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TRAPPING OF WATER WAVES BY SUBMERGED CYLINDERS AT HIGH FREQUENCIES

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Abstract

As is well-known, submerged horizontal cylinders can serve as waveguides for surface water waves. For large values of the wave number k in the direction of the cylinders, there is only one trapped wave. We construct asymptotics of these trapped modes and their frequencies as $k \to \infty$ in the case of n submerged cylinders by means of reducing the initial problem to a system of integral equations on the boundaries and then solving them using the Neumann series.

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1. Introduction

It is well-known that submerged horizontal cylinders can serve as waveguides for water waves. The first result in this direction was obtained by Ursell [10] for a cylinder of circular cross-section. Later, it was discovered that horizontal "bumps" on the bottom (underwater ridges) can also trap waves (see [2, 5]). In [2], Bonnet-Ben Dhia and Joly proved that for large values of the wave number k in the direction of the ridge, there is only one trapped mode. Their proof can be straightforwardly carried over to the case of one or several parallel submerged cylinders. They also showed that the distance of the frequency of this mode to the cut-off frequency is exponentially small in k and that the corresponding eigenfunction decays exponentially slowly in the direction orthogonal to the ridge. In [7], we have constructed explicitly this trapped mode for large values of k in the case of a ridge and also indicated the formula for the frequency in the case of one submerged cylinder. In [8], we obtained this trapped mode for large values of k in the case of one or two submerged cylinders.

In the present paper, we obtain a generalization of those results to the case of n submerged cylinders. We note that the limit $k \to \infty$ is to some extent analogous to the limit of small height of the underwater ridge: surface water waves decay exponentially with depth h as $\exp(-kh)$, so the influence of an object submerged at a finite distance from the surface is small, just as the influence of a small bump on the bottom. The problem of the ridge of small height was treated in [11], where a close analogy of the problem of water waves and small perturbations of the one-dimensional Schrödinger equation is established. The latter problem was studied by a number of authors (we mention, for example, [3, 6, 9], and, in the context of water waves, [4]). In our case, a technique similar to that of [11] yields the desired result. We note that in contrast to [11] the asymptotics turns out to be exponential, i.e., the distance of the trapped wave frequency to the cut-off frequency is exponentially small in k. This fact seemingly could have rendered the problem quite complicated from the point of view of asymptotic expansions, but, since in fact, we construct an exact convergent expansion, no additional difficulties arise.

2. Mathematical Formulation and Main Result

The geometry of the problem is as follows: we assume that

$$\Gamma_i = \{x = x_i(t),\ y = y_i(t),\ t \in [-\pi,\ \pi]\}$$

with smooth $x_i(t)$ and $y_i(t)$, $y_i(t) < 0$,

$$(x_i')^2 + (y_i')^2 \neq 0, \tag{2.1}$$

and $\max y_i(t) = y_i(0)$, $y_i''(0) < 0$, $x_i'(0) > 0$, where y is the vertical coordinate, x is the horizontal coordinate orthogonal to the direction of the cylinders, $\Gamma_1, \Gamma_2, ..., \Gamma_n$ describe curves bounding their cross-sections. We assume that these cross-sections are disjoint and

$$\sqrt{(x_i(t) - x_j(t'))^2 + (y_i(t) - y_j(t'))^2} \geq d > 0, \quad i \neq j. \tag{2.2}$$

$\Gamma_F = \{(x, 0) : x \in \mathbb{R}\}$ is the free surface. The water layer Ω is the domain exterior to $\Gamma_1, \Gamma_2, ..., \Gamma_n$ and lying below Γ_F (see Figure 1).

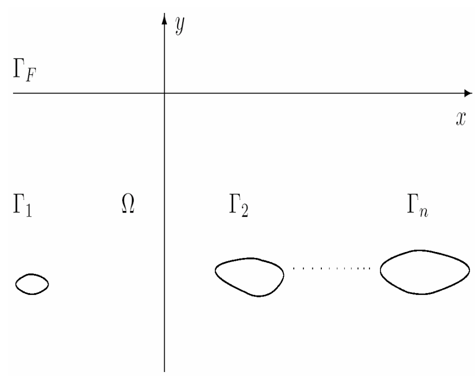

Figure 1

Looking for the velocity potential in the form $\exp\{i(\omega t - kz)\}\Phi(x, y)$, where z is the horizontal coordinate along the cylinders, ω is the frequency, we come to the problem

$$\Phi_y = \lambda\Phi, \quad y = 0, \tag{2.3}$$

$$\Phi_{xx} + \Phi_{yy} - k^2\Phi = 0 \quad \text{in} \quad \Omega, \tag{2.4}$$

$$\partial\Phi/\partial\vec{n}_{\Gamma_i} = 0 \quad \text{on} \quad \Gamma_i, \ i = 1, ..., n, \tag{2.5}$$

for the function Φ; here $\lambda = \omega^2/g$. Solutions of this problem from the Sobolev space $H_1(\Omega)$ are called *trapped waves* and exist only for certain values of λ (the eigenparameter) for k fixed. It is known that essential spectrum of (2.3)-(2.5) coincides with the interval $[k, \infty)$. There exists only one eigenvalue λ below the essential spectrum for large values of k. Our goal is to construct an asymptotic of this eigenvalue. Our main result is as follows.

Theorem 2.1. *The unique eigenvalue* $\lambda(k)$ *of* (2.3)-(2.5) *has the form*

$$\lambda = k - \beta^2, \tag{2.6}$$

where

$$\beta = \left(\sum_{i=1}^{n} k \sqrt{\frac{2\pi}{|y_i''(0)|}} e^{2y_i(0)k} x_i'(0) \right) (1 + O(k^{-1})). \tag{2.7}$$

Remark 2.2. Clearly, if $|y_i(0)| < |y_j(0)|$ for some j and all $i \neq j$, then all the summands with $i \neq j$ in (2.7) are negligible, and the result, in fact is, the same as in the case of one cylinder.

In the next section, we construct the corresponding eigenfunction.

3. Reduction to Integral Equations and their Solution

We reduce (2.3)-(2.5) to $n + 1$ integral equations on $\Gamma_F, \Gamma_1, \Gamma_2, \ldots, \Gamma_n$ for the functions $\varphi = \Phi|_{y=0}$, $\theta_i = \Phi|_{\Gamma_i}$. We have, by the Green formula,

$$\begin{aligned}\Phi(\xi, \eta) = {} & \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} K_0(k\sqrt{(x-\xi)^2 + \eta^2})\varphi(x)\,dx \\ & + \frac{k\eta}{2\pi} \int_{-\infty}^{\infty} \frac{K_0'(k\sqrt{(x-\xi)^2 + \eta^2})}{\sqrt{(x-\xi)^2 + \eta^2}} \varphi(x)\,dx \\ & - \sum_{i=1}^{n} \frac{k}{2\pi} \int_{-\pi}^{\pi} \frac{K_0'(k\sqrt{(x_i(t)-\xi)^2 + (y_i(t)-\eta)^2})}{\sqrt{(x_i(t)-\xi)^2 + (y_i(t)-\eta)^2}} \\ & \times [y_i'(t)(x_i(t)-\xi) - x_i'(t)(y_i(t)-\eta)]\theta_i(t)\,dt \quad (\xi, \eta) \in \Omega, \end{aligned} \tag{3.1}$$

(here $K_0(r)$ is the Macdonald function so that $-\frac{1}{2\pi} K_0(kr)$ is the fundamental solution to (2.4)). Passing in equation (3.1) to the limit when $\eta \to 0^-$, $\xi \to x_i(t)$,

$\eta \to y_i(t)$, and taking into account the jump formulas for the potentials, we obtain the following integral equations:

$$\pi\varphi(\xi) = \lambda\int_{-\infty}^{\infty} K_0(k|x-\xi|)\varphi(x)\,dx$$

$$-k\sum_{i=1}^{n}\int_{-\pi}^{\pi}\frac{K_0'(k\sqrt{(x_i(t)-\xi)^2+y_i(t)^2})}{\sqrt{(x_i(t)-\xi)^2+y_i(t)^2}}$$

$$\times[y_i'(t)(x_i(t)-\xi)-x_i'(t)y_i(t)]\theta_i(t)\,dt, \qquad (3.2)$$

$$\pi\theta_i(t) = \lambda\int_{-\infty}^{\infty} K_0(k\sqrt{(x-x_i(t))^2+y_i(t)^2})\varphi(x)\,dx$$

$$+ky_i(t)\int_{-\infty}^{\infty}\frac{K_0'(k\sqrt{(x-x_i(t))^2+y_i(t)^2})}{\sqrt{(x-x_i(t))^2+y_i(t)^2}}\varphi(x)\,dx$$

$$-k\sum_{j=1}^{n}\int_{-\pi}^{\pi}\frac{K_0'(k\sqrt{(x_j(t')-x_i(t))^2+(y_j(t')-y_i(t))^2})}{\sqrt{(x_j(t')-x_i(t))^2+(y_j(t')-y_i(t))^2}}$$

$$\times[y_j'(t')(x_j(t')-x_i(t))-x_j'(t')(y_j(t')-y_i(t))]\theta_j(t')\,dt'. \qquad (3.3)$$

In order to apply the technique of [11], it is necessary to pass to the Fourier transform $\tilde{\varphi}$ of the function φ,

$$\mathcal{F}_{\xi\to p}[\varphi(\xi)](p) \equiv \tilde{\varphi}(p) = \int_{-\infty}^{\infty} e^{-ip\xi}\varphi(\xi)\,d\xi.$$

Using the formulas (see [1]; here $h > 0$ is a parameter):

$$K_0'(z) = -K_1(z), \quad \mathcal{F}_{\xi\to p}[K_0(k|\xi|)](p) = \frac{\pi}{\sqrt{k^2+p^2}},$$

$$\mathcal{F}_{\xi\to p}\left[\frac{K_1(k\sqrt{\xi^2+h^2})}{\sqrt{\xi^2+h^2}}\right](p) = \frac{\pi}{kh}e^{-h\sqrt{k^2+p^2}},$$

$$\mathcal{F}_{\xi\to p}[K_0(k\sqrt{\xi^2+h^2})](p) = \frac{\pi}{\sqrt{k^2+p^2}}e^{-h\sqrt{k^2+p^2}}, \qquad (3.4)$$

and passing to the Fourier transform $\tilde{\varphi}$ of the function φ, we come to the following system for $\tilde{\varphi}(p)$, $\theta_i(t)$:

$$\left(1 - \frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p) = \sum_{i=1}^{n} \int_{-\pi}^{\pi} e^{-ipx_i(t) + y_i(t)\tau(p)} \left(x_i'(t) + \frac{ipy_i'(t)}{\tau(p)}\right)\theta_i(t)\,dt, \tag{3.5}$$

$$\theta_i(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip'x_i(t) + y_i(t)\tau(p')}\left(1 + \frac{\lambda}{\tau(p')}\right)\tilde{\varphi}(p')\,dp'$$

$$- \frac{k}{\pi}\sum_{j=1}^{n}\int_{-\pi}^{\pi} \frac{K_0'(k\sqrt{\varrho_{ij}(t, t')})}{\sqrt{\varrho_{ij}(t, t')}}\sigma_{ij}(t, t')\theta_j(t')\,dt', \tag{3.6}$$

where

$$\tau(p) := \sqrt{k^2 + p^2},$$

$$\varrho_{ij}(t, t') := (x_j(t') - x_i(t))^2 + (y_j(t') - y_i(t))^2,$$

$$\sigma_{ij}(t, t') := y_j'(t')(x_j(t') - x_i(t)) - x_j'(t')(y_j(t') - y_i(t)).$$

Rewriting system (3.5) as

$$\left(1 - \frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p) = (\hat{M}\theta)(p), \tag{3.7}$$

$$[(1 - \hat{O})\theta](t) = (\hat{N}\tilde{\varphi})(t), \tag{3.8}$$

where

$$\theta = (\theta_1, \ldots, \theta_n), \quad \hat{N} = (\hat{N}_1, \ldots, \hat{N}_n), \quad \hat{O} = (\hat{O}_{ij})_{i,\, j=1,\ldots,n}, \quad \hat{M}\theta = \sum_{i=1}^{n}\hat{M}_i\theta_i,$$

$$(\hat{M}_i\theta_i)(p) = \int_{-\pi}^{\pi} M_i(p, t)\theta_i(t)\,dt, \quad (\hat{N}_i\tilde{\varphi})(t) = \int_{-\infty}^{\infty} N_i(t, p)\tilde{\varphi}(p)\,dp,$$

$$(\hat{O}_{ij}\theta_j)(t) = \int_{-\pi}^{\pi} O_{ij}(t, t')\theta_j(t')\,dt'$$

with

$$M_i(p,\, t) = e^{-ipx_i(t)+y_i(t)\tau(p)}\left(x_i'(t) + \frac{ipy_i'(t)}{\tau(p)}\right),$$

$$N_i(t,\ p) = \frac{1}{2\pi} e^{ipx_i(t)+y_i(t)\tau(p)}\left(1 + \frac{\lambda}{\tau(p)}\right),$$

$$O_{ij}(t,\, t') = -\frac{k}{\pi}\frac{K_0'(k\sqrt{\varrho_{ij}(t,\, t')})}{\sqrt{\varrho_{ij}(t,\, t')}}\sigma_{ij}(t,\, t'). \tag{3.9}$$

4. Solution of the System of Integral Equations

Consider equation (3.8). It is not hard to see, using the asymptotics of $K_1(z)$ for small and large z, that the operator $\hat{O}$ is bounded by const. $k^{-1/2}$. In fact, the following lemma holds.

Lemma 4.1. *We have*

$$\left|\int_{-\pi}^{\pi} O_{ii}(t,\, t')\theta_i(t')dt'\right| \le Ck^{-1/2}\| \theta_i \|,$$

where C is a constant and $\| \theta_i \| = \sup_{t\in[-\pi,\, \pi]}| \theta_i(t) |.$ *Furthermore,*

$$\left|\int_{-\pi}^{\pi} O_{ij}(t,\, t')\theta_j(t')dt'\right| \le Ck^{-1/2}\exp(-dk)\| \theta_j \|, \quad i \ne j.$$

Proof. For a given $\delta > 0,$ we divide the interval of integration in two domains, $k| t - t' | < \delta$ and $k| t - t' | > \delta$. In the first domain, we use the asymptotics $K_1(z) \sim \frac{1}{z}$, and in the second, the asymptotics $K_1(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}$. For $k| t - t' | < \delta,$ we have

$$| O_{ii}(t,\, t') | \le C_1 \frac{| \sigma_{ii}(t,\, t') |}{\varrho_{ii}(t',\, t')}.$$

The numerator here is $O((t - t')^2),$ and by (2.1),

$$\sqrt{\varrho_{ii}(t,\, t')} \ge c| t - t' |, \quad c > 0. \tag{4.1}$$

Hence $O_{ii}(t, t')$ is bounded in this domain. For $k|t - t'| > \delta,$ we have

$$|O_{ii}(t, t')| \le C_2 k^{1/2} e^{-k\sqrt{\varrho_{ii}(t,t')}} \frac{|\sigma_{ii}(t, t')|}{(\varrho_{ii}(t, t'))^{3/4}}.$$

The last factor is bounded by virtue of (4.1), and by the same inequality, we obtain

$$|O_{ii}(t, t')| \le C_3 k^{1/2} e^{-ck|t-t'|}. \tag{4.2}$$

Since $e^{-k|t-t'|} \ge e^{-\delta} = \text{const}$ for $|t - t'| < \delta/k,$ we see that (4.2) holds for all t, t'. Now

$$\left| \int_{-\pi}^{\pi} O_{ii}(t, t')\theta_i(t')dt' \right| \le \text{const} \int_{-\pi}^{\pi} k^{1/2} e^{-ck|t-t'|} dt_1 \| \theta_i \| \le Ck^{-1/2} \| \theta_i \|$$

as claimed. The second inequality of the lemma follows from the fact that $O_{ij}(t, t')$ are exponentially small, since $\sqrt{\varrho_{ij}(t, t')} \ge d > 0,\ i \ne j,$ by (2.2). □

By Lemma 4.1, $\hat{O}$ is small (e.g., in the Cartesian product of n copies of $C[-\pi, \pi]$). Hence we can invert $(1 - \hat{O})$ in (3.8) by the Neumann series, obtaining

$$\theta(t) = [(1 - \hat{O})^{-1}\hat{N}]\tilde{\varphi}(t), \tag{4.3}$$

where $(1 - \hat{O})^{-1} = \sum_{m=0}^{\infty} \hat{O}^m$. Substituting in (3.5), we finally come to

$$\left(1 - \frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p) = [\hat{Q}\tilde{\varphi}](p), \tag{4.4}$$

where $\hat{Q} = \hat{M}(1 - \hat{O})^{-1}\hat{N}$.

We apply the reasoning of [11]. We look for the eigenvalue λ in the form (2.6) and we know that β is exponentially small in k [2]. Hence the first factor in the left-hand side of (4.4),

$$L(p) := 1 - \frac{\lambda}{\tau(p)} = 1 - \frac{k - \beta^2}{\sqrt{k^2 + p^2}}, \tag{4.5}$$

is exponentially small in k for $p = 0$. In fact, the roots of $L(p) = 0$ which tend to zero as $k \to \infty$, as it is not hard to see, are simple and given by

$$p = p_{\pm} = \pm \frac{i\sqrt{2}\beta}{\sqrt{\varepsilon}} + O(\varepsilon^{1/2}\beta^3), \quad \varepsilon = \frac{1}{k}. \tag{4.6}$$

We look for $\tilde{\varphi}$ in the form $\tilde{\varphi}(p) = A(p)/L(p)$. Assuming that $A(p)$ is analytic and using the fact that $N_j(t, p)$ are analytic in a strip containing the real axis, and we can change the contour of integration in the integrals

$$\int_{-\infty}^{\infty} N_j(t, p) \frac{A(p)}{L(p)} dp,$$

to the one shown in Figure 2 (with a suitable $a > 0$ such that in the disc $|p| < a$ there are no zeros of $L(p)$ apart from $p_{\pm}$): we have, by the residue theorem,

$$\int_{-\infty}^{\infty} N_j(t, p) \frac{A(p)}{L(p)} dp = \int_{\gamma} N_j(t, p) \frac{A(p)}{L(p)} dp + 2\pi i \frac{N_j(t, p_+) A(p_+)}{\frac{d}{dp} L(p)|_{p=p_+}}. \tag{4.7}$$

Thus (4.4) transforms into

$$A(p) = \hat{Q}^{\gamma} A(p) + g(p) A(p_+), \tag{4.8}$$

where

$$\hat{Q}^{\gamma} = \hat{M}(1 - \hat{O}_{ji})^{-1} \hat{N}^{\gamma},$$

$$[\hat{N}^{\gamma} A](t) = \int_{\gamma} N(t, p') \frac{A(p')}{L(p')} dp',$$

$$g(p) = \hat{M}(1 - \hat{O})^{-1} f(t), \quad f(t) = 2\pi i \frac{N(t, p_+)}{\frac{d}{dp} L(p)|_{p=p_+}}.$$

Note that now the operator $\hat{Q}^{\gamma}$ is small in ε, since $|L(p)| \geq \text{const}\, k^{-2}$ along γ and $N(t, p)$ is exponentially small. Indeed, on the arc we have up to $O(k^{-\infty})$,

$$|L(p)| = \left| 1 - \frac{1}{\sqrt{1 + p^2/k^2}} \right| = \frac{a^2}{2k^2} + O(k^{-4}), \tag{4.9}$$

and on the part of the contour which lies on the real axis, the minimum of $|L(p)|$ is attained at the points $p = \pm a$, hence, the above estimate still holds.

Rewriting (4.8) as

$$[(1-\hat{Q}^{\gamma})A](p) = g(p)A(p_+), \tag{4.10}$$

we see that $(1-\hat{Q}^{\gamma})$ is invertible and

$$A(p) = [(1-\hat{Q}^{\gamma})^{-1}g](p)A(p_+). \tag{4.11}$$

Let us show that $A(p)$ is indeed analytic in a strip containing the real axis.

Lemma 4.2. *Let* $f_j(t)$ *be continuous in* $t \in [-\pi, \pi]$. *Then* $g(p) = \hat{M}(1-\hat{O})^{-1}f(t)$ *is analytic in a strip containing* γ, *and*

$$|g(p)| \le Ce^{-h_0 \Re\tau/2} \max_{i=1,\dots,n} \| f_i \|,$$

$$p \in \gamma, \quad h_0 = \max_{i=1,\dots,n} |y_i(0)|.$$

Remark 4.3. Note that $\Re\tau = k + O(k^{-1})$ for finite p and $\Re\tau = \tau$ for p real.

Proof. By Lemma 4.1, $(1-\hat{O})^{-1}$ is bounded on $(C[-\pi, \pi])^n$. The assertion now follows directly from the explicit formula for $M_j(p, t)$, since $M_j(p, t)$ are analytic in p for $|\Im p| < k$. □

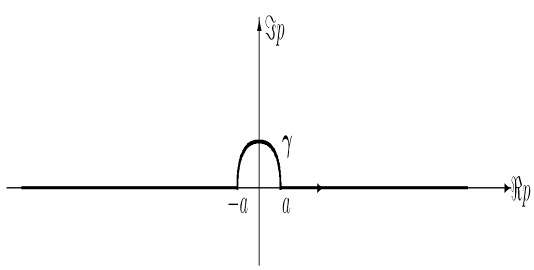

Figure 2

Lemma 4.4. *Let* $g(p)$ *be analytic in a strip containing* γ *and* $\| g \| = \sup_{p\in\gamma} |g(p)| < \infty$. *Then* $A(p) = (1-\hat{Q}^{\gamma})^{-1}g(p)$ *is analytic in* p *in a strip containing* γ *and* $\| A \| \le C\| g \|$.

Proof. We have $(1-\hat{Q}^{\gamma})^{-1} = \sum_{n=0}^{\infty}(\hat{Q}^{\gamma})^n$ and

$$|\hat{Q}^{\gamma}g| = |\hat{M}(1-\hat{O})^{-1}\hat{N}^{\gamma}g| \leq Ce^{-h_0\Re\tau - h_0k/2}\|g\| \tag{4.12}$$

by Lemma 4.1, (4.9), and the explicit form of $\hat{M}_j$, $\hat{N}_j$. $\hat{Q}^{\gamma}g$ is analytic in the strip, since $M_j(p, t)$ are. Iterating (4.12), we see that

$$|(\hat{Q}^{\gamma})^n g| \leq C^n e^{-nh_0k/2 - h_0\Re\tau}\|g\|,$$

hence, the series $\sum_{n=0}^{\infty}(\hat{M}^{\gamma})^n g$ converges uniformly for sufficiently large k and its sum is analytic by the Weierstrass theorem. □

Putting $p = p_+$ in (4.11) and dividing by $A(p_+)$, we obtain an equation for β:

$$1 = [(1-\hat{Q}^{\gamma})^{-1}g](p)|_{p=p_+}. \tag{4.13}$$

A standard application of the Laplace method of asymptotic evaluation of integrals to the leading term in (4.13) yields formula (2.7). In fact, from the leading term in (4.13),

$$\beta \sim \frac{\sqrt{2}\pi}{\varepsilon^{3/2}}\int_{-\pi}^{\pi}\sum_{i=1}^{n} M_i(p_+, t)N_i(t, p_+)dt, \tag{4.14}$$

with $M_i(p, t)$, $N_i(t, p)$ defined in (3.9). We have $\lambda = k - \beta^2$ and $\tau(p_+) = k(1+O(\varepsilon\beta^2))$, hence $1 + \frac{\lambda}{\tau(p_+)} = 2 + O(\varepsilon\beta^2)$. Then from (4.14), we have for large values of k,

$$\beta \sim \frac{\sqrt{2}}{\varepsilon^{3/2}}\int_{-\pi}^{\pi}\sum_{i=1}^{n} e^{2ky_i(t)}x_i'(t)dt.$$

Applying the Laplace method to the last integral, we obtain (2.7).

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