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NONPARAMETRIC ESTIMATION OF SIBUYA'S MEASURE OF LOCAL DEPENDENCE FOR TIME SERIES

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Abstract

The measures of local dependence are calculated on a function of the support of the variables, and may provide more information about the structure of dependence than the global coefficients which result in a single numerical value. In this paper, we study the function of Sibuya [11] in the context of stationary stochastic process both in the univariate and bivariate cases. We rewrite this function in terms of copula studying its properties. Two kernel smoothed estimators are proposed and their properties are derived. Monte Carlo experiments considering a stationary vector autoregressive process of order one and a Gaussian process are

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performed. Empirical illustrations are done using the series of daily returns of CAC 40 and DAX and of IBOVESPA (the Brazilian index).

1. Introduction

Some measures of local dependence in the context of random variables found in the literature are the function of local dependence of Holland and Wang [5], the curve of correlation of Bjerve and Doksum [2], the function of local dependence of Bairamov et al. [1], and also the copula and copula density (Nelsen [9]). Another measure of interest is the function of dependence of Sibuya [11] for which Kolev et al. [6] developed additional properties and Latif and Morettin [7] proposed kernel estimators verifying their weak convergence and finite sample properties.

In the context of stochastic processes, few studies have been developed to assess measures of local dependence. Among them, we can mention the study of copulas for time series addressed by Fermanian and Scaillet [3] who used kernel estimators, and by Morettin et al. [8] who used estimators through wavelets.

The purpose of this paper is to study the function of Sibuya [11] for time series considering univariate and bivariate stationary processes. This function is written in terms of copula and its properties are verified. We propose two kernel smoothed estimators, one through distribution functions and other through copula for which the weak convergence are derived and the consistency is verified for the first one. The behaviour of this local measure is assessed through simulations and some information on finite sample performance are evaluated through a Monte Carlo study. Empirical illustrations are provided considering the daily returns of CAC 40 (French stock market index) and DAX (German stock index), and also the IBOVESPA (Brazilian stock index).

The plan of this paper is as follows: in Section 2, we describe the function of Sibuya and its properties. In Section 3, we study the function of Sibuya for a bivariate stationary process, where its properties, kernel estimators and asymptotic behaviors are derived. Also, we show some Monte Carlo results considering a VAR(1) model with Gaussian innovations and an empirical illustration considering the daily returns of CAC 40 and DAX. A similar study is done in Section 4 for a univariate stationary process, where the Monte Carlo simulations are performed considering a Gaussian process and the empirical illustration concerns the IBOVESPA index. Section 5 contains the conclusions.

2. Definitions

Aiming to extend the concept of extreme statistics from the univariate to bivariate case, Sibuya [11] proposed a function of dependence between two continuous random variables, which relates the joint distribution with their corresponding marginal distributions.

Let X and Y be continuous random variables with joint distribution function $F(x, y) = P[X \leq x, Y \leq y]$ and marginals $F_1(x) = P[X \leq x]$ and $F_2(y) = P[Y \leq y]$. Thus, the function of dependence of Sibuya, $\Lambda = \Lambda(F_1(x), F_2(y))$, is given by

$$\Lambda(F_1(x), F_2(y)) = \frac{F(x, y)}{F_1(x)F_2(y)}, \quad \forall (x, y) \in \mathbb{R}^2, \tag{1}$$

where $F_1(x) > 0$ and $F_2(y) > 0$. If $F_1(x) = 0$ or $F_2(y) = 0$, then $\Lambda(F_1(x), F_2(y))$ will be defined if the limit

$$\lim_{F_1(x)\to 0} F(x, y)/(F_1(x)F_2(y)) \quad \text{or} \quad \lim_{F_2(y)\to 0} F(x, y)/(F_1(x)F_2(y))$$

exists, respectively.

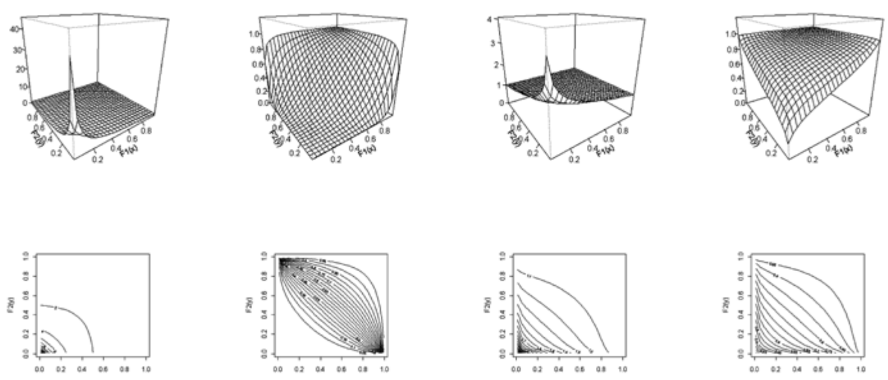

Figure 1. Plots (top panel) and contour curves (bottom panel) of the Sibuya's measure of local dependence given by formula (1) for (X, Y) with standard normal distribution and coefficient of correlation equals to $\rho = 0.80$, $\rho = -0.80$, $\rho = 0.20$ and $\rho = -0.20$, respectively.

Since this function of dependence corresponds to the association between the events $(X \leq x)$ and $(Y \leq y)$ with (x, y) belonging to the support of the distribution of (X, Y) (see Sibuya [11]), we call it as *local*. That is, we are not referring to the concept of local in terms of conditional distributions on the point (x, y).

The theoretical behaviour of this measure of local dependence for a bivariate random vector with standard normal distribution and correlation coefficient equal to +0.80, –0.80, +0.20 and –0.20, are shown, respectively, in the plots (top panel) and contour curves (bottom panel) of Figure 1.

The properties for Λ defined by Sibuya [11] are given below, see also Kolev et al. [6]. First, let us remember that X and Y are PQD-positively quadrant dependent (NQD-negatively quadrant dependent), if for all $(x, y) \in \mathbb{R}^2$, $F(x, y) \geq (\leq)$ $F_1(x)F_2(y)$.

(i)

$$\max\left(0, \frac{F_1(x) + F_2(y) - 1}{F_1(x)F_2(y)}\right) \leq \Lambda(F_1(x), F_2(y))$$

$$\leq \min\left(\frac{1}{F_1(x)}, \frac{1}{F_2(y)}\right), \quad \forall (x, y) \in \mathbb{R}^2;$$

(ii) $\Lambda(F_1(x), F_2(y)) = 1, \ \forall (x, y) \in \mathbb{R}^2$ if and only if X and Y are independent;

(iii) If X and Y are PQD(NQD)[(1)], then $\Lambda(F_1(X), F_2(Y)) \geq 1 (\leq 1)$ almost surely;

(iv) Let $\varphi(\cdot)$ and $\psi(\cdot)$ be arbitrary functions of X and Y, respectively, such that $\varphi^{-1}(\cdot)$ and $\psi^{-1}(\cdot)$ exist. Using the notation

$$\overline{F}_1(x) = 1 - F_1(x), \ \ \overline{F}_2(y) = 1 - F_2(y), \ \ F_{\varphi(X)}(x) = P[\varphi(X) \leq x],$$

$$F_{\psi(Y)}(y) = P[\psi(Y) \leq y], \ \ \Lambda_{\varphi(X)\psi(Y)}(x, y) = \Lambda(F_{\varphi(X)}(x), F_{\psi(Y)}(y))$$

and $\Lambda_{XY}(\varphi^{-1}(x), \psi^{-1}(y)) = \Lambda(F_1(\varphi^{-1}(x)), F_2(\psi^{-1}(y)))$, with S being the support of (X, Y), then:

(a) If $\varphi(\cdot)$ and $\psi(\cdot)$ are increasing functions on the support of X and Y, respectively, then

$$\Lambda_{\varphi(X)\psi(Y)}(x, y) = \Lambda_{XY}(\varphi^{-1}(x), \psi^{-1}(y)), \quad \forall (x, y) \in S;$$

(b) If $\varphi(\cdot)$ is a decreasing function on the support of X and $\psi(\cdot)$ is an increasing function on the support of Y, then

$$\Lambda_{\varphi(X)\psi(Y)}(x, y) = \frac{1}{\overline{F}_1(\varphi^{-1}(x))} - \frac{F_1(\varphi^{-1}(x))}{\overline{F}_1(\varphi^{-1}(x))}\Lambda_{XY}(\varphi^{-1}(x), \psi^{-1}(y)), \quad \forall(x, y) \in S;$$

(c) If $\varphi(\cdot)$ is an increasing function on the support of X and $\psi(\cdot)$ is a decreasing function on the support of Y, then

$$\Lambda_{\varphi(X)\psi(Y)}(x, y) = \frac{1}{\overline{F}_2(\psi^{-1}(y))} - \frac{F_2(\psi^{-1}(y))}{\overline{F}_2(\psi^{-1}(y))}\Lambda_{XY}(\varphi^{-1}(x), \psi^{-1}(y)), \ \forall(x, y) \in S;$$

(d) If $\varphi(\cdot)$ and $\psi(\cdot)$ are decreasing functions on the support of X and Y, respectively, then

$$\Lambda_{\varphi(X)\psi(Y)}(x, y) = \frac{1 - F_1(\varphi^{-1}(x)) - F_2(\psi^{-1}(y))}{\overline{F}_1(\varphi^{-1}(x))\overline{F}_2(\psi^{-1}(y))}$$

$$+ \frac{F_1(\varphi^{-1}(x))F_2(\psi^{-1}(y))}{\overline{F}_1(\varphi^{-1}(x))\overline{F}_2(\psi^{-1}(y))}\Lambda_{XY}(\varphi^{-1}(x), \psi^{-1}(y)), \ \forall(x, y) \in S;$$

(v) If $\rho_{XY} = 0\ (> 0, < 0)$, then $\Lambda(F_1(X), F_2(Y)) = 1(> 1, < 1)$ in the bivariate normal case.

Kolev et al. [6] observed that the property (iv) shows that Λ at point (x, y) is not ordinally invariant under monotone transformations, and this is because it depends on the functions $\varphi^{-1}(\cdot)$ and $\psi^{-1}(\cdot)$, and possible functions of the marginals.

These authors also suggested the empirical estimator Λ_n of Λ which is built by the plug-in method using the empirical joint distribution and empirical marginal distributions. However, some method of smoothing is recommended to obtain better results for small samples and also to facilitate the graphical display.

We rewrite formula (1) in the following equivalent expression:

$$\Lambda(u, v) = \frac{C(u, v)}{uv}, \quad \forall(u, v) \in (0, 1]^2, \tag{2}$$

where $C(u, v)$ denotes the copula of (X, Y) on the point $(u, v) = (F_1(x), F_2(y))$.

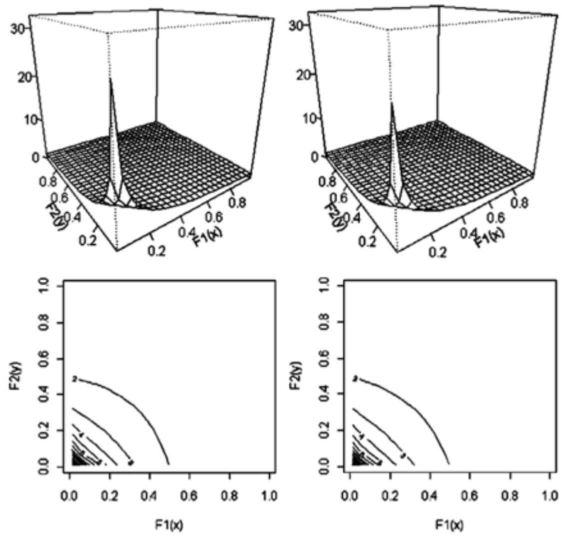

Figure 2. Considering the VAR(1) model with $\mu = (3.05,\ 6.44)'$, $vec(\mathbf{\Gamma}(0)) = (1.13,\ 1.49,\ 1.49,\ 3.99)'$ (correlation 0.70) and Gaussian innovations, we have the plots and contour curves of Λ_0 (formula (3) or (4)) and of $\hat{\Lambda}_0$ (formula (5)), respectively.

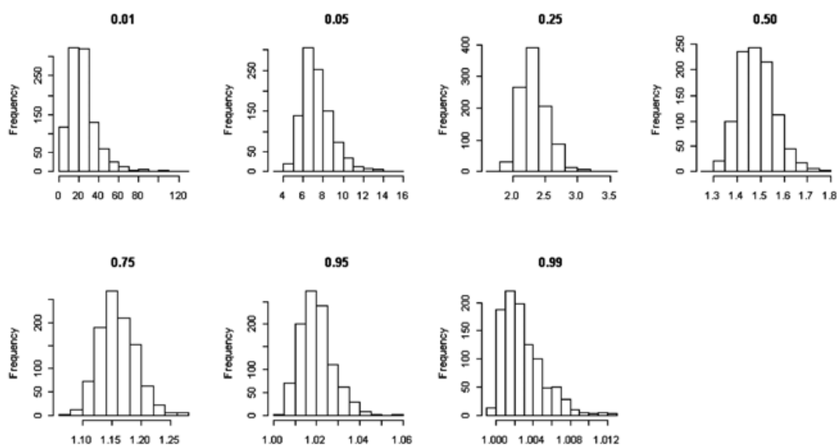

Figure 3. Considering the VAR(1) model with $\mu = (3.05,\ 6.44)'$, $vec(\mathbf{\Gamma}(0)) = (1.13,\ 1.49,\ 1.49,\ 3.99)'$ (correlation 0.70) and Gaussian innovations, we have on some bivariate points of the secondary diagonal the histograms of $\hat{\Lambda}_0$ (formula (5)).

The properties (i) to (v) of Λ written as formula (1) remain valid for formula (2), with the necessary adjustments. Considering formula (2), we observe that $\Lambda_{\varphi(X)\psi(Y)}(u, v) = \Lambda_{XY}(u, v)$ if $\varphi(\cdot)$ and $\psi(\cdot)$ are strictly increasing.

For Λ given through formula (1) or (2), Latif and Morettin [7] suggested the use of kernel smoothed estimators whose weak convergence were obtained for large samples and also for finite samples via bootstrap procedure, and their finite sample performance was studied through simulations.

3. Contemporaneous Function of Local Dependence

Let $\{(X_t, Y_t), t \in \mathbb{Z}\}$ be a strictly stationary process with continuous values. Let $F(x, y)$ be the joint distribution of X_t and Y_t, $F_1(x)$ and $F_2(y)$ their marginal distributions, and $C(u, v)$ the corresponding copula, $\forall t \in \mathbb{Z}$. Thus, the Sibuya's measure of local dependence given by formula (1) can be represented by

$$\Lambda_0(F_1(x), F_2(y)) = \frac{F(x, y)}{F_1(x)F_2(y)}, \quad \forall (x, y) \in \mathbb{R}^2, \forall t \in \mathbb{Z}, \tag{3}$$

with $F_1(x) > 0$ and $F_2(y) > 0$, and for formula (2), we used the notation

$$\Lambda_0(u, v) = \frac{C(u, v)}{uv}, \quad \forall (u, v) \in (0, 1]^2, \forall t \in \mathbb{Z}, \tag{4}$$

where

$$C(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)),$$

$$F_1^{-1}(u) = \inf\{x \in \mathbb{R} : F_1(x) \geq u\},$$

$$F_2^{-1}(v) = \inf\{y \in \mathbb{R} : F_2(y) \geq v\}.$$

For this process, the properties (i) to (v) are satisfied by formula (3) and also by formula (4) with the necessary adjustments. We call the formulas (3) and (4) as contemporaneous function of local dependence through distribution functions and copula, respectively.

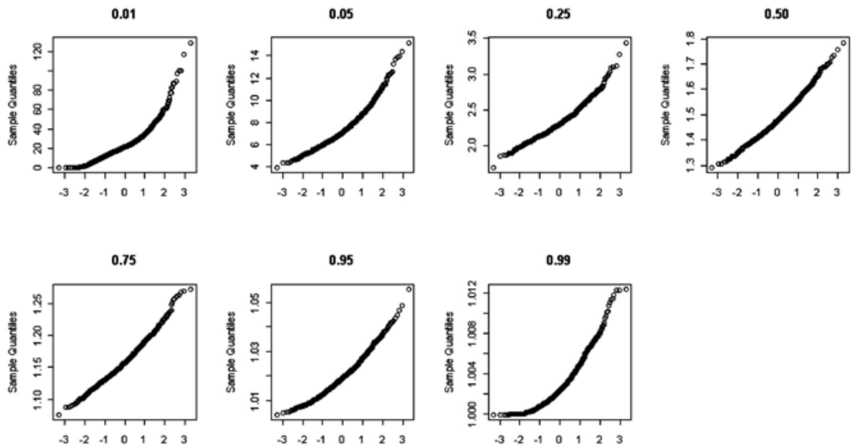

Figure 4. Considering the VAR(1) model with $\mu = (3.05,\ 6.44)'$, $vec(\mathbf{\Gamma}(0)) = (1.13,\ 1.49,\ 1.49,\ 3.99)'$ (correlation 0.70) and Gaussian innovations, we have for some bivariate points of the secondary diagonal the qqplot of $\hat{\Lambda}_0$ (formula (5)).

3.1. Estimators and their properties

We propose for time series the same kind of smoothed estimators for random variables (Latif and Morettin [7]). Let $((X_1,\ Y_1),\ ...,\ (X_T,\ Y_T))$ be observed from the process under study, then formula (3) can be estimated by the plug-in method using kernels, that is

$$\hat{\Lambda}_0(\hat{F}_1(x),\ \hat{F}_2(y)) = \frac{\hat{F}(x,\ y)}{\hat{F}_1(x)\hat{F}_2(y)}, \quad \forall (x,\ y) \in \mathbb{R}^2, \tag{5}$$

with $\hat{F}_1(x),\ \hat{F}_2(y) > 0,$ where

$$\hat{F}(x,\ y) = \frac{1}{T}\sum_{t=1}^{T} K\left(\frac{x - X_t}{h_1},\ \frac{y - Y_t}{h_2}\right),$$

$$\hat{F}_1(x) = \frac{1}{T}\sum_{t=1}^{T} K_1\left(\frac{x - X_t}{h_1}\right),$$

$$\hat{F}_2(y) = \frac{1}{T}\sum_{t=1}^{T} K_2\left(\frac{y - Y_t}{h_2}\right),$$

with K such that $K(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} k(u, v)dudv$, $K(x, y; h_1, h_2) = K\left(\frac{x}{h_1}, \frac{y}{h_2}\right)$ where $k : \mathbb{R}^2 \to \mathbb{R}$ is a kernel function such that $\int\int k(u, v)dudv = 1$, $k(x, y; h_1, h_2) = \frac{1}{h_1 h_2} k\left(\frac{x}{h_1}, \frac{y}{h_2}\right)$, and $h_i > 0$, $i = 1, 2$, are functions of T such that $h_i \to 0$ as $T \to \infty$. Also, $K_i(w; h_i)$, $i = 1, 2$, are the univariate versions of K. For formula (4), a smoothed kernel estimator is

$$\tilde{\Lambda}_0(u, v) = \frac{\hat{C}(u, v)}{uv}, \quad \forall (u, v) \in (0, 1]^2, \tag{6}$$

where $\hat{C}(u, v)$, $\hat{F}_1^{-1}(u)$ and $\hat{F}_2^{-1}(v)$ are the corresponding kernel estimators of copula and quantiles.

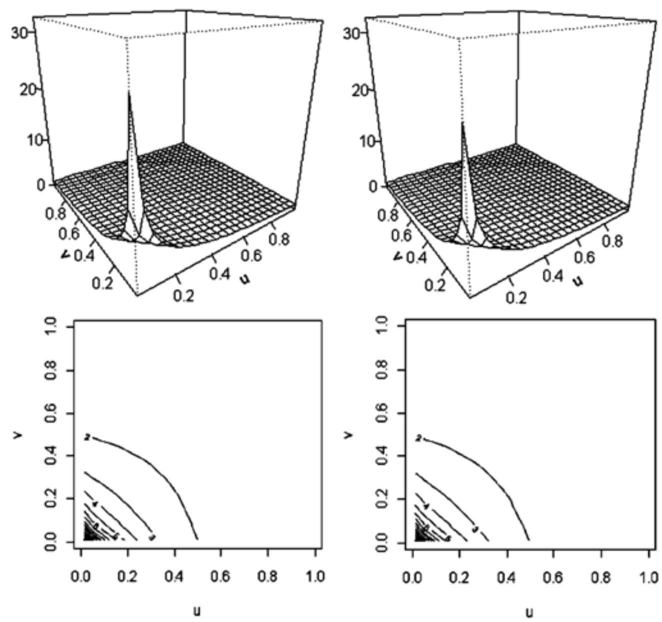

Figure 5. Considering the VAR(1) model with $\mu = (3.05, 6.44)'$, $vec(\mathbf{\Gamma}(0)) = (1.13, 1.49, 1.49, 3.99)'$ (correlation 0.70) and Gaussian innovations, we have the plots of perspective and contour curves of Λ_0 (formula (3) or (4)) and of $\tilde{\Lambda}_0$ (formula (6)), respectively.

In order to establish the properties of the estimators, consider the following regularity conditions where $h_* = \max(h_1, h_2)$:

(**C1**) $Th_*^2 \to 0$ as $T \to \infty$;

(**C1′**) $Th_*^4 \to 0$ as $T \to \infty$ and the bivariate kernel k is even;

(**C2**) the kernel k has a compact support;

(**C3**) the process (X_t, Y_t) is α-mixing with coefficients $\alpha_T = o(T^{-a})$ for some $a > 1$ as $T \to \infty$;

(**C4**) the marginal distributions F_i, $i = 1, 2$, are continuously differentiable on the intervals $[F_i^{-1}(a) - \varepsilon; F_i^{-1}(b) + \varepsilon]$ for every $0 < a < b < 1$ and some $\varepsilon > 0$, with positive derivatives f_i. Moreover, the first partial derivatives of F exist and are Lipschitz continuous on the product of these intervals.

As pointed out by Fermanian and Scaillet [3], the condition (C2) can be weakened by controlling the tails of k, for example, by assuming that $\sup_j |k_j(x)| \leq (1 + |x|)^{-\alpha}$ for every x and some $\alpha > 0$. This type of assumption is satisfied by most kernels, in particular by the Gaussian kernel (see Robinson [10]).

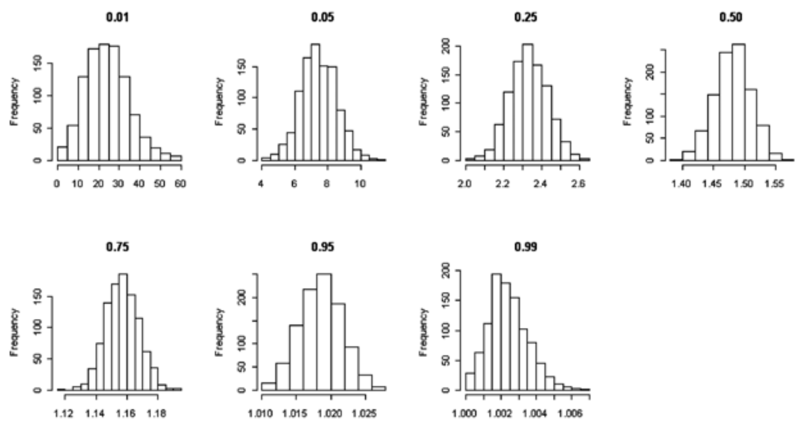

Figure 6. Considering the VAR(1) model with $\mu = (3.05, 6.44)'$, $vec(\mathbf{\Gamma}(0)) = (1.13, 1.49, 1.49, 3.99)'$ (correlation 0.70) and Gaussian innovations, on some bivariate points of the secondary diagonal we have the histograms of $\tilde{\Lambda}_0$ (formula (6)).

Theorem 1. *Let* $\{(X_t, Y_t), t \in \mathbb{Z}\}$ *be a strictly stationary process with continuous values*. *Under assumptions* (C1) (*or* (C1′)) *to* (C4), *we have*

$$\{W^F(x, y) \equiv \sqrt{T}(\hat{\Lambda}_0 - \Lambda_0)(x, y), x, y \in \mathbb{R} : F_1(x), F_2(y), \hat{F}_1(x), \hat{F}_2(y) > 0\}$$

converges weakly to a centered Gaussian process in $l^\infty(\mathbb{R}^2)$.

Proof. Assuming the conditions (C1) (or (C1′)) to (C4) valid for the process $\{(X_t, Y_t), t \in \mathbb{Z}\}$, Theorem 2 of Fermanian and Scaillet [3] is valid, i.e., $T^{1/2}(\hat{F} - F)$ tends weakly to a centered Gaussian process $\mathbb{G}$ in $l^\infty(\mathbb{R}^2)$ (the space of almost surely bounded functions on $\mathbb{R}^2$), endowed with the sup-norm, with covariance function

$$Cov(\mathbb{G}(x, y), \mathbb{G}(x', y')) = \sum_{t \in \mathbb{Z}} Cov(\mathbf{1}\{(X_0, Y_0) \leq (x, y)\}, \mathbf{1}\{(X_t, Y_t) \leq (x', y')\}),$$

and also $T^{1/2} \sup|\hat{F}(x, y) - F(x, y)| = o_P(1)$. Then, $\hat{F}_1(x) \xrightarrow{P} F_1(x)$ and $\hat{F}_2(y) \xrightarrow{P} F_2(y)$ as $T \to \infty$. Therefore, we can write

$$\hat{\Lambda}_0(\hat{F}_1(x), \hat{F}_2(y)) - \Lambda_0(F_1(x), F_2(y))$$

$$= [\hat{F}(x, y)/(\hat{F}_1(x)\hat{F}_2(y))] - [F(x, y)/(F_1(x)F_2(y))]$$

asymptotically as $(\hat{F}(x, y) - F(x, y))/(F_1(x)F_2(y))$, and then the process W^F converges weakly to a centered Gaussian process. □

Theorem 2. *Let* $\{(X_t, Y_t), t \in \mathbb{Z}\}$ *be a strictly stationary process with continuous values*. *Assuming the conditions* (C1) (*or* (C1′)) *to* (C4) *are valid*, *we have*

$$\hat{\Lambda}_0(\hat{F}_1(x), \hat{F}_2(y)) \xrightarrow[T \to \infty]{P} \Lambda_0(F_1(x), F_2(y)) \textit{ for every } (x, y) \in \mathbb{R}^2,$$

with $F_1(x), F_2(y), \hat{F}_1(x), \hat{F}_2(y) > 0$.

Proof. By Theorem 2 of Fermanian and Scaillet (see the proof of the previous theorem), we have that $\hat{F}(x, y) \xrightarrow{P} F(x, y)$, $\hat{F}_1(x) \xrightarrow{P} F_1(x)$ and $\hat{F}_2(y) \xrightarrow{P} F_2(y)$ as $T \to \infty$, and the result follows. □

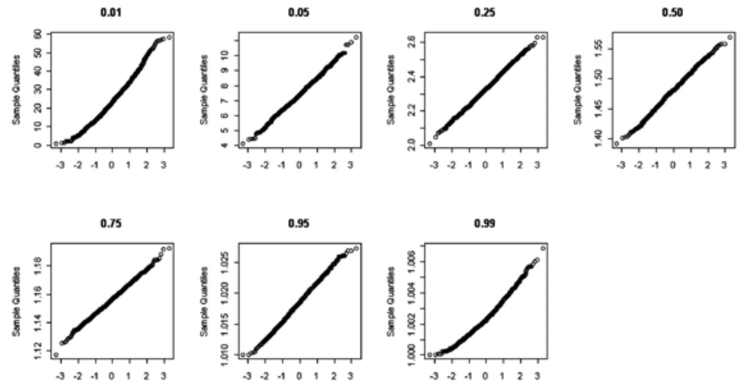

Figure 7. Considering the VAR(1) model with $\mu = (3.05, 6.44)'$, $vec(\mathbf{\Gamma}(0)) = (1.13, 1.49, 1.49, 3.99)'$ (correlation 0.70) and Gaussian innovations for some bivariate points of the secondary diagonal we have the qqplot of $\tilde{\Lambda}_0$ (formula (6)).

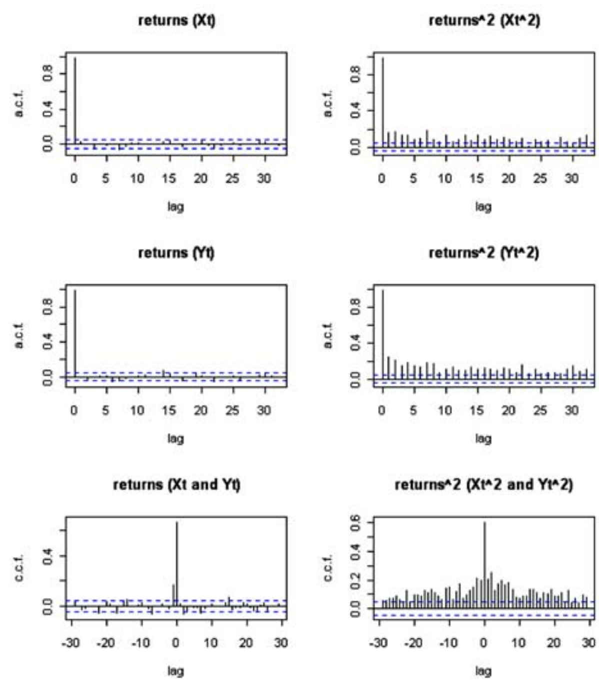

Figure 8. Autocorrelation functions of returns and squared returns of CAC 40 (X_t) and DAX (Y_t) (from 3/01/94 to 08/08/00), and their cross correlation function.

Theorem 3. *Let* $\{(X_t, Y_t), t \in \mathbb{Z}\}$ *be a strictly stationary process with continuous values. Under assumptions* (C1) (*or* (C1′)) *to* (C4),

$$\{W^C(u, v) \equiv \sqrt{T}(\tilde{\Lambda}_0 - \Lambda_0)(u, v), 0 < u, v \leq 1\}$$

converges weakly to a centered Gaussian process in $l^\infty((0, 1]^2)$.

Proof. Theorem 3 of Fermanian and Scaillet is valid, i.e., $T^{1/2}(\hat{C} - C)$ tends weakly to a centered Gaussian process $\phi'(\mathbb{G})$ in $l^\infty([0, 1]^2)$ endowed with the sup-norm (where $\mathbb{G}$ is the Gaussian process described in the proof of Theorem 1 and ϕ is a Hadamard-differentiable map), whose limiting process is given by

$$\phi'(\mathbb{G})(u, v) = \mathbb{G}(F_1^{-1}(u), F_2^{-1}(v)) - \frac{\partial C(u, v)}{\partial u}\mathbb{G}(u, \infty) - \frac{\partial C(u, v)}{\partial v}\mathbb{G}(\infty, v).$$

Using the continuous mapping theorem, the weakly convergence of the process W^C is obtained. □

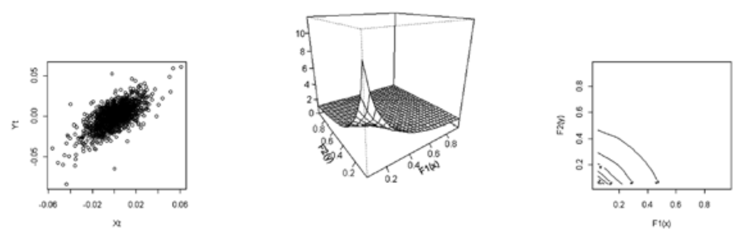

Figure 9. Scatter plot, plots and contour curves of $\hat{\Lambda}_0$ (formula (5)) for the daily returns of CAC 40 (X_t) and DAX (Y_t) from 3/01/94 to 8/08/00.

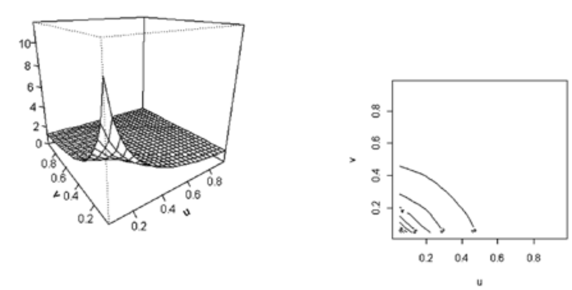

Figure 10. Plots and contour curves of $\tilde{\Lambda}_0$ (formula (6)) for the daily returns of CAC 40 (X_t) and DAX (Y_t) from 3/01/94 to 8/08/00.

3.2. Simulation results

The simulations are based on a bivariate stationary vector autoregressive model of order 1, VAR(1), with contemporary correlation 0.70. Thus, the contemporaneous function of local dependence Λ_0 is calculated (by formula (3) or (4)) and then estimated firstly by formula (5) and then by formula (6), using 1,000 replications of Monte Carlo (with series of sizes 500 and 1,000) from which the biases, mean squared errors and histograms are computed. The simulations are done on a bivariate grid with 25×25 points, 98% of the central data, using the product of two Gaussian kernels and optimal bandwidth according to Hansen [4].

Consider the stationary VAR(1) process

$$\mathbf{Z}_t = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{Z}_{t-1} + \varepsilon_t,$$

where $\mathbf{Z}_t = (X_t, Y_t)$, $\mathbf{\Phi}_0 = (1, 1)'$, $vec(\mathbf{\Phi}_1) = (0.25, 0.2, 0.2, 0.75)'$ and $\varepsilon_t \sim N(\mathbf{0}, \mathbf{\Sigma})$ with $vec(\mathbf{\Sigma}) = (0.75, 0.5, 0.5, 1.25)'$, i.e., $\mathbf{Z}_t$ has dependent components. Thus, the parameters of the Gaussian stationary distribution are $\mu = (3.05, 6.44)'$ and $vec(\mathbf{\Gamma}(0)) = (1.13, 1.49, 1.49, 3.99)'$ (correlation 0.70).

The plots and contour curves on the left of Figure 2 refer to Λ_0, and the others refer to $\hat{\Lambda}_0$. We can see that the left plots are very similar to the right ones.

The theoretical values, the biases and mean squared errors of $\hat{\Lambda}_0$ are shown in Table 1 of the Appendix. The estimator computed on the lower quantiles (0.01 and 0.05) exhibits the higher biases and mean squared errors. For some bivariate grid points on the secondary diagonal, we can see the histograms (Figure 3) and qqplots (Figure 4) of $\hat{\Lambda}_0$, which show non-normality as we move away from the central point. Comparing the results of simulations for series of sizes 500 (not shown) and 1,000 (Table 1 of the Appendix), we see that with increasing T, the biases and the mean squared errors decrease.

Next, we evaluate $\tilde{\Lambda}_0$ given through (6). For the 1,000 series of size 1,000 considered previously, we see in Figure 5 that $\tilde{\Lambda}_0$ is very similar to $\hat{\Lambda}_0$ (Figure 2). Table 2 (see Appendix) shows that the biases and the mean squared errors have small values, except at the first two points of the bivariate grid. In Figures 6 and 7, we see some histograms and qqplots of $\tilde{\Lambda}_0$ which show that their behaviour is closer to normality than $\hat{\Lambda}_0$.

The estimator $\tilde{\Lambda}_0$ when compared with $\hat{\Lambda}_0$ shows similar biases, and smaller mean squared errors and less departure from normality. One possible explanation of this behaviour is that $\hat{\Lambda}_0$ is a ratio between estimators whereas $\tilde{\Lambda}_0$ is not, and the inverse $\hat{F}_i^{-1}$, $i = 1, 2,$ used for the calculation of $\hat{C}$ and then of $\tilde{\Lambda}_0$, was calculated on a very fine grid.

3.3. Empirical illustration

To illustrate the implementation of the proposed estimators, we consider the daily log returns of CAC 40 (Cotation Assistée en Continu) and DAX (Deutscher Aktien IndeX) from 03/01/94 to 8/08/00, i.e., 1722 observations.

The returns of the CAC 40 (X_t) and DAX (Y_t) have contemporaneous correlation coefficient 0.67, Spearman's rho 0.60 and Kendalls tau 0.44. The a.c.f. and c.c.f. of squared returns for these series are shown in Figure 8. In Figure 9, we see their scatter plot which shows positive linear relationship, plots and contour curves of the estimated contemporaneous function of local dependence given by formula (5) indicating positive dependence. Applying the estimator $\tilde{\Lambda}_0$ to the series under study, we obtain the plots of Figure 10, whose behaviour is very similar to those of Figure 9.

4. Function of Local Autodependence

Let $\{X_t, t \in \mathbb{Z}\}$ be a strictly stationary process with continuous values. We know that the univariate distribution is time invariant and the distribution of $(X_t, X_{t+\tau})$, $\forall t, \tau \in \mathbb{Z}$, only depends on the lag τ. Thus, let $F(x_1, x_2; \tau) = P[X_t \leq x_1, X_{t+\tau} \leq x_2]$ be the joint distribution of $(X_t, X_{t+\tau})$ with marginal distributions $F_1(x)$. Then the contemporaneous function of local dependence given by (3) can be rewritten as

$$\Lambda_\tau(F_1(x_1), F_1(x_2)) = \frac{F(x_1, x_2; \tau)}{F_1(x_1)F_1(x_2)}, \quad \forall (x_1, x_2) \in \mathbb{R}^2, \forall \tau \in \mathbb{Z}, \tag{7}$$

with $F_1(x_i) > 0$, $i = 1, 2$, and the function (4) as

$$\Lambda_\tau(u_1, u_2) = \frac{C(u_1, u_2)}{u_1 u_2}, \quad \forall (u_1, u_2) \in (0, 1]^2, \forall \tau \in \mathbb{Z}, \tag{8}$$

where

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_1^{-1}(u_2); \tau),$$

$$F_1^{-1}(u_i) = \inf\{x_i \in \mathbb{R} : F_1(x_i) \geq u_i\}, \quad i = 1, 2.$$

The properties (i), (iii), (iv)-a and (iv)-d $(\text{where } \varphi(\cdot) = \psi(\cdot))$, besides (v) (in this case, if $\{X_t\}$ is a Gaussian process) of Section 2 are satisfied by (7) and (8), with the necessary adjustments. Also, the following properties are valid for these functions, which are written in terms of (7) only for convenience. The proofs are immediate.

(a) $\Lambda_\tau(F_1(x_1), F_1(x_2)) \xrightarrow[\tau \to \infty]{} 1$, $\forall (x_1, x_2) \in \mathbb{R}^2$, $\tau \in \mathbb{Z}$, if the process is α-mixing;

(b) $\Lambda_{-\tau}(F_1(x_1), F_1(x_2)) = \Lambda_\tau(F_1(x_1), F_1(x_2))$, $\forall (x_1, x_2) \in \mathbb{R}^2$, $\forall \tau \in \mathbb{Z}$.

We call the formulas (7) and (8) as function of local autodependence written through distribution functions and copula, respectively.

4.1. Estimators and their properties

One smoothed estimator for the formula (7) using the plug-in method is

$$\hat{\Lambda}_\tau(\hat{F}_1(x_1), \hat{F}_1(x_2)) = \frac{\hat{F}(x_1, x_2; \tau)}{\hat{F}_1(x_1)\hat{F}_1(x_2)}, \quad \forall (x_1, x_2) \in \mathbb{R}^2, \forall \tau \in \mathbb{Z}, \tag{9}$$

with $\hat{F}_1(x_i) > 0$, where

$$\hat{F}(x_1, x_2; \tau) = \frac{1}{T - \tau} \sum_{t=1}^{T-\tau} K\left(\frac{x_1 - X_t}{h}, \frac{x_2 - X_{t+\tau}}{h}\right),$$

$$\hat{F}_1(x) = \frac{1}{T} \sum_{t=1}^{T} K_1\left(\frac{x - X_t}{h}\right).$$

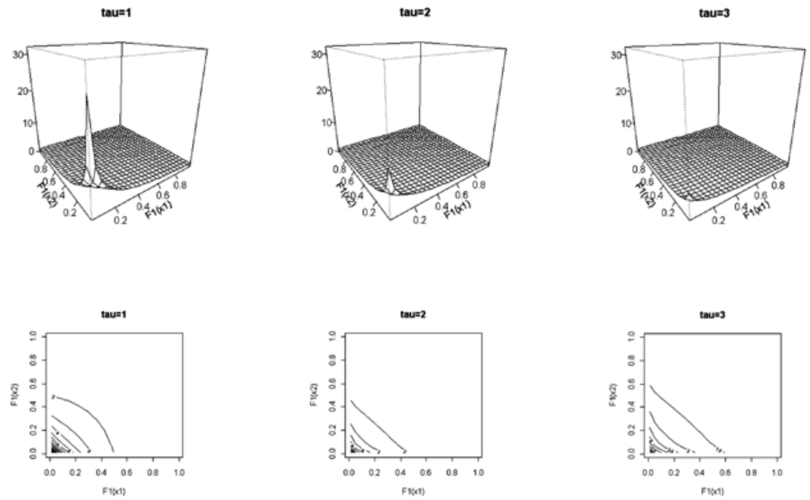

Figure 11. The first three plots refer to Λ_τ (formula (7) or (8)), $\tau = 1, 2, 3,$ and the next three plots are their contour curves, considering a Gaussian process with zero mean, unit variance and autocorrelation structure from an AR(1) model with $\phi_1 = 0.70$.

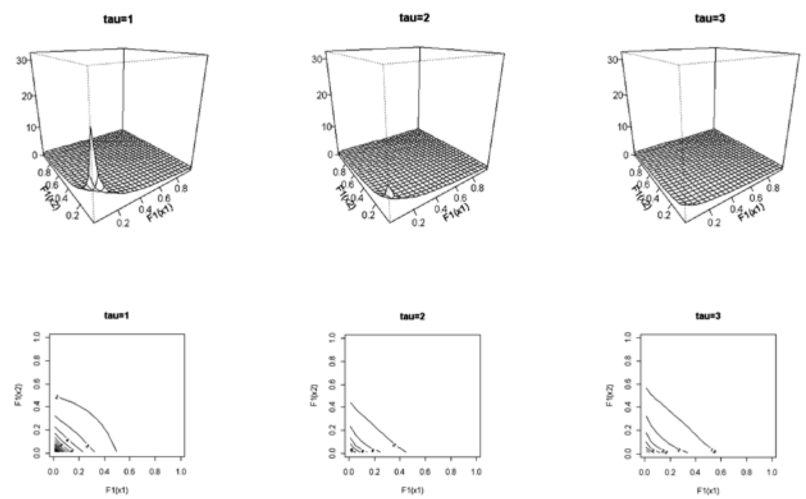

Figure 12. The first three plots refer to $\hat{\Lambda}_\tau$ (formula (9)), $\tau = 1, 2, 3,$ and the next three plots are their contour curves, considering 1,000 series with $T = 1{,}000$ observed from a Gaussian process with zero mean, unit variance and autocorrelation structure from an AR(1) model with $\phi_1 = 0.70$.

Here $h = h(T)$ denote the bandwidth such that $h \to 0$ as $T \to \infty$, and K and K_1 are as before. To estimate the formula (8), we suggest

$$\tilde{\Lambda}_\tau(u_1, u_2) = \frac{\hat{C}(u_1, u_2)}{u_1 u_2}, \quad \forall (u_1, u_2) \in (0, 1]^2, \forall \tau \in \mathbb{Z}, \tag{10}$$

where

$$\hat{C}(u_1, u_2) = \hat{F}(\hat{F}_1^{-1}(u_1), \hat{F}_1^{-1}(u_2); \tau),$$

$$\hat{F}_1^{-1}(u) = \inf\{x \in \mathbb{R} : \hat{F}_1(x) \geq u\}$$

are kernel estimators of the corresponding functions.

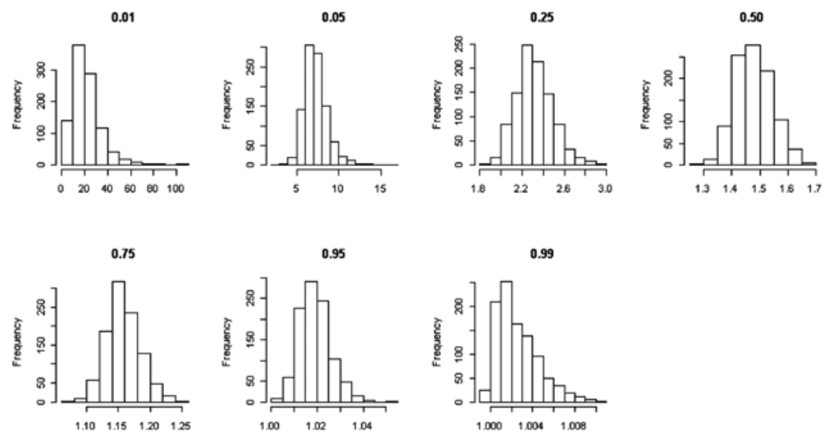

Figure 13. Histograms of $\hat{\Lambda}_\tau$ (formula (9)), $\tau = 1$, for a Gaussian process with zero mean, unit variance and autocorrelation structure from an AR(1) model with $\phi_1 = 0.70$.

Consider the following regularity conditions:

(**C1**) $Th^2 \to 0$ as $T \to \infty$;

(**C1′**) $Th^4 \to 0$ as $T \to \infty$, and the bivariate kernel k is even;

(**C2**) the kernel k has a compact support;

(**C3**) the process (X_t) is α-mixing with coefficients $\alpha_T = o(T^{-a})$ for some $a > 1$ as $T \to \infty$;

(**C4**) the marginal distribution F_1 is continuously differentiable on the interval $[F_1^{-1}(a) - \varepsilon;\ F_1^{-1}(b) + \varepsilon]$ for every $0 < a < b < 1$ and some $\varepsilon > 0$, with positive derivatives *f*. Moreover, the first partial derivatives of *F* exist and are Lipschitz continuous on the product of these intervals.

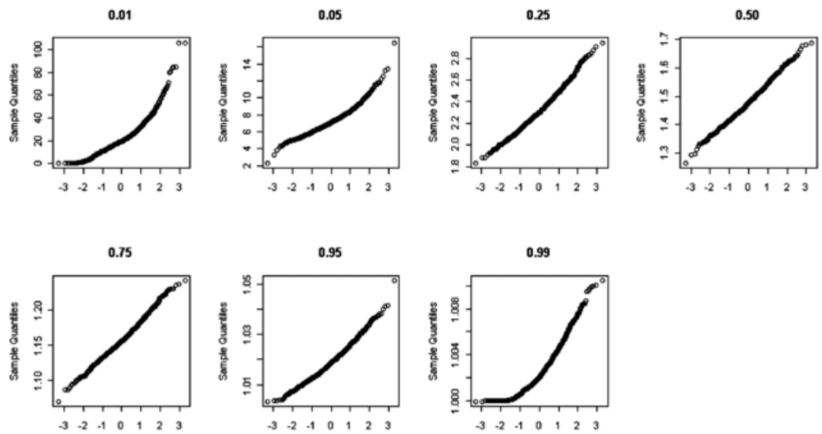

Figure 14. qqplots of $\hat{\Lambda}_\tau$ (formula (9)), $\tau = 1$, for a Gaussian process with zero mean, unit variance and autocorrelation structure from an AR(1) model with $\phi_1 = 0.70$.

Theorem 4. *Let* $\{X_t,\ t \in \mathbb{Z}\}$ *be a strictly stationary process with continuous values*. *Assuming* (C1) (*or* (C1′)) *to* (C4) *valid, we have*

$$\{W^F(x_1,\ x_2) \equiv \sqrt{T}(\hat{\Lambda}_\tau - \Lambda_\tau)(x_1,\ x_2),\ x_1,\ x_2 \in \mathbb{R} : F_1(x),\ \hat{F}_1(x) > 0,\ \tau \in \mathbb{Z}\}$$

converges weakly to a centered Gaussian process in $l^\infty(\mathbb{R}^2)$.

Theorem 5. *Let* $\{X_t,\ t \in \mathbb{Z}\}$ *be a strictly stationary process with continuous values*. *Under assumptions* (C1) (*or* (C1′)) *to* (C4), *we have*

$$\hat{\Lambda}_\tau(\hat{F}_1(x_1),\ \hat{F}_1(x_2)) \xrightarrow[T\to\infty]{P} \Lambda_\tau(F_1(x_1),\ F_1(x_2)) \text{ for every } (x_1,\ x_2) \in \mathbb{R}^2,\ \tau \in \mathbb{Z}$$

such that $F_1(x),\ \hat{F}_1(x) > 0$.

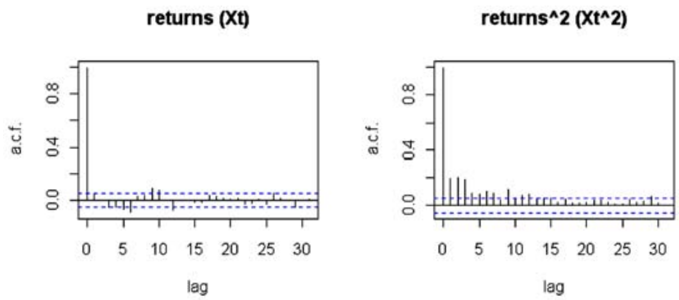

Figure 15. Autocorrelation function of returns and squared returns of IBOVESPA (X_t), from 3/01/95 to 27/12/00.

Theorem 6. *Let* $\{X_t,\ t \in \mathbb{Z}\}$ *be a strictly stationary process with continuous values. Assuming* (C1) (*or* (C1′)) *to* (C4) *are valid, we have*

$$\{W^C(u_1, u_2) \equiv \sqrt{T}(\tilde{\Lambda}_\tau - \Lambda_\tau)(u_1, u_2),\ 0 < u_1, u_2 \leq 1\}$$

converges weakly to a centered Gaussian process in $l^\infty((0, 1]^2)$.

The proofs of the previous three theorems are similar to those of Theorems 1 to 3.

4.2. Simulations

The functions of local autodependence given by (7) or (8) were calculated and their corresponding estimators (9) and (10) were evaluated through 1,000 replications of Monte Carlo with series of sizes 500 and 1,000 observed from a stationary Gaussian process with zero mean, unit variance and autocorrelation structure from an AR(1) model with $\phi_1 = 0.70$. Some statistics and graphics of each estimator are obtained. The remaining specifications of these simulations were the same as the Subsection 3.2.

First, we evaluated $\hat{\Lambda}_\tau$ (formula (9)) with series of size 1,000. In Figure 11, we see the plots and contour curves of Λ_τ, $\tau = 1, 2, 3$, which show the decay of the dependence along lags. The same types of plots for $\hat{\Lambda}_\tau$, $\tau = 1, 2, 3$, are shown in Figure 12, in which we see a similar behaviour. Considering this estimator on lag 1, we see in Table 3 of the Appendix that the biases and mean squared errors are small, except for the quantiles 0.01 and 0.05. In Figures 13 and 14, we see the distribution of the estimator on some bivariate points of the grid through histograms and qqplots, which show a behaviour near to normality for most of central points. For all lags, the biases and the mean squared errors generally decrease with increasing *T*. Comparing the estimator for these three lags, we verify that increasing the lag, the biases and mean squared errors generally decrease.

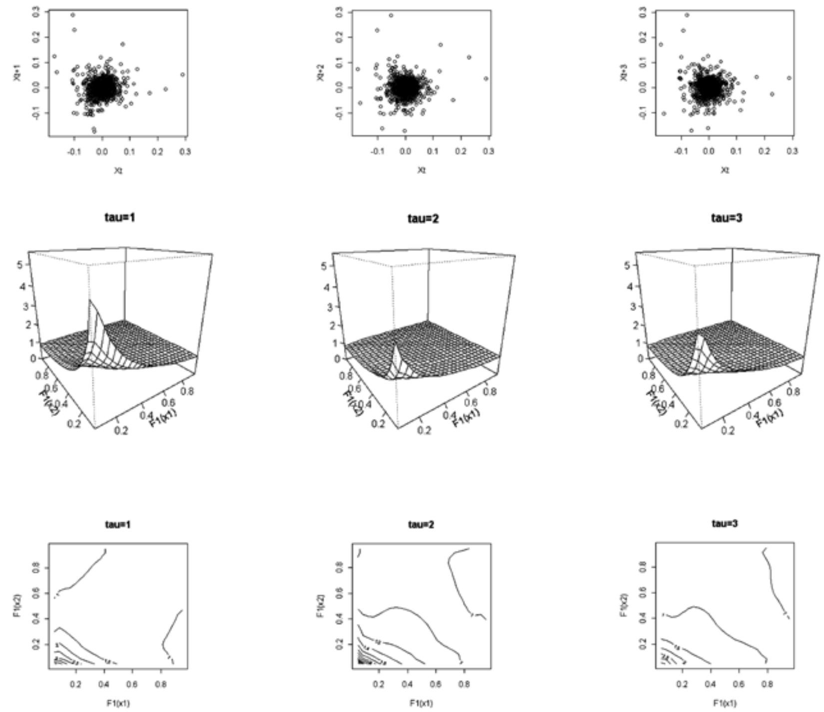

Figure 16. Scatter plot, plots of the function of local autodependence estimated through distribution functions (formula (9)) and their contour curves for the daily returns of IBOVESPA (from 3/01/95 to 27/12/00) on the lags 1 to 3.

The behaviour of $\tilde{\Lambda}_\tau$ formula (10), $\tau = 1, 2, 3,$ are similar to their theoretical functions (Figure 11) and also to the estimator $\hat{\Lambda}_\tau$ (see Figure 12).

Comparing the performance of the two estimators, $\tilde{\Lambda}_\tau$ presented lower bias on the quantile $(0.01, 0.01)$ (and greater frequency of slight higher bias on the remaining bivariate grid points), lower mean squared error and greater proximity to normality. Also, it can be seen that as *T* increases, both statistics tend to be less different.

4.3. Empirical illustration

Next, we estimate the function of local autodependence on the first three lags considering the daily log returns of IBOVESPA from 3/01/95 to 27/12/00, i.e., 1,498 observations.

In Figure 15, we see the autocorrelation function of the returns and squared returns, which shows dependence along the lags. For the first three lags of the returns, we see in Figure 16 the scatter plots, the plots and contour curves of $\hat{\Lambda}_\tau$ (given by formula (9)). We see that the positive dependence presents a smooth decay from lag 1 to lag 2. A similar result holds for the estimator given by formula (10).

5. Further Remarks

In the context of stationary time series, we study the estimation of the Sibuya's function which can be seen as a measure of local dependence of quadrants.

For this measure, two smoothed nonparametric estimators using kernels were proposed, one written in terms of distribution functions and other in terms of copulas, for both bivariate and univariate strictly stationary processes. Using the results of the functional limit theorem of copulas for these processes and the continuous mapping theorem, we obtained the weak convergence of the estimators and also the consistence for the estimator written in terms of distribution functions.

The finite sample properties of the estimators were verified through Monte Carlo simulations considering series observed from a VAR(1) model with contemporary correlation 0.70 and from a Gaussian process with autocorrelation structure from an AR(1) model with $\phi_1 = 0.70.$ In general, in both cases, the estimator through copula presents smaller bias on the quantile $(0.01, 0.01),$ smaller mean squared errors and less departure from normality.

Some applications of the proposed estimators to real data were done.

Appendix

Table 1. Actual value of Λ_0 (formula (3) or (4)), bias and mean squared error of $\hat{\Lambda}_0$ (formula (5)) at some points of the bivariate grid considering 1,000 series of size 1,000 observed from the VAR(1) model with $\mu = (3.05, 6.44)'$, $vec(\mathbf{\Gamma}(0)) = (1.13, 1.49, 1.49, 3.99)'$ (correlation 0.70) and Gaussian innovations

		0.01	**0.05**	**0.25**	**0.50**	**0.75**	**0.95**	**0.99**
Actual	**0.01**	27.083	12.293	3.728	1.989	1.342	1.054	1.010
Bias		–3.136	–0.839	–0.054	0.003	0.006	0.003	0.001
MSE		222.144	14.851	0.299	0.030	0.004	0.000	0.000
Actual	**0.05**	12.293	7.820	3.342	1.947	1.339	1.054	1.010
Bias		–0.858	–0.437	–0.082	–0.006	0.005	0.003	0.001
MSE		13.695	2.545	0.179	0.026	0.004	0.000	0.000
Actual	**0.25**	3.728	3.342	2.385	1.742	1.312	1.053	1.010
Bias		–0.088	–0.094	–0.058	–0.020	–0.001	0.002	0.001
MSE		0.218	0.133	0.047	0.014	0.003	0.000	0.000
Actual	**0.50**	1.989	1.947	1.742	1.497	1.254	1.051	1.010
Bias		–0.002	–0.011	–0.019	–0.013	–0.003	0.002	0.001
MSE		0.018	0.016	0.011	0.006	0.002	0.000	0.000
Actual	**0.75**	1.342	1.339	1.312	1.254	1.162	1.043	1.009
Bias		0.005	0.004	–0.001	–0.004	–0.003	0.001	0.001
MSE		0.002	0.002	0.002	0.002	0.001	0.000	0.000
Actual	**0.95**	1.054	1.054	1.053	1.051	1.043	1.020	1.006
Bias		0.003	0.003	0.003	0.002	0.001	0.000	0.000
MSE		0.000	0.000	0.000	0.000	0.000	0.000	0.000
Actual	**0.99**	1.010	1.010	1.010	1.010	1.009	1.006	1.003
Bias		0.001	0.001	0.001	0.001	0.001	0.000	0.000
MSE		0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 2. Actual value of Λ_0 (formula (3) or (4)), bias and mean squared error of $\tilde{\Lambda}_0$ (formula (6)) at some points of the bivariate grid considering 1,000 series of size 1,000 observed from the VAR(1) model with $\mu = (3.05, 6.44)'$, $vec(\mathbf{\Gamma}(0)) = (1.13, 1.49, 1.49, 3.99)'$ (correlation 0.70) and Gaussian innovations

		0.01	**0.05**	**0.25**	**0.50**	**0.75**	**0.95**	**0.99**
Actual	**0.01**	27.083	12.293	3.728	1.989	1.342	1.054	1.010
Bias		–3.082	–0.828	–0.071	–0.010	–0.003	–0.002	–0.002
MSE		119.657	7.574	0.055	0.002	0.000	0.000	0.000
Actual	**0.05**	12.293	7.820	3.342	1.947	1.339	1.054	1.010
Bias		–0.907	–0.396	–0.083	–0.016	–0.003	–0.001	–0.001
MSE		7.979	1.331	0.039	0.002	0.000	0.000	0.000
Actual	**0.25**	3.728	3.342	2.385	1.742	1.312	1.053	1.010
Bias		–0.088	–0.084	–0.054	–0.024	–0.006	0.000	0.000
MSE		0.071	0.039	0.013	0.002	0.000	0.000	0.000
Actual	**0.50**	1.989	1.947	1.742	1.497	1.254	1.051	1.010
Bias		–0.011	–0.016	–0.022	–0.016	–0.007	–0.001	0.000
MSE		0.002	0.002	0.002	0.001	0.000	0.000	0.000
Actual	**0.75**	1.342	1.339	1.312	1.254	1.162	1.043	1.009
Bias		–0.003	–0.003	–0.006	–0.007	–0.005	–0.001	0.000
MSE		0.000	0.000	0.000	0.000	0.000	0.000	0.000
Actual	**0.95**	1.054	1.054	1.053	1.051	1.043	1.020	1.006
Bias		–0.002	–0.001	0.000	–0.001	–0.001	–0.001	0.000
MSE		0.000	0.000	0.000	0.000	0.000	0.000	0.000
Actual	**0.99**	1.010	1.010	1.010	1.010	1.009	1.006	1.003
Bias		–0.002	–0.001	0.000	0.000	0.000	0.000	0.000
MSE		0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 3. Actual value of Λ_τ (formula (7) or (8)), bias and mean squared error of $\hat{\Lambda}_\tau$ (formula (9)) for $\tau = 1$ at some points of the bivariate grid considering 1,000 series of size 1,000 observed from a Gaussian process with zero mean, unit variance and autocorrelation structure from an AR(1) model with $\phi_1 = 0.70$

		0.01	**0.05**	**0.25**	**0.50**	**0.75**	**0.95**	**0.99**
Actual	**0.01**	26.684	12.167	3.718	1.989	1.342	1.054	1.010
Bias		–4.954	–0.994	–0.078	–0.006	0.005	0.002	0.001
MSE		197.255	12.619	0.182	0.016	0.002	0.000	0.000
Actual	**0.05**	12.167	7.743	3.327	1.945	1.339	1.054	1.010
Bias		–1.016	–0.508	–0.090	–0.013	0.003	0.002	0.001
MSE		11.908	2.161	0.112	0.014	0.002	0.000	0.000
Actual	**0.25**	3.718	3.327	2.374	1.737	1.311	1.053	1.010
Bias		–0.074	–0.088	–0.057	–0.022	–0.002	0.002	0.001
MSE		0.183	0.110	0.033	0.009	0.002	0.000	0.000
Actual	**0.50**	1.989	1.945	1.737	1.494	1.252	1.051	1.010
Bias		–0.003	–0.012	–0.021	–0.015	–0.004	0.001	0.001
MSE		0.016	0.014	0.009	0.004	0.001	0.000	0.000
Actual	**0.75**	1.342	1.339	1.311	1.252	1.161	1.043	1.009
Bias		0.005	0.003	–0.001	–0.005	–0.004	0.000	0.000
MSE		0.002	0.002	0.002	0.001	0.001	0.000	0.000
Actual	**0.95**	1.054	1.054	1.053	1.051	1.043	1.019	1.006
Bias		0.002	0.002	0.002	0.001	0.000	0.000	0.000
MSE		0.000	0.000	0.000	0.000	0.000	0.000	0.000
Actual	**0.99**	1.010	1.010	1.010	1.010	1.009	1.006	1.003
Bias		0.001	0.001	0.001	0.001	0.000	0.000	0.000
MSE		0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 4. Actual value of Λ_τ (formula (7) or (8)), bias and mean squared error of $\tilde{\Lambda}_\tau$ (formula (10)) for $\tau = 1$ at some points of the bivariate grid considering 1,000 series of size 1,000 observed from a Gaussian process with zero mean, unit variance and autocorrelation structure from an AR(1) model with $\phi_1 = 0.70$

		0.01	**0.05**	**0.25**	**0.50**	**0.75**	**0.95**	**0.99**
Actual	**0.01**	26.684	12.167	3.718	1.989	1.342	1.054	1.010
Bias		–4.602	–0.985	–0.079	–0.012	–0.003	–0.002	–0.002
MSE		135.016	7.850	0.062	0.002	0.000	0.000	0.000
Actual	**0.05**	12.167	7.743	3.327	1.945	1.339	1.054	1.010
Bias		–0.950	–0.449	–0.084	–0.018	–0.003	–0.001	–0.001
MSE		7.815	1.419	0.042	0.002	0.000	0.000	0.000
Actual	**0.25**	3.718	3.327	2.374	1.737	1.311	1.053	1.010
Bias		–0.080	–0.080	–0.053	–0.023	–0.006	0.000	0.000
MSE		0.062	0.040	0.012	0.002	0.000	0.000	0.000
Actual	**0.50**	1.989	1.945	1.737	1.494	1.252	1.051	1.010
Bias		–0.009	–0.017	–0.022	–0.017	–0.008	–0.001	0.000
MSE		0.001	0.002	0.002	0.001	0.000	0.000	0.000
Actual	**0.75**	1.342	1.339	1.311	1.252	1.161	1.043	1.009
Bias		–0.003	–0.003	–0.006	–0.008	–0.006	–0.001	0.000
MSE		0.000	0.000	0.000	0.000	0.000	0.000	0.000
Actual	**0.95**	1.054	1.054	1.053	1.051	1.043	1.019	1.006
Bias		–0.002	–0.001	0.000	–0.001	–0.002	–0.001	–0.001
MSE		0.000	0.000	0.000	0.000	0.000	0.000	0.000
Actual	**0.99**	1.010	1.010	1.010	1.010	1.009	1.006	1.003
Bias		–0.002	–0.001	0.000	0.000	0.000	–0.001	0.000
MSE		0.000	0.000	0.000	0.000	0.000	0.000	0.000

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