

5	1	4	0	5	0	5
6	0	6	0	6	0	6
7	1	6	0	7	0	7
8	0	8	0	8	0	8
9	1	8	0	9	0	9
10	0	10	0	10	0	10
11	1	10	0	11	0	11
12	0	12	0	12	0	12
13	1	12	0	13	0	13

In Table 1, we choose $q = 1/2$.

We calculated an approximate solution satisfying $E_{n,q,w}(x)$, $x \in \mathbb{C}$. The results are given in Table 2.

Table 2. Approximate solutions of $E_{n,q,w}(x) = 0$

degree n	x
1	-1.0000
2	$-1.0000 - 1.4142i$, $-1.0000 + 1.4142i$
3	-1.8846, $-0.5577 - 2.5665i$, $-0.5577 + 2.5665i$
4	$-2.076 - 1.256i$, $-2.076 + 1.256i$, $0.0756 - 3.5686i$, $0.0756 + 3.5686i$
5	-2.739, $-1.951 - 2.402i$, $-1.951 + 2.402i$, $0.820 - 4.468i$, $0.820 + 4.468i$
6	$-2.999 - 1.188i$, $-2.999 + 1.188i$, $-1.640 - 3.461i$, $-1.640 + 3.461i$, $1.640 - 5.290i$, $1.640 + 5.290i$

In Table 2, we choose $q = 1/2$ and $w = e^{\pi i}$.

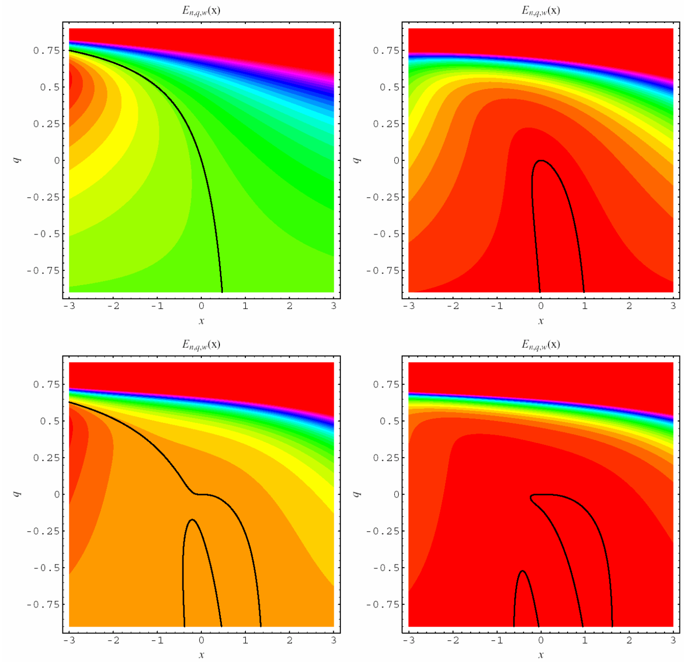

Figure 3. Zero contour of $E_{n,q,w}(x)$.

The plot above shows $E_{n,q,w}(x)$ for real $-9/10 \leq q \leq 9/10$ and $-3 \leq x \leq 3$, with the zero contour indicated in black (Figure 3). In Figure 3(top-left), we choose $n = 1$ and $w = e^{\pi i}$. In Figure 3(top-right), we choose $n = 2$ and $w = e^{\pi i}$. In Figure 3(bottom-left), we choose $n = 3$ and $w = e^{\pi i}$. In Figure 3(bottom-right), we choose $n = 4$ and $w = e^{\pi i}$.

We shall consider the more general open problem. How many roots do $E_{n,q,w}(x)$ have? Prove or disprove: $E_{n,q,w}(x)$ has n distinct solutions. Find the numbers of complex zeros $C_{E_{n,q,w}(x)}$ of $E_{n,q,w}(x)$, $\operatorname{Im}(x) \neq 0$. Prove or give a counterexample: Since n is the degree of the polynomial $E_{n,q,w}(x)$, the number of real zeros $R_{E_{n,q,w}(x)}$ lying on the real plane $\operatorname{Im}(x) = 0$ is then $R_{E_{n,q,w}(x)} = n - C_{E_{n,q,w}(x)}$, where $C_{E_{n,q,w}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,q,w}(x)}$ and $C_{E_{n,q,w}(x)}$. The theoretical prediction on the zeros of $E_{n,q,w}(x)$ is await for further study. For related topics the interested reader is referred to [1-5].

References

[1] T. Kim, On *p*-adic interpolating function for *q*-Euler numbers and its derivatives, J. Math. Anal. Appl. 339 (2008), 598-608.

[2] T. Kim, M.-S. Kim, L. C. Jang and S.-H. Rim, New *q*-Euler numbers and polynomials associated with *p*-adic *q*-integrals, Adv. Stud. Contemp. Math. 15 (2007), 243-252.

[3] C. S. Ryoo and T. Kim, A note on the *q*-extension of Bernoulli numbers and Bernoulli polynomials, Discrete Dynamics in Nature and Society 2010 (2010), Art. ID 807176, 11pp.

[4] C. S. Ryoo, Calculating zeros of the second kind Euler polynomials, J. Comput. Anal. Appl. 12 (2010), 828-833.

[5] C. S. Ryoo, On the generalized Barnes type multiple *q*-Euler polynomials twisted by ramified roots of unity, Proc. Jangjeon Math. Soc. 13 (2010), 255-263.