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A NOTE ON (AMPLY) δ_M -SUPPLEMENTED MODULES

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Abstract

Let M be a right module over a ring R. In the present paper, several properties of (amply) δ_M -supplemented modules are proved. It is shown, among others, that (1) The class of (amply) δ_M -supplemented modules is closed under taking homomorphic images; (2) Every π-projective module in $\sigma[M]$ is an amply δ_M -supplemented module; (3) Every module in $\sigma[M]$ is an amply δ_M -supplemented module if and only if every module in $\sigma[M]$ is a δ_M -supplemented module; and (4) A projective module in $\sigma[M]$ is an amply δ_M -supplemented module if and only if it is a δ_M -supplemented module. We also investigate the interconnections between δ_M -supplemented modules and δ_M -semiperfect modules.

1. Introduction and Preliminaries

Throughout this article, R denotes an associative ring with unity and modules M are unitary right R-modules. $Mod\text{-}R$ denotes the category of all right R-modules.

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Let M be any R-module. Any R-module N is M-generated (or generated by M) if there exists an epimorphism $f : M^{(\Lambda)} \to N,$ for some indexed set Λ. An R-module N is said to be *subgenerated* by M if N is isomorphic to a submodule of an M-generated module. We denote by $\sigma[M]$ the full subcategory of the right R-modules whose objects are all right R-modules subgenerated by M. In case of $M = R,$ $\sigma[M] = Mod\text{-}R.$ Any module $N \in \sigma[M]$ is said to be *M-singular* if $N \cong L/K,$ for some $L \in \sigma[M]$ and K is essential in L. The class of all M-singular modules is closed under submodules, homomorphic images, and direct sums. The concept of small submodule has been generalized to δ-small submodule by Zhou [8]. Zhou called a submodule N of a module M is δ-*small* in M (notation $N \leq_\delta M$) if, whenever $N + X = M$ with M/X singular, we have $X = M$. Özcan and Alkan considered this notation in $\sigma[M]$. For a module N in $\sigma[M]$, Özcan and Alkan [4] called a submodule L of N is a δ-M *-small submodule*, written $L \ll_{\delta_M} N,$ in N if $L + K \neq N,$ for any proper submodule K of N with N/K is M-singular. We write δ_M -small for δ-M -small. Clearly, if L is δ-small, then L is a δ_M -small submodule. Hence δ_M -small submodules are the generalization of δ-small submodules in the category $Mod\text{-}R.$ Let L, K be submodules of M. L is called a δ-*supplement* of K in M if $M = L + K$ and $L \cap K \ll_\delta L.$ L is called a δ-*supplement submodule* of M if L is a δ-supplement of some submodule of M. M is called a δ-*supplemented module* if every submodule of M has a δ-supplement. If for every submodules L, K of M with $M = L + K,$ there exists a δ-supplement N of L such that $N \leq K,$ then M is called an *amply* δ-*supplemented module*. For the other definitions and notation in this paper, we refer to [1] and [7].

We write below some known results which are observed in [4] and [5].

Lemma 1.1. *Let* $N \in \sigma[M]$.

(1) *For modules K and L with,* $K \leq L \leq N,$ *we have* $L \ll_{\delta_M} N$ *if and only if* $K \ll_{\delta_M} N$ *and* $L/K \ll_{\delta_M} N/K.$

(2) *For submodules K and L of N,* $K + L \ll_{\delta_M} N$ *if and only if* $K \ll_{\delta_M} N$ *and* $L \ll_{\delta_M} N.$

(3) *If* $K \ll_{\delta_M} N$ *and* $f : N \to L$ *is a homomorphism*, *then* $f(K) \ll_{\delta_M} L$. *In particular*, *if* $K \ll_{\delta_M} L$, $L \leq N$, *then* $K \ll_{\delta_M} N$.

(4) *If* $K \leq L \leq^{\oplus} N$ *and* $K \ll_{\delta_M} N$, *then* $K \ll_{\delta_M} L$.

Also, Özcan and Alkan in [4] considered the following submodule of a module *N* in $\sigma[M]$ (see also Zhou [8]):

$$\delta_M(N) = \bigcap\{K \leq N : N/K \text{ is } M\text{-singular simple}\}.$$

Lemma 1.2. $\delta_M(N) = \sum\{L \leq N : L \ll_{\delta_M} N\}$.

Lemma 1.3. *Let K be a submodule of a module N in* $\sigma[M]$. *Then* $K \ll_{\delta_M} N$ *if and only if* $N = X \oplus Y$ *for a projective semisimple submodule Y in* $\sigma[M]$ *with* $Y \leq K$, *whenever* $X + K = N$.

2. (Amply) δ_M -supplemented Modules

Let $N \in \sigma[M]$ and $L, K \leq N$. L is called a δ_M *-supplement* of K in N if $N = K + L$ and $K \cap L \ll_{\delta_M} L$. *L* is called a δ_M *-supplement submodule* of *N* if *L* is a δ_M -supplement of some submodule of *N*. *N* is called a δ_M *-supplemented module* if every submodule of *N* has a δ_M -supplement in *N*. N is called an *amply* δ_M *-supplemented module* if for every submodules *L*, *K* of *N* with $N = L + K$, there exists a δ_M -supplement *X* of *L* such that $X \leq K$. It is clear that every amply δ_M -supplemented module is δ_M -supplemented. But the converse is not true.

Proposition 2.1. *Let* $N \in \sigma[M]$ *and* $L, K \leq N$. *Then the following are equivalent*:

(a) *L is a* δ_M *-supplement of K in N*.

(b) $N = K + L$ *and for each* $X \leq L$ *with* $N = K + X$ *and* L/X *M-singular*, $X = L$.

Proof. (a) $\Rightarrow$ (b) By the hypothesis, $N = K + L$ and $K \cap L \ll_{\delta_M} L$. If $X \leq L$ with $N = K + X$ and L/X is *M*-singular, then $L = (K \cap L) + X$. Hence, since $K \cap L \ll_{\delta_M} L$, $X = L$.

(b) $\Rightarrow$ (a) Here we have to prove only $K \cap L \ll_{\delta_M} L$. Suppose that $X \leq L$ such that $L = (L \cap K) + X$ and L/X is M-singular. Then $N = K + L = K + (L \cap K) + X = K + X$. Hence, by (b), $X = L$. This shows $K \cap L \ll_{\delta_M} L$. □

Proposition 2.2. *Let $N \in \sigma[M]$ and $L, K, H \leq N$. If K is a δ_M-supplement of L in N and L is a δ_M-supplement of H in N, then L is a δ_M-supplement of K in N.*

Proof. Let K be a δ_M-supplement of L in N and L is a δ_M-supplement of H in N. Then $N = K + L = L + H$, $K \cap L \ll_{\delta_M} K$ and $H \cap L \ll_{\delta_M} L$. We have to prove $K \cap L \ll_{\delta_M} L$. Let $X \leq L$ such that $L = (K \cap L) + X$ and L/X is M-singular. Hence $N = (L \cap K) + X + H$. Since $K \cap L \ll_{\delta_M} K$, $K \cap L \ll_{\delta_M} N$ and so by Lemma 1.3, $N = (X + H) \oplus Y$, for projective semisimple module Y in $\sigma[M]$ with $Y \subseteq L \cap K$. Hence, $L = (X \oplus Y) + (L \cap H)$. Since $L/(X + Y)$ is M-singular and $L \cap H \ll_{\delta_M} L$, $L = X \oplus Y$. This implies $Y = 0$ because Y is semisimple projective in $\sigma[M]$ and $Y \cong L/X$ is M-singular. Thus $K \cap L \ll_{\delta_M} L$. □

Lemma 2.3. *Let $N \in \sigma[M]$ be δ_M-supplemented module. Then*

(1) $N/\delta_M(N)$ *is semisimple.*

(2) *If $L \leq N$ with $L \cap \delta_M(N) = 0$, then L is semisimple.*

Proof. (1) We show that every submodule of $N/\delta_M(N)$ is a direct summand. Let K be a submodule of N containing $\delta_M(N)$. Since N is δ_M-supplemented, there exists a submodule U of N such that $N = K + U$ and $K \cap U \ll_{\delta_M} U$. Then $N/\delta_M(N) = U/\delta_M(N) + (K + \delta_M(N))/\delta_M(N)$ and $K \cap U \ll_{\delta_M} N$. It follows that $N/\delta_M(N) = U/\delta_M(N) \oplus (K + \delta_M(N))/\delta_M(N)$ as $U \cap (K + \delta_M(N)) = (U \cap K) + \delta_M(N) = \delta_M(N)$.

(2) Since $L \cong (L \oplus \delta_M(N))/\delta_M(N) \leq N/\delta_M(N)$, L is semisimple by (1). □

Proposition 2.4. *Let* $N \in \sigma[M]$ *be* δ_M*-supplemented. Then there is a decomposition* $N = N_1 \oplus N_2$ *with* N_1 *is semisimple module and* N_2 *is a module with* $\delta_M(N_2) \leq_e N_2$.

Proof. For $\delta_M(N)$, there exists $N_1 \leq N$ such that $\delta_M(N) \cap N_1 = 0$ and $N_1 \oplus \delta_M(N) \leq_e N$. By the hypothesis, there is $N_2 \leq N$ such that $N = N_1 + N_2$ and $N_1 \cap N_2 \ll_{\delta_M} N_2$. Hence $N = N_1 \oplus N_2$ because $N_1 \cap N_2 \leq N_1 \cap \delta_M(N)$ $= 0$ and so by Lemma 2.3(2), N_1 is semisimple. The rest part of the proposition follows directly from [1, Proposition 5.20], since $\delta_M(N) = \delta_M(N_1) \oplus \delta_M(N_2)$ and $N_1 \oplus \delta_M(N) \leq_e N = N_1 \oplus N_2$. □

The following example shows that the converse of Proposition 2.4 is not hold.

Example 2.5. *Let* $R = \mathbb{Z}$. If $N = \bigoplus_{i=1}^{\infty} N_i$ with each $N_i = \mathbb{Z}_{p^\infty}$, where p is a prime number, then $N \in \sigma[\mathbb{Z}]$, $N = N \oplus 0$ and $\delta_{\mathbb{Z}}(N) = \bigoplus_{i=1}^{\infty} \delta_{\mathbb{Z}}(N_i) = \bigoplus_{i=1}^{\infty} N_i = N$ $\leq_e N$. But N in not a δ_M -supplemented module (see [3, Example 2.14]).

Lemma 2.6. *Let* $N \in \sigma[M]$ *and let* $L, K \leq N$ *such that* L *is a* δ_M *-supplemented module. If* $L + K$ *has a* δ_M *-supplement in N, then so does K.*

Proof. Let U be a δ_M -supplement of $L + K$ in N. Then $N = (L + K) + U$ and $(L + K) \cap U \ll_{\delta_M} U$. Since L is a δ_M -supplemented module, $L \cap (U + K)$ has a δ_M -supplement in N, there exists a submodule W of L such that $L =$ $(L \cap (U + K)) + W$ and $(U + K) \cap W \ll_{\delta_M} W$. Thus $N = (K + U) + W$ and $(U + K)$ $\cap W \ll_{\delta_M} N$. Now we show that $U + W$ is a δ_M -supplement of K in N. It is clear that $N = K + (U + W)$, hence it remains to prove $K \cap (U + W) \ll_{\delta_M} U + W$. Since $K + W \leq K + L$, $U \cap (K + W) \leq U \cap (L + K) \ll_{\delta_M} U$. Then, by Lemma 1.1(2), $U \cap (K + W) + W \cap (U + K) \ll_{\delta_M} U + W$ and so $K \cap (U + W) \ll_{\delta_M} U + W$ as $K \cap (U + W) \leq U \cap (K + W) + W \cap (U + K)$. □

Proposition 2.7. *The homomorphic images of a* δ_M *-supplemented module are* δ_M *-supplemented modules*.

Proof. Let N be a δ_M-supplemented module and $f : N \to L$ be an epimorphism. We have to show that L is δ_M-supplemented. Let $K \leq L$. Then $f^{-1}(K) \leq N$ and so, by the hypothesis, there exists a submodule U of N with $N = f^{-1}(K) + U$ and $f^{-1}(K) \cap U \ll_{\delta_M} U$. It is clear that $L = K + f(U)$, hence it suffices to show that $K \cap f(U) \ll_{\delta_M} f(U)$. Since $f^{-1}(K) \cap U \ll_{\delta_M} U$, by Lemma 1.1(3), $f(f^{-1}(K) \cap U) \ll_{\delta_M} f(U)$ and so $K \cap f(U) = f(f^{-1}(K) \cap U)$ $\ll_{\delta_M} f(U)$. Thus L is a δ_M-supplemented module. □

Corollary 2.8. *Direct summands and factor modules of a* δ_M *-supplemented module are* δ_M *-supplemented modules*.

Lemma 2.9. *Let n be any positive integer*. *If* $N = N_1 + \cdots + N_n \in \sigma[M]$ *with each* N_i *is a* δ_M *-supplemented module*, *then N is a* δ_M *-supplemented module*.

Proof. To show that N is δ_M *-supplemented*, it is sufficient by induction on n to prove this in the case when $n = 2$. Thus suppose that $n = 2$ and let $K \leq N$. Then $N = N_1 + (N_2 + K)$. By Lemma 2.6, $N_2 + K$ has δ_M-supplement in N as N_1 is δ_M *-supplemented* and N has a δ_M-supplement in N. Again, by Lemma 2.6, K has a δ_M-supplement in N. So N is δ_M-supplemented. □

The following proposition follows immediately from Lemma 2.9 and Proposition 2.7.

Proposition 2.10. *Let* $N \in \sigma[M]$ *be* δ_M *-supplemented*. *Then every finitely N-generated module is* δ_M *-supplemented module*.

Proposition 2.11. *The homomorphic images of amply* δ_M *-supplemented modules are amply* δ_M *-supplemented modules*.

Proof. Let $N \in \sigma[M]$ be an amply δ_M-supplemented module and let $f : N \to K$ be an epimorphism. We have to show that K is an amply δ_M-

supplemented module. Let K_1 and K_2 be submodules of K such that $K = K_1 + K_2$. Then $N = f^{-1}(K_1 + K_2) = f^{-1}(K_1) + f^{-1}(K_2)$. Since N is amply δ_M -supplemented, there exists a submodule X of $f^{-1}(K_2)$ such that $N = f^{-1}(K_1) + X$ and $f^{-1}(K_1) \cap X \ll_{\delta_M} X \leq K_2$. Hence $N = K_1 + f(X)$ and $K_1 \cap f(X) = f(f^{-1}(K_1) \cap X) \ll_{\delta_M} f(X) \leq K_2$. This shows that $f(X)$ is a δ_M -supplement of K_1 in N. Therefore, K is amply δ_M -supplemented. □

Corollary 2.12. *Direct summands and factor modules of an amply* δ_M *-supplemented module are amply* δ_M *-supplemented modules.*

Proposition 2.13. *A module* $N \in \sigma[M]$ *whose all submodules are* δ_M *-supplemented modules is an amply* δ_M *-supplemented module.*

Proof. Let $L, K \leq N$ such that $N = L + K$. Since L is δ_M -supplemented, $L \cap K$ has δ_M -supplement H in L. Hence $L = (L \cap K) + H$ and $L \cap K \cap H \ll_{\delta_M} H$ imply that $N = L + K = L \cap K + H + K = K + H$ and $H \cap K \ll_{\delta_M} H$, that is, L contains a δ_M *-supplement* of K in N. So N is an amply δ_M -supplemented module. □

Corollary 2.14. *The following are equivalent for any module M*:

(a) *Every module in* $\sigma[M]$ *is an amply* δ_M *-supplemented module*.

(b) *Every module in* $\sigma[M]$ *is a* δ_M *-supplemented module*.

Proof. (a) $\Rightarrow$ (b) This obvious as every *amply* δ_M *-supplemented module* is δ_M *-supplemented module*.

(b) $\Rightarrow$ (a) Since $\sigma[M]$ is closed under submodules, (a) follows directly from Proposition 2.13. □

If we take $M = R$ in Corollary 2.14, then we get the following corollary which proved in [6]:

Corollary 2.15. *The following are equivalent for any ring R*:

(a) *Every R-module is an amply* δ*-supplemented module*.

(b) *Every R-module is a* δ*-supplemented module*.

Following [7 p. 359], a module M is called π-*projective* if for every two submodules L, K of M with $K + L = M$, there exists $f \in End(M)$ with $\operatorname{Im}(f) \subseteq K$ and $\operatorname{Im}(1 - f) \subseteq L$. This is obviously true if and only if the epimorphism $f : K \oplus L \to K + L = M$ given by $f((k, l)) = k + l$, splits.

Proposition 2.16. *If N is a* π-*projective* δ_M *-supplemented module in* $\sigma[M]$, *then N is an amply* δ_M *-supplemented module*.

Proof. Let $K, L \leq N$ such that $N = K + L$. Since N is π-projective, there exists $f \in End(N)$ such that $f(N) \leq K$ and $(1 - f)(N) \leq L$. Note that $(1 - f)(K) \leq K$ and $N = f(N) + (1 - f)(N)$. If H is a δ_M-supplement of K in N, then $(1 - f)(N) = (1 - f)(K) + (1 - f)(H) \leq K + (1 - f)(H)$. So $N = K + (1 - f)(H)$. We claim that $K \cap (1 - f)(H) \ll_{\delta_M} (1 - f)(H)$. Clearly, $(1 - f)(H) \leq L$. Since $K \cap H \ll_{\delta_M} K$, by Lemma 1.1(3), $(1 - f)(K \cap H) \ll_{\delta_M} (1 - f)(H)$. Now, let $k \in K \cap (1 - f)(H)$. Then $k \in K$ and $k = h - f(h)$, for some $h \in H$. Hence $h = k + f(h) \in K$ as $k, f(h) \in K$. So $k \in (1 - f)(K \cap H)$. Thus $K \cap (1 - f)(H) \leq (1 - f)(K \cap H) \ll_{\delta_M} (1 - f)(H)$. It follows that N is an amply δ_M-supplemented module. □

Proposition 2.17. *Let N be a* π-*projective module in* $\sigma[M]$. *If L*, *K are* δ_M *-supplements to each other in N*, *then* $K \cap L$ *is semisimple and projective in* $\sigma[M]$. *If*, *in addition*, *N is projective in* $\sigma[M]$, *then L and K are both projective in* $\sigma[M]$.

Proof. Since N is π-projective, the epimorphism $f : L \oplus K \to L + K$ given by $f((l, k)) = l + k$ for each l in L and each k in K splits and so $N = L \oplus K = Ker(f) \oplus H$, where $H \cong N$. As $\ker(f) = \{(l, -l) : l \in K \cap L\}$, $Ker(f) \cong K \cap L$. Hence there exists a submodule Y of $Ker(f)$ which is semisimple and projective in $\sigma[M]$ and $L \oplus K = Y \oplus H$. This implies $Y = Ker(f) \cong L \cap K$. So $L \cap K$ is semisimple and projective in $\sigma[M]$. Further, if N is projective in $\sigma[M]$, then $L \oplus K$ is projective and hence L and K are projective in $\sigma[M]$. □

3. Applications

In this section, we investigate the interconnection between (amply) δ_M-supplemented modules and δ_M-semiperfect modules. Following Özcan and Alkan [4], a module N in $\sigma[M]$ is said to be a δ_M-*semiperfect module* if for every submodule K of N, there is a decomposition $K = A \oplus B$ such that A is a projective summand of N in $\sigma[M]$ and $B \leq \delta_M(N)$.

Let $N, P \in \sigma[M]$. An epimorphism $f : P \to N$ is called a δ_M-*cover* if $Ker(f) \ll_{\delta_M} P$. A δ_M-cover $f : P \to N$ is called a *projective* δ_M-*cover* in case P is projective in $\sigma[M]$. We start this section by proving the composite of δ_M-covers is also a δ_M-cover.

Lemma 3.1. *If* $f : P \to K$, $g : K \to N$ *are* δ_M-*covers, then* $gof : P \to N$ *is also a* δ_M-*cover*.

Proof. Clearly that *gof* is an epimorphism. We have to prove $Ker(gof) \ll_{\delta_M} P$. Suppose that $L \leq P$ such that $P = Ker(gof) + L$ and P/L is M-singular. Then $f(P) = f(Ker(gof)) + f(L)$ and so, $N = Ker(g) + f(L)$. Since P/L is M-singular, $f(P)/f(L)$ is M-singular implying $N = f(L)$ as $Kerg \ll_{\delta_M} N$. Thus $gof : P \to N$ is a δ_M-cover. □

Proposition 3.2. *Let* N *be in* $\sigma[M]$ *and* $K \leq N$. *Then the following are equivalent*:

(a) N/K *has a projective* δ_M-*cover*.

(b) *If* $L \leq N$ *and* $N = K + L$, *then* K *has a* δ_M-*supplement* $K^{\backslash} \leq L$ *such that* $K^{\backslash}$ *has a projective* δ_M-*cover*.

(c) K *has a* δ_M-*supplement* $K^{\backslash}$ *which has a projective* δ_M-*cover*.

Proof. (a) $\Rightarrow$ (b) Let $L \leq N$ such that $N = K + L$. Suppose that $f : P \to N$ is a projective δ_M-*cover*. Then the map $g : L \to N/K$ given by $g(v) = \bar{v} = v + K$ is an epimorphism. As P is projective in $\sigma[M]$, there is a map $h : P \to L$ such that

$f = g \circ h.$ We have $N/K = f(P) = g(h(P)) = (h(P) + K)/K$ and so $N = h(P) + K.$ Now $K \cap h(P) \ll_{\delta_M} h(P)$ because $h(Ker(f)) \ll_{\delta_M} h(P)$ and $h(Ker(f)) = K \cap h(P).$ Therefore, $h(P)$ is a δ_M-supplement of K in N. Moreover, $Ker(h) \subseteq Ker(f) \ll_{\delta_M} P.$ Hence $h : P \to h(P)$ is a projective δ_M-cover.

(b) $\Rightarrow$ (a) Since $N = K + N,$ (c) follows directly from (b).

(c) $\Rightarrow$ (a) Let $f : P \to K^{\backslash}$ be a projective δ_M-cover. Since $K^{\backslash}$ is a δ_M-supplement of K in $N,$ $N = K + K^{\backslash}$ and $K \cap K^{\backslash} \ll_{\delta_M} K^{\backslash}$ imply that $K^{\backslash}/K \cap K^{\backslash} \cong N/K$ and the natural homomorphism $g : K^{\backslash} \to K^{\backslash}/K \cap K^{\backslash}$ is a δ_M-cover. So, if h is an isomorphism from $K^{\backslash}/K \cap K^{\backslash}$ to $N/K,$ hence, by Lemma 3.1, $h \circ g \circ f : P \to N/K$ is a projective δ_M-cover. □

As an immediate consequence of Proposition 3.2, we get the following:

Theorem 3.3. *Let N be a module in* $\sigma[M]$. *Then the following are equivalent*:

(a) *Every factor module of N has a projective* δ_M*-cover*.

(b) *N is amply* δ_M*-supplemented by* δ_M*-supplements which have a projective* δ_M*-cover*.

(c) *N is* δ_M*-supplemented by* δ_M*-supplements which have a projective* δ_M*-cover*.

Wang [6] called a module *N* is δ-*semiperfect module* if every factor module of *N* has a projective δ-cover. Putting $M = R$ in Theorem 3.3, we get the following corollary which was proved in [6].

Corollary 3.4. *Let N be any R-module*. *Then the following are equivalent*:

(a) *N is* δ-*semiperfect module*.

(b) *N is amply* δ-*supplemented by* δ-*supplements which have a projective* δ-*cover*.

(c) *N is* δ-*supplemented by* δ-*supplements which have a projective* δ-*cover*.

Next, we show that the concepts amply δ_M -supplemented and δ_M -supplemented modules are equivalent for projective modules in $\sigma[M]$.

Theorem 3.5. *Let* $N \in \sigma[M]$ *be projective in* $\sigma[M]$. *Then the following are equivalent*:

(a) *Every factor module of N has a projective* δ_M *-cover*.

(b) *N is an amply* δ_M *-supplemented module*.

(c) *N is a* δ_M *-supplemented module*.

Proof. (a) $\Rightarrow$ (b) is by Theorem 3.3 and (b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) Let *K* be a submodule of *N*. Then there exists a submodule *L* of *N* such that $N = K + L$ and $K \cap L \ll_{\delta_M} L$. Consider the natural epimorphism $\pi : N \to N/K$. Let $\pi_L^\setminus : L \to N/K$ be the restriction of π to L. Since N is projective in $\sigma[M]$, there exists a homomorphism $g : N \to L$ such that $\pi = \pi_L^\setminus \circ g$. Then $N/K = \pi(N) = \pi_L^\setminus(g(N)) = (g(N) + K)/K$. This gives $N = g(N) + K$. Hence $L = g(N) + (L \cap K)$ and so, by Lemma 1.3, $L = g(N) \oplus Y$ for a projective semisimple module Y in $\sigma[M]$ with $Y \subseteq K \cap L$. We claim that $g(N) \cap K \ll_{\delta_M} g(N)$. Let $g(N) = (g(N) \cap K) + X$ with $g(N)/X$ is *M*-singular. Since $L = ((g(N) \cap K) + X) \oplus Y$, $L/(X \oplus Y) \cong ((g(N) \cap K) + X)/X = g(N)/X$ is *M*-singular. Then, since $L = (g(N) \cap K) + (X \oplus Y)$ and $g(N) \cap K \leq L \cap K \ll_{\delta_M} L$, we have $L = X \oplus Y$. Then $X = g(N)$ as $L = g(N) \oplus Y$. Hence $g(N)K \ll_{\delta_M} g(N)$. This shows that $g(N)$ is a δ_M -supplement of *K* in *N*. Next, we show $g(N)$ is projective in $\sigma[M]$. As N is δ_M -supplemented, $g(N)$ has a δ_M -supplement Q in N. Hence, by Proposition 2.2, $g(N)$ is also a δ_M -supplement of *Q* in *N* and so $g(N)$ is projective in $\sigma[M]$ (see Proposition 2.17). Thus $\pi_{g(N)}^\setminus : g(N) \to N/K$ is a protective δ_M -cover. □

We conclude this paper by giving a characterization of a δ_M -semiperfect modules in $\sigma[M]$ for a certain class of modules.

Theorem 3.6. *Let* $N \in \sigma[M]$ *be projective in* $\sigma[M]$ *and* $\delta_M(N) \ll_{\delta_M} N$. *Then the following are equivalent*:

(a) *N is a* δ_M *-semiperfect module*.

(b) *Every factor module of N has a projective* δ_M *-cover*.

(c) *N is amply* δ_M *-supplemented by* δ_M *-supplements which have a projective* δ_M *-cover*.

(d) *N is* δ_M *-supplemented by* δ_M *-supplements which have a projective* δ_M *-cover*.

(e) *N is a* δ_M *-supplemented module*.

(f) *N is amply* δ_M *-supplemented module*.

Proof. It follows by [6, Theorem 2.19], Theorem 3.3 and Theorem 3.5. □

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References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York-Heidelberg-Berlin, 1974.

[2] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, Extending modules, Pitman Research Notes in Mathematics Series 313, 1994.

[3] M. T. Kosan, δ-lifting and δ-supplemented modules, Alg. Coll. 14(1) (2007), 53-60.

[4] A. Ç. Özcan and M. Alkan, Semiperfect modules with respect to a preradical, Comm. Alg. 34(2) (2006), 841-856.

[5] D. K. Tütüncü, F. Kaynarca and M. T. Koşan, On non-δ-M-cosingular completely $\oplus$-δ_M -supplemented modules, Ring Theory, Proceeding of the Fifth China-Japan-Korea Conference, Tokyo, Japan, 10-15 September 2007.

[6] Y. Wang, δ-small submodules and δ-supplemented modules, Int. J. Math. Math. Sci. (2007), 1-8.

[7] R. Wisbauer, Foundation of Module and Ring Theory, Gordon and Breach, 1991.

[8] Y. Zhou, Generalizations of perfect, semiperfect, and semiregular, Alg. Coll. 7(3) (2000), 305-318.