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ON A NEW SUBCLASS OF MULTIVALENT MEROMORPHIC FUNCTION WITH NEGATIVE COEFFICIENT FOR OPERATOR ON HILBERT SPACE

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Abstract

In this paper, we investigate a new subclass of normalized multivalent analytic meromorphic p-valent functions with negative coefficients in the unit disc $U = \{z : |z| < 1\}$. We aim to study this new subclass for operator on Hilbert space and the results obtained for different properties generalize many earlier results in the literature. All the results obtained in this paper are found to be sharp for this new subclass.

1. Introduction and Motivation

Let $S_w(p)$ denote the class of functions $f(z)$ normalized by

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$$f(z) = \frac{1}{(z-w)^p} + \sum_{n=1}^{\infty} a_{p+n}(z-w)^{p+n}, \quad p \in N, \quad a_{p+n} \geq 0, \quad z \neq w \tag{1.1}$$

which is analytic and meromorphic p-valent in the open disc $U = \{z : |z| < 1\}$ and w is a fixed point in U.

A function $f(z) \in S_w(p)$ is said to be *p-valent starlike* of order δ and *p-valent convex* of order δ in U if it satisfies the conditions, respectively:

$$-\mathrm{Re}\left\{(z-w)\frac{f'(z)}{f(z)}\right\} > \delta \quad (z \in U;\ 0 \leq \delta < p;\ p \in N), \tag{1.2}$$

$$-\mathrm{Re}\left\{1 + \frac{(z-w)f''(z)}{f'(z)}\right\} > \delta \quad (z \in U;\ 0 \leq \delta < p;\ p \in N). \tag{1.3}$$

Let $S_w(1) = S_w$ which studied by Acu and Owa [6] and also studied by Kanas and Ronning [9].

Also, $S_{wc}(p)$ and $S_{ws}(p)$ are the classes of all p-valent convex and starlike functions of order δ, respectively.

This concept is motivated by Ghanim and Darus [1], Goodman [2], Fan [5], Duren [8], Owa and Srivastava [13], Khairnar et al. [15, 16] and Schild and Silverman [17].

Now, for the function $f(z)$ in the class $S_w(p)$, we define

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = (z-w) f'(z) + \frac{1+p}{(z-w)^p},$$

$$D^2 f(z) = (z-w)(D' f(z))' + \frac{1+p}{(z-w)^p}$$

and for $k = 1, 2, 3, \ldots,$

$$D^k f(z) = (z-w)(D^{k-1} f(z))' + \frac{1+p}{(z-w)^p}$$

$$= \frac{1}{(z-w)^p} + \sum_{n=1}^{\infty} (n, p)^k a_{p+n}(z-w)^{p+n}.$$

If we put $p = 1$ and $w = 0$, then differential operator D^k is reduced to Frasia and Darus [3].

Definition. A function $f(z) \in S_w(p)$ is said to be in the class $S_w^*(p)$ if and only if

$$\left| \frac{(z-w)\dfrac{(D^k f(z))'}{D^k f(z)} + p}{(\alpha-\beta)\gamma - \beta\left[(z-w)\dfrac{(D^k f(z))'}{D^k f(z)} + p\right]} \right| < \mu, \tag{1.4}$$

for $-1 \leq \beta < \alpha \leq 1$, $0 < \mu \leq 1$ and $0 \leq \gamma < 1$.

Let $T_w(p)$ denote the subclass of $S_w(p)$ consisting function of the form

$$f(z) = \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} a_{p+n}(z-w)^{p+n}, \quad p \in N, \quad a_{p+n} \geq 0. \tag{1.5}$$

Now, we define

$$S_w^{**}(p) = S_w^*(p) \bigcap T_w(p). \tag{1.6}$$

Such type of classes has been studied by Lee et al. [11], Owa and Srivastava [13] and Ghanim and Darus [1].

Let H be a Hilbert space on the complex field. Let A be an operator on H. Then a complex analytic function $f(z)$ on the disc U is denoted by $f(A)$ and operator on H is defined by Riesz-Dunford integral [7]:

$$f(A) = \frac{1}{2\pi i}\int_c f(z)(zI - A)^{-1} dz,$$

where I is the identity operator on H. C is a positively oriented simple closed rectifiable contour lying in U and containing spectrum of A in its interior domain [5]. The conjugate of A is denoted by A^*.

A function $f(z)$ given by (1.5) is in the class $S_w^{**}(p)$ if it satisfies the condition

$$|\, A(D^k f(A))' + pD^k f(A) \,| < \mu |\, (\alpha-\beta)\gamma D^k f(A) - \beta[A(D^k f(A))' + pD^k f(A)] \,|$$

for $-1 \leq \beta < \alpha \leq 1$, $0 < \mu \leq 1$, $0 \leq \gamma < 1$ and all operator A with $|\, A \,| < 1$ and $A \neq 0$, 0 is zero operator on H.

In this paper, we define and study new class $S_w^{**}(p)$ and obtain sharp results for the properties like coefficient inequality, growth and distortion, radius of starlikeness and convexity, extreme points, closure theorem, Hadamard product and inclusion property. Such type of work was carried out by Joshi [10], Xiaopei [14], Khairnar et al. [15, 16] and Schild and Silverman [17].

2. Main Result

In this section, we obtain necessary and sufficient condition for the function $f(z)$ to be in the class $S_w^{**}(p)$.

Theorem 2.1. *A function* $f(z)$ *given by*

$$f(z) = \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} a_{p+n}(z-w)^{p+n}, \quad a_{p+n} \geq 0, \quad p \in N$$

is in the class $S_w^{**}(p)$ *if and only if*

$$\sum_{n=1}^{\infty} [(2p+n)(1-\mu\beta) + \mu\gamma(\alpha-\beta)](n, p)^k a_{p+n} \leq \mu\gamma(\alpha-\beta). \tag{2.1}$$

The result is sharp for the function

$$f(z) = \frac{1}{(z-w)^p} - \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n, p)^k}(z-w)^{p+n}.$$

Proof. Assuming that (2.1) holds, we have

$$| A(D^k f(A))' + pD^k f(A) | - \mu | (\alpha-\beta)\gamma D^k f(A) - \beta[A(D^k f(A))' + pD^k f(A)] |$$

$$= \left| \sum_{n=1}^{\infty} (2p+n) a_{p+n} A^{p+n} (n, p)^k \right|$$

$$- \mu \left| (\alpha-\beta)\gamma A^p - \sum_{n=1}^{\infty} [(\alpha-\beta)\gamma - \beta(2p+n)] a_{p+n} A^{p+n} (n, p)^k \right|$$

$$\leq \sum_{n=1}^{\infty} [(2p+n)(1-\mu\beta) + \mu\gamma(\alpha-\beta)](n, p)^k a_{p+n} - \mu\gamma(\alpha-\beta)$$

$$\leq 0.$$

Hence $f(z) \in S_w^{**}(p)$.

Conversely, suppose that $f(z) \in S_w^{**}(p)$,

$$
\begin{aligned}
& |A(D^k f(A))' + pD^k f(A)| \\
& < \mu |(\alpha - \beta)\gamma D^k f(A) - \beta[A(D^k f(A))' + pD^k f(A)]| \\
& = \left| \sum_{n=1}^{\infty} (2p+n)(n,\ p)^k a_{p+n} A^{p+n} \right| \\
& < \mu \left| (\alpha - \beta)\gamma A^p - \sum_{n=1}^{\infty} [(\alpha - \beta)\gamma - \beta(2p+n)](n,\ p)^k a_{p+n} A^{p+n} \right|.
\end{aligned}
$$

Selecting $A = eI$ $(0 < e < 1)$ in above equality, we have

$$
\begin{aligned}
& \sum_{n=1}^{\infty} (2p+n)(n,\ p)^k a_{p+n} e^{p+n} \\
& \leq \mu \left[(\alpha - \beta)\gamma e^{p+n} - \sum_{n=1}^{\infty} [(\alpha - \beta)\gamma - \beta(2p+n)](n,\ p)^k a_{p+n} e^{p+n} \right].
\end{aligned}
$$

Letting $e \to 1$ $(0 < e < 1)$, we get

$$
\sum_{n=1}^{\infty} (2p+n)(n,\ p)^k a_{p+n} \leq \mu\gamma(\alpha - \beta) - \mu \sum_{n=1}^{\infty} [(\alpha - \beta)\gamma - \beta(2p+n)](n,\ p)^k a_{p+n},
$$

that is,

$$
\sum_{n=1}^{\infty} [(2p+n)(1 - \mu\beta) + \mu\gamma(\alpha - \beta)](n,\ p)^k a_{p+n} \leq \mu\gamma(\alpha - \beta).
$$

Corollary. *If* $f(z)$ *given by*

$$
f(z) = \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} a_{p+n}(z-w)^{p+n}, \quad a_{p+n} \geq 0, \quad p \in N
$$

is in the class $S_w^{**}(p)$, *then*

$$
a_{p+n} \leq \frac{\mu\gamma(\alpha - \beta)}{[(2p+n)(1 - \mu\beta) + \mu\gamma(\alpha - \beta)](n,\ p)^k}, \quad p,\ n \in N.
$$

In the following theorem, we study growth and distortion property:

Theorem 2.2. *If the function* $f(z) \in S_w^{**}(p)$, *then*

$$\frac{1}{|A|^p} - \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n}$$

$$\leq |f(A)| \leq \frac{1}{|A|^p} + \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n}.$$

The result is sharp for the function

$$f(z) = \frac{1}{(z-w)^p} - \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}(z-w)^{p+n},$$

where $n,\ p \in N$, $|A| < 1$ *and* $A \neq 0$.

Proof.

$$|f(z)| = \left|\frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} a_{p+n}(z-w)^{p+n}\right| \leq \frac{1}{|z-w|^p} + \sum_{n=1}^{\infty} a_{p+n}|z-w|^{p+n},$$

$$|f(A)| \leq \frac{1}{|A|^p} + |A|^{p+n}\sum_{n=1}^{\infty} a_{p+n}, \quad |A| < 1, \quad A \neq 0.$$

By Theorem 2.1,

$$\sum_{n=1}^{\infty} a_{p+n} \leq \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}. \tag{2.2}$$

Thus

$$|f(A)| \leq \frac{1}{|A|^p} + \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n}. \tag{2.3}$$

Also,

$$|f(z)| \geq \left|\frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} a_{p+n}(z-w)^{p+n}\right|$$

$$\geq \frac{1}{|z-w|^p} - \sum_{n=1}^{\infty} a_{p+n}|z-w|^{p+n},$$

$$| f(A)| \geq \frac{1}{|A|^p} - |A|^{p+n} \sum_{n=1}^{\infty} a_{p+n}$$

$$\geq \frac{1}{|A|^p} - \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n}. \tag{2.4}$$

Using (2.2), $p, n \in N$, $|A| < 1$, $A \neq 0$.

Using (2.3) and (2.4), we have

$$\frac{1}{|A|^p} - \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n}$$

$$\leq | f(A)| \leq \frac{1}{|A|^p} + \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n}.$$

Theorem 2.3. *If the function* $f(z) \in S_w^{**}(p)$, *then*

$$\frac{p}{|A|^{p+1}} - \frac{\mu\gamma(\alpha-\beta)(p+n)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n-1}$$

$$\leq | f'(A)| \leq \frac{p}{|A|^{p+1}} + \frac{\mu\gamma(\alpha-\beta)(p+n)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n-1}.$$

The result is sharp for the function

$$f(z) = \frac{1}{(z-w)^p} - \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}(z-w)^{p+n},$$

where $n,\ p \in N$, $|A| < 1$ *and* $A \neq 0$.

Proof.

$$| f'(z)| \leq \frac{p}{|z-w|^{(p+1)}} + \sum_{n=1}^{\infty}(p+n)a_{p+n}|z-w|^{p+n-1},$$

$$| f'(A)| \leq \frac{p}{|A|^{(p+1)}} + \sum_{n=1}^{\infty}(p+n)a_{p+n}|A|^{p+n-1},\ \text{since } |A| < 1,\ A \neq 0.$$

Using (2.2), we have

$$|f'(A)| \leq \frac{p}{|A|^{p+1}} + \frac{\mu\gamma(\alpha-\beta)(p+n)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n-1}. \quad (2.5)$$

On the other side,

$$|f'(z)| \geq \frac{p}{|z-w|^{p+1}} - \sum_{n=1}^{\infty}(p+n)a_{p+n}|z-w|^{p+n-1},$$

$$|f'(A)| \geq \frac{p}{|A|^{p+1}} - \sum_{n=1}^{\infty}(p+n)a_{p+n}|A|^{p+n-1}, \text{ since } |A|<1,\ A \neq 0.$$

Using (2.2), we have

$$|f'(A)| \geq \frac{p}{|A|^{p+1}} - \frac{\mu\gamma(\alpha-\beta)(p+n)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n-1}. \quad (2.6)$$

Using (2.5) and (2.6),

$$\frac{p}{|A|^{p+1}} - \frac{\mu\gamma(\alpha-\beta)(p+n)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n-1}$$

$$\leq |f'(A)| \leq \frac{p}{|A|^{p+1}} + \frac{\mu\gamma(\alpha-\beta)(p+n)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}|A|^{p+n-1}.$$

3. Radius of Starlikeness and Convexity

Theorem 3.1. *If* $f(z) \in S^{**}(p)$, *then* $f(z)$ *is p-valently starlike in* $0 < |A| < R$ *of order in* $\delta,\ 0 \leq \delta < p,$ *where*

$$R = \inf_n \left\{\frac{(p-\delta)[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)(3p+n-\delta)}\right\}^{\frac{1}{p+n}},$$

$|A| < 1,\ \ A \neq 0.$

Proof. $f(z)$ is p-valently starlike of order $\delta\,(0 \leq \delta < p)$, if $-R_e\left\{(z-w)\frac{f'(z)}{f(z)}\right\}$ $> \delta$, that is, if $\left|(z-w)\frac{f'(z)}{f(z)} + p\right| < p - \delta$ which simplifies to

$$\sum_{n=1}^{\infty} \frac{(3p+n-\delta)}{p-\delta} a_{p+n} |A|^{p+n} \leq 1, \tag{3.1}$$

$|A| < 1, \quad A \neq 1.$

By using Theorem 2.1, we have

$$a_{p+n} \leq \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)]} (n, p)^k. \tag{3.2}$$

Using (3.1) and (3.2), we get

$$\sum_{n=1}^{\infty} \frac{(3p+n-\delta)\mu\gamma(\alpha-\beta)}{(p-\delta)[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n, p)^k} |A|^{p+n} \leq 1,$$

that is,

$$|A|^{p+n} \leq \frac{(p-\delta)[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n, p)^k}{\mu\gamma(\alpha-\beta)(3p+n-\delta)}.$$

Thus

$$|A| \leq R = \inf_n \left\{ \frac{(p-\delta)[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n, p)^k}{\mu\gamma(\alpha-\beta)(3p+n-\delta)} \right\}^{\frac{1}{p+n}},$$

$|A| < 1, \quad A \neq 0.$

Theorem 3.2. *If* $f(z) \in S_w^{**}(p)$, *then* $f(z)$ *is p-valently convex in* $0 < |A| < R$ *of order* δ, $0 \leq \delta < p$,

$$R = \inf_n \left\{ \frac{p(p-\delta)[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n, p)^k}{(p+n)(p+n+\delta)\mu\gamma(\alpha-\beta)} \right\}^{\frac{1}{p+n-1}},$$

$|A| < 1, \quad A \neq 0.$

Proof. $f(z)$ is p-valently starlike of order δ $(0 \leq \delta < p)$, if

$$-\mathrm{Re}\left\{1 + \frac{(z-w) f''(z)}{f'(z)}\right\} > \delta$$

that is if

$$\left| (z-w)\frac{f''(z)}{f'(z)} + 1 + p \right| \le p - \delta,$$

that is if

$$\left| \frac{\sum_{n=1}^{\infty} (p+n)(2p+n) a_{p+n} (z-w)^{p+n-1}}{p(z-w)^{-p+1} + \sum_{n=1}^{\infty} (p+n) a_{p+n} (z-w)^{p+n-1}} \right| \le p - \delta$$

which simplifies to

$$\sum_{n=1}^{\infty} \frac{(p+n)(p+n+\delta)}{p(p-\delta)} a_{p+n} |A|^{p+n-1} \le 1, \tag{3.3}$$

$|A| < 1,\ \ A \ne 1.$

By using Theorem 2.1, we have

$$a_{p+n} \le \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta) + \mu\gamma(\alpha-\beta)](n,\ p)^k}. \tag{3.4}$$

Using (3.3) and (3.4), we get

$$\sum_{n=1}^{\infty} \frac{(p+n)(p+n+\delta)\mu\gamma(\alpha-\beta)}{p(p-\delta)[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k} |A|^{p+n-1} \le 1,$$

that is,

$$|A|^{p+n-1} \le \frac{p(p-\delta)[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{(p+n)(p+n+\delta)\mu\gamma(\alpha-\beta)}.$$

Thus

$$|A| \le R = \inf_n \left\{ \frac{p(p-\delta)[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{(p+n)(p+n+\delta) - \mu\gamma(\alpha-\beta)} \right\}^{\frac{1}{p+n-1}},$$

$|A| < 1,\ \ A \ne 0.$

4. Extreme Points

Theorem 4.1. *Let* $f_0(z) = \dfrac{1}{(z-w)^p}$ *and*

$$f_{p+n}(z) = \frac{1}{(z-w)^p} - \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}(z-w)^{p+n}.$$

Then $f(z) \in S_w^{**}(p)$ *if and only if it can be expressed in the form*

$$f(z) = \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_{p+n} f_{p+n}(z),$$

where $\lambda_{p+n} \geq 0$ *and* $\lambda_p + \sum_{n=1}^{\infty} \lambda_{p+n} = 1.$

Proof. Let us assume that

$$f(z) = \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_{p+n} f_{p+n}(z)$$

$$= \frac{1}{(z-w)^p} - \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}(z-w)^{p+n}.$$

Then we have

$$\sum_{n=1}^{\infty} \frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)}$$

$$\cdot \lambda_{p+n} \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k} = \sum_{n=1}^{\infty} \lambda_{p+n} \leq 1.$$

Hence $f(z) \in S_w^{**}(p).$

Conversely, we assume that $f(z)$ given by (1.5) is in the class $S_w^{**}(p).$

From Corollary, $a_{p+n} \leq \dfrac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}.$

Setting $\lambda_{p+n} = a_{p+n} \dfrac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)}$, $p, n \in N$, and

$\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_{p+n}$, we have $f(z) = \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_{p+n} f_{p+n}(z)$.

5. Closure Theorem

In continuation of our study, we shall prove that the class $S_w^{**}(p)$ is closed under convex linear combination.

Theorem 5.1. *Let* $f_j(z) \in S_w^{**}(p)$, $j = 1, 2, \ldots, m$. *Then*

$$g(z) = \sum_{j=1}^{m} c_j f_j(z) \in S_w^{**}(p).$$

For

$$f_j(z) = \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} a_{p+n}(z-w)^{p+n},\ \textit{where}\ \sum_{j=1}^{m} c_j = 1.$$

Proof. Now

$$g(z) = \sum_{j=1}^{m} c_j f_j(z)$$

$$= \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty}\sum_{j=1}^{m} c_j a_{p+n,\ j}(z-w)^{p+n}$$

$$= \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} \xi_{p+n}(z-w)^{p+n},\ \text{where}\ \xi_{p+n} = \sum_{j=1}^{m} c_j a_{p+n,\ j}.$$

Thus $g(z) \in S_w^{**}(p)$ if

$$\sum_{n=1}^{\infty} \frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)} \xi_{p+n} \le 1,$$

that is if

$$\sum_{n=1}^{\infty}\sum_{j=1}^{m}\frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^{k}}{\mu\gamma(\alpha-\beta)}c_{j}a_{p+n,\ j}$$

$$\leq \sum_{j=1}^{m}c_{j}\sum_{n=1}^{\infty}\frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^{k}}{\mu\gamma(\alpha-\beta)}a_{p+n,\ j}$$

$$\leq \sum_{j=1}^{m}c_{j}\leq 1,\ \text{since}\ \frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^{k}}{\mu\gamma(\alpha-\beta)}a_{p+n,\ j}\leq 1.$$

That is $g(z)\in S_{w}^{**}(p)$ if $f_{j}(z)\in S_{w}^{**}(p)$.

Theorem 5.2. *Let*

$$f_{j}(z)=\frac{1}{(z-w)^{p}}-\sum_{n=1}^{\infty}a_{p+n}(z-w)^{p+n},\ a_{p+n}\geq 0,\ j=1,\ 2,\ \ldots,\ m$$

be in the class $S_{w}^{**}(p)$. *Then the function* $h(z)=\frac{1}{m}\sum_{j=1}^{m}f_{j}(z)$ *also belongs to the class* $S_{w}^{**}(p)$.

Proof. We have

$$h(z)=\frac{1}{m}\sum_{j=1}^{m}f_{j}(z)$$

$$=\frac{1}{(z-w)^{p}}-\sum_{n=1}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m}a_{p+n,\ j}\right)(z-w)^{p+n}$$

$$=\frac{1}{(z-w)^{p}}-\sum_{n=1}^{\infty}e_{k}(z-w)^{p+n},\ \text{where}\ e_{k}=\frac{1}{m}\sum_{j=1}^{m}a_{p+n,\ j},$$

since $f_{j}(z)\in S_{w}^{**}(p)$ from Theorem 2.1, we have

$$\sum_{n=1}^{\infty}\frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^{k}}{\mu\gamma(\alpha-\beta)}a_{p+n,\ j}\leq 1. \tag{5.1}$$

Now, $h(z) \in S_w^{**}(p)$.

Since

$$\sum_{n=1}^{\infty} \frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)} e_k$$

$$= \sum_{n=1}^{\infty} \frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)} \frac{1}{m} \sum_{j=1}^{m} a_{p+n,\ j}$$

$$= \frac{1}{m} \sum_{j=1}^{m} \sum_{n=1}^{\infty} \frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)} a_{p+n,\ j}$$

$$\leq \frac{1}{m} \sum_{j=1}^{m} 1, \text{ using (5.1)}$$

$$\leq 1.$$

6. Hadamard Product and Inclusion Property

If

$$f(z) = \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} a_{p+n}(z-w)^{p+n}$$

and

$$g(z) = \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} b_{p+n}(z-w)^{p+n}$$

in $S_w^{**}(p)$, then convolution $(f * g)(z)$ is defined by

$$(f * g)(z) = \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} a_{p+n} b_{p+n}(z-w)^{p+n}. \tag{6.1}$$

Theorem 6.1. *Let $f(z)$ and $g(z)$ belong to $S_w^{**}(p)$. Then $(f * g)(z) \in S_w^{**}(p)$ for*

$$\xi \geq \frac{\mu^2\gamma(\alpha-\beta)(2p+n)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)]^2(n,\ p)^k+\mu^2\gamma(\alpha-\beta)[2p+n-\gamma(\alpha-\beta)]}.$$

Proof. $f(z),\ g(z) \in S_w^{**}(p)$ and so

$$\sum_{n=1}^{\infty} \frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)} a_{p+n} \leq 1 \tag{6.2}$$

and

$$\sum_{n=1}^{\infty} \frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)} b_{p+n} \leq 1. \tag{6.3}$$

We need to find the smallest number ξ such that

$$\sum_{n=1}^{\infty} \frac{[(2p+n)(1-\xi\beta)+\xi\gamma(\alpha-\beta)](n,\ p)^k}{\xi\gamma(\alpha-\beta)} a_{p+n}b_{p+n} \leq 1. \tag{6.4}$$

Using Cauchy Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)} \sqrt{a_{p+n}b_{p+n}} \leq 1. \tag{6.5}$$

Thus it is enough to show that

$$\frac{[(2p+n)(1-\xi\beta)+\xi\gamma(\alpha-\beta)]}{\xi(\alpha-\beta)} \sqrt{a_{p+n}b_{p+n}}$$

$$\leq \frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)]}{\mu(\alpha-\beta)}$$

$$\sqrt{a_{p+n}b_{p+n}} \leq \frac{[\xi(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)]}{\mu[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)]}. \tag{6.6}$$

From (6.5), we have

$$\sqrt{a_{p+n}b_{p+n}} \leq \frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}. \tag{6.7}$$

In view of (6.6) and (6.7), it is enough to show that

$$\frac{\mu\gamma(\alpha-\beta)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k} \leq \frac{\xi[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)]}{\mu[(2p+n)(1-\xi\beta)+\xi\gamma(\alpha-\beta)]}.$$

Simplifying, we get

$$\xi \geq \frac{\mu^2\gamma(\alpha-\beta)(2p+n)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)]^2(n,\ p)^k + \mu^2\gamma(\alpha-\beta)[2p+n-\gamma(\alpha-\beta)]}.$$

Next, we study inclusion theorem for the class $S_w^{**}(p)$.

Theorem 6.2. *Let* $f(z),\ g(z) \in S_w^{**}(p)$. *Then*

$$h(z) = \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty}(a_{p+n}^2 + b_{p+n}^2)(z-w)^{p+n}$$

is in $S_w^{**}(p)$, *where*

$$\delta \geq \frac{2\mu\gamma(\alpha-\beta)(2p+n)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)]^2(n,\ p)^k + 2\mu\gamma(\alpha-\beta)[\beta-\gamma(\alpha-\beta)]}.$$

Proof. Let $f(z),\ g(z) \in S_w^{**}(p)$. Then

$$\sum_{n=1}^{\infty}\left\{\frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)}\right\}^2 a_{p+n}^2$$

$$\leq \sum_{n=1}^{\infty}\left\{\frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)}a_{p+n}\right\}^2 \leq 1. \tag{6.8}$$

Similarly,

$$\sum_{n=1}^{\infty}\left\{\frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)}\right\}^2 b_{p+n}^2$$

$$\leq \sum_{n=1}^{\infty}\left\{\frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)}b_{p+n}\right\}^2 \leq 1. \tag{6.9}$$

We have to show that $h(z) \in S_w^{**}(p)$ by replacing μ by δ,

$$\sum_{n=1}^{\infty}\frac{[(2p+n)(1-\delta\beta)+\delta\gamma(\alpha-\beta)](n,\ p)^k}{\delta\gamma(\alpha-\beta)}(a_{p+n}^2 + b_{p+n}^2) \leq 1. \tag{6.10}$$

Adding (6.8) and (6.9), we get

$$\sum_{n=1}^{\infty} \frac{1}{2}\left[\frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)]^2 (n,\ p)^k}{\mu\gamma(\alpha-\beta)}\right]^2 (a_{p+n}^2 + b_{p+n}^2) \leq 1. \qquad (6.11)$$

In view of (6.10) and (6.11), it is enough to show that

$$\frac{[(2p+n)(1-\delta\beta)+\delta\gamma(\alpha-\beta)](n,\ p)^k}{\delta\gamma(\alpha-\beta)}$$

$$\leq \frac{1}{2}\left[\frac{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)](n,\ p)^k}{\mu\gamma(\alpha-\beta)}\right]^2.$$

This simplifies to

$$\delta \geq \frac{2\mu\gamma(\alpha-\beta)(2p+n)}{[(2p+n)(1-\mu\beta)+\mu\gamma(\alpha-\beta)]^2 (n,\ p)^k + 2\mu\gamma(\alpha-\beta)[\beta-\gamma(\alpha-\beta)]}.$$

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