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INVERSE OPTIMAL CONTROL FOR NONLINEAR SYSTEMS WITH AN INPUT CONSTRAINT

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Abstract

This paper studies the inverse optimal control problem for nonlinear system with an input constraint. Based on the control Lyapunov function (CLF), the proposed controller is designed such that the small control property (SCP) is satisfied. This implies the existence of a smooth CLF and then a smooth stabilizing feedback control providing global asymptotical stability of the closed-loop system is designed. Using the inverse approach, a robust optimal controller is developed to minimize a derived performance index and make the system globally asymptotically stable. Moreover, some relations between the resulting controller and sliding mode controller are given. An example is presented and simulation results are included to verify the usefulness of the developed controller.

1. Introduction

The concepts of CLF have attracted much attention in nonlinear control theory [1-3]. In previous works [4-7], some stabilizing controllers were studied for input constrained nonlinear systems by using CLFs. However, the main drawback of the concept of CLF as a design tool is that there is no systematic technique for finding

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CLFs for general nonlinear systems and these control laws do not ensure robustness against structural uncertainty.

Inverse optimal control [8-11] is an alternative approach to solve the nonlinear optimal control problem without solving the Hamilton-Jacobi-Bellman (HJB) equation. By finding a CLF, which can be shown also to be a value function, an optimal controller that optimizes a derived cost can be developed. In most cases, an inverse optimal control requires exact knowledge of the nonlinear dynamics, however, inverse optimal adaptive control [12-14] techniques have been developed for systems with linear in the constant parameter uncertainty. Other works on robust inverse optimal control approach were proposed in [15-19], but these works did not consider the input constraints.

In this paper, our controller is designed such that the SCP [1, 5] is satisfied. We then obtain an almost smooth stabilizing feedback control providing global asymptotical stability of the closed-loop system. Furthermore, the inverse optimal control problem for nonlinear systems with an input constraint is considered. Hence the contribution in this work is based on incorporating inverse optimal control elements with improvements of robustness performance. Some theoretical concepts will be derived to ensure that our proposed control law globally asymptotically stabilizes the equilibrium point of the closed-loop system. We also use the derived theorem to design an inverse optimal controller for a class of nonlinear systems with an input constraint. On the other hand, sliding mode control (SMC) has been shown to be an effective approach when applied to a system with disturbances which satisfy the matched uncertainty condition [20-23]. This paper presents connections between sliding mode control and inverse optimal control with an input constraint.

This paper is organized as follows: In Section 2, a nonlinear system with structural uncertainty is described. Some preliminary definitions of CLFs and smooth feedback stabilizer are restated. In Section 3, we present the design of a smooth feedback stabilizer which is proven to satisfy the SCP and makes the system globally asymptotically stable. Section 4 addresses the inverse optimal control technique that gives optimality and robustness performance. A controller minimizing the derived performance index is proposed and the stability proof of this controller is given. In Section 5, relations between inverse optimal control and sliding mode control are illustrated. In Section 6, an example of spacecraft manoeuvres is presented to explain the design procedure and to verify the usefulness of the proposed method. In Section 7, we present conclusions.

2. System Description and Preliminaries

A nonlinear system with structural uncertainty can be described by

$$\dot{x} = f(x) + g(x)u + h(x), \tag{2.1}$$

where $x \in \mathfrak{R}^n$ denotes the state of the system. The mappings $f : \mathfrak{R}^n \to \mathfrak{R}^n$, with $f(0) = 0$, $g : \mathfrak{R}^n \to \mathfrak{R}^n \times \mathfrak{R}^m$ are assumed to be smooth. $u \in \mathfrak{R}^m$ is the control vector with a following constraint:

$$u \in \mathcal{U} = \{u \in \mathfrak{R}^n \,|\, \| u \| \leq 1\}, \tag{2.2}$$

where $\| \cdot \|$ denotes the Euclidean norm. $h : \mathfrak{R}^n \to \mathfrak{R}^n$, with $h(0) = 0$, represents a structural uncertainty characterized by

$$h(x) = e(x)\delta(x), \tag{2.3}$$

where $e : \mathfrak{R}^n \to \mathfrak{R}^n \times \mathfrak{R}^m$ is a matrix whose entries are given smooth functions, and $\delta : \mathfrak{R}^n \to \mathfrak{R}^m$ is an unknown, vector valued function. It is assumed that $\delta(x)$ is constraint to a given smooth function $N : \mathfrak{R}^n \to \mathfrak{R}^+$ with $N(0) = 0$, i.e.,

$$\Gamma = \{\delta(x) : \| \delta(x) \| \leq N(x)\}. \tag{2.4}$$

If, for any $x \in \mathfrak{R}^n$, $\delta(x)$ satisfies (2.3), then $\delta(x)$ or $h(x)$ is said to be *admissible*.

Let $V : \mathfrak{R}^n \to \mathfrak{R}^+$ be a continuous function. Then V is said to be *positive definite* if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$; V is said to be *proper* if $V(x) \to \infty$ as $\| x \| \to \infty$.

The Lie derivative of V with respect to f is defined as the inner product of the gradient of V with f, i.e., $L_f V(x) = \frac{\partial V}{\partial x} f(x)$.

Throughout the paper, we follow the definitions presented in [8] and [19].

Definition 2.1. A smooth, proper and positive definite function $V : \mathfrak{R}^n \to \mathfrak{R}^+$ is a CLF for the system (2.1) if it satisfies

$$\inf_u \{L_f V(x) + L_g V(x)u\} \leq -\| L_e V(x) \| N(x) \tag{2.5}$$

for each $x \neq 0$.

Definition 2.2. $V(x)$ is said to satisfy the SCP with respect to the system (2.1) if, for each $\varepsilon > 0,$ there is a $\sigma > 0$ such that $x \neq 0$ satisfies $\| x \| < \sigma,$ then there is some u with $\| u \| < \varepsilon$ such that inequality (2.5) holds.

Definition 2.3. Let $k : \mathfrak{R}^n \to \mathfrak{R}$ be a function with $k(0) = 0.$ Then $u = k(x)$ is said to be *almost smooth* on $\mathfrak{R}^n,$ if it is not only smooth on $\mathfrak{R}^n - \{0\},$ but also continuous on all of $\mathfrak{R}^n.$

3. Smooth Feedback Stabilization

In this section, a controller is designed such that a CLF for the system (2.1) satisfies a sufficient condition for globally asymptotical stabilization. If a CLF satisfies the SCP, then an almost smooth feedback stabilizer [5] that ensures global asymptotical stability of the closed-loop system will be obtained. For bounded control case, our design can be considered as an extension of the work by [5], but the nonlinear model in this research includes the structural uncertainty and has more complicated formula.

Let

$$\psi(x) = L_f V(x), \quad \beta(x) = (L_g V(x))^T, \quad C(x) = L_e V(x) \quad \text{and} \quad \beta'(x) = L_g V(x). \tag{3.1}$$

The construction of our control law can be illustrated using the following theorem:

Theorem 3.1. *Let V be a CLF for the system* (2.1) *and* $P(x)$ *be defined by*

$$P(x) = \frac{\psi(x) + \| C(x) \| N(x)}{\| \beta(x) \|}, \tag{3.2}$$

and

$$\Xi = [| \beta'_1(x) | sign(\beta'_1(x)) \;\; | \beta'_2(x) | sign(\beta'_2(x)) \;\; \cdots \;\; | \beta'_m(x) | sign(\beta'_m(x))]^T.$$

Then the input

$$u = k(x) = \begin{cases} \dfrac{P(x) + \sqrt{P^2(x) + \| \beta(x) \|^2}}{\| \beta(x) \|} \Xi & for \;\; \beta(x) \neq 0 \\ 0 & for \;\; \beta(x) = 0 \end{cases} \tag{3.3}$$

globally asymptotically stabilizes the origin.

Note that, by simple multiplications, we obtain $\|\beta'(x)\|^2 = \|\beta(x)\|^2 = \|\Xi\|^2$. This implies $\|\beta(x)\| = \|\Xi\|$. Next, we present the proof of the above theorem.

Proof. Using (2.5), we obtain

$$\beta'(x) = 0, \quad x \neq 0 \Rightarrow \psi(x) < -\|C(x)\| N(x). \tag{3.4}$$

Consider an open subset of $\Re^n$ as follows:

$$S = \{(a, b) \in \Re^2 \mid b \neq 0 \text{ or } a < 0\}. \tag{3.5}$$

Let $q(b) = b$. Then the statement is found by [5] that the function defined by

$$\phi(a, b) = \begin{cases} \dfrac{a + \sqrt{a^2 + bq(b)}}{b} & \text{for } b \neq 0 \\ 0 & \text{for } b = 0 \end{cases}$$

is smooth on S.

Using (3.4) for any $x \neq 0$ then $\left(\dfrac{\psi(x) + \|C(x)\| N(x)}{\|\beta(x)\|}, \|\beta(x)\|\right) \in S$. For any $x \neq 0$, $u = k(x)$ is smooth on S. Moreover, at nonzero x, we have

$$\begin{aligned}
\dot{V} &= L_f V(x) + L_g V(x) u + L_h V(x) \\
&\leq \psi(x) + \|C(x)\| N(x) - \beta'(x) \frac{(P(x) + \sqrt{P^2(x) + \|\beta(x)\|^2})}{\|\beta(x)\|} \Xi \\
&\leq \left[\frac{\psi(x) + \|C(x)\| N(x)}{\|\beta(x)\|} - \beta'(x) \begin{bmatrix} |\beta_1'| \operatorname{sign}(\beta_1') \\ |\beta_2'| \operatorname{sign}(\beta_2') \\ \vdots \\ |\beta_m'| \operatorname{sign}(\beta_m') \end{bmatrix} \frac{(P(x) + \sqrt{P^2(x) + \|\beta(x)\|^2})}{\|\beta(x)\|^2} \right] \|\beta(x)\| \\
&\leq \left[\frac{\psi(x) + \|C(x)\| N(x)}{\|\beta(x)\|} - \|\beta(x)\|^2 \frac{(P(x) + \sqrt{P^2(x) + \|\beta(x)\|^2})}{\|\beta(x)\|^2} \right] \|\beta(x)\| \\
&\leq (P(x) - (P(x) + \sqrt{P^2(x) + \|\beta(x)\|^2})) \|\beta(x)\| \\
&\leq -(\sqrt{P^2(x) + \|\beta(x)\|^2}) \|\beta(x)\| < 0.
\end{aligned} \tag{3.6}$$

Thus, V decreases along trajectories of the corresponding closed-loop system. Now we can say that the control $u = k(x)$ is continuous on $\mathfrak{R}^n \backslash \{0\}$. To improve this result, it is required to show that the function $V(x)$ satisfies the SCP. If this is the case, then the control law $u = k(x)$ is continuous on all of $\mathfrak{R}^n$ and completely characterizes almost smooth stabilizability [5]. Clearly, we always obtain $k(x) = 0$ whenever $\beta(x) = 0$, so it is sufficient to consider the case that $\beta(x) \neq 0$. $V(x)$ satisfies the SCP. This means that there is a $\sigma > 0$ such that, if x satisfies $x \neq 0$ and $\| x \| < \sigma$, then there is some u with $\| u \| < \varepsilon$ so that

$$\psi(x) + \beta'(x) < -\| C(x) \| N(x). \tag{3.7}$$

Since V is positive definite, it has a minimum at zero. This implies $\frac{\partial V}{\partial x}(0) = 0.$ Since the gradient is continuous, we obtain

$$\| \beta(x) \| < \varepsilon, \tag{3.8}$$

when x is sufficiently small. By the Cauchy-Schwartz inequality, one has

$$\psi(x) + \| C(x) \| N(x) < -\beta'(x)u \leq \| \beta(x) \| \| u \|. \tag{3.9}$$

Dividing both the sides of (3.9) by $\| \beta(x) \|$, we obtain

$$P \leq \| u \| \leq \varepsilon. \tag{3.10}$$

For $P(x) > 0$, since

$$P(x) + \sqrt{P(x)^2 + \| \beta(x) \|^2} \leq 2| P(x) | + \| \beta(x) \|, \tag{3.11}$$

$$\begin{aligned} \| u \| = \| k(x) \| &\leq (2| P(x) | + \| \beta(x) \|)[| \beta_1' | \ \ | \beta_2' | \ \ \cdots \ \ | \beta_m' |]^T \\ &\leq (2\varepsilon + \| \beta(x) \|) \| \beta(x) \| \\ &\leq (2\varepsilon + \varepsilon)\varepsilon = 3\varepsilon^2 = \bar{\varepsilon}. \end{aligned} \tag{3.12}$$

On the other hand, for $P(x) \leq 0$ it implies that

$$\begin{aligned} \| k(x) \| &\leq (P(x) + \sqrt{P(x)^2 + \| \beta(x) \|^2})[| \beta_1' | \ \ | \beta_2' | \ \ \cdots \ \ | \beta_m' |]^T \\ &\leq (P(x) + | P(x) | + \| \beta(x) \|) \| \beta(x) \| \leq \| \beta(x) \|^2 \leq \varepsilon^2 = \frac{\bar{\varepsilon}}{3}. \end{aligned} \tag{3.13}$$

Clearly, $u = k(x)$ satisfies the SCP. That is to say, $u = k(x)$ is continuous at the origin. Hence the control law (3.3) is an almost smooth control and globally asymptotically stabilizes the equilibrium $x = 0$ of the closed-loop system. □

In this paper, we find u_0 that solves the system (2.1) and minimizes the performance index:

$$J(u, x, x_0) = \int_0^{\infty} (l(x) + R(x)\| u \|^2) dt. \tag{3.14}$$

Let u_0 be the solution to the optimal control problem and

$$V(x_0) = \min_u \int_0^{\infty} (l(x) + R(x)\| u \|^2) dt. \tag{3.15}$$

The HJB equation with the function V of (3.15) is given by

$$-\frac{\partial V}{\partial t} = \min_u \{l(x) + R(x)\| u \|^2 + V_x^T (f(x) + g(x)u - e(x)\delta(x))\}. \tag{3.16}$$

Therefore, the controller u_0 satisfies

$$0 = l(x) + R(x)\| u_0 \|^2 + V_x^T (f(x) + g(x)u - e(x)\delta(x)) \tag{3.17}$$

and

$$0 = 2R(x)\| u_0 \|^T \operatorname{sign}(u_0) + L_g V(x). \tag{3.18}$$

Using (3.18), one obtains the control input

$$u_0 = -\frac{1}{2} R^{-1}(x)\Xi \tag{3.19}$$

which is the solution to the above optimal control problem. In Section 4, we give the design and analysis of this control law.

4. Inverse Optimal Control

In this section, the design of an inverse optimal controller is examined. The proposed controller is designed to globally asymptotically stabilize the equilibrium $x = 0$ and minimize the performance index (3.14). Using the inverse approach, it is shown that $l(x) \geq 0$ and $R(x) > 0$ such that $u = k(x)$ optimizes (3.14).

Pick any $\eta > 0$ and let

$$R^{-1}(x) = \begin{cases} 2\eta + 2\left(\dfrac{P(x) + \sqrt{P^2(x) + \|\beta(x)\|^2}}{\|\beta(x)\|}\right) & \text{for } \beta(x) \neq 0, \\ 2\eta & \text{for } \beta(x) = 0. \end{cases} \tag{4.1}$$

Theorem 4.1. *Let V be a CLF for the system* (2.1) *and the control input be chosen as* (3.19) *with $R^{-1}(x)$ determined by* (4.1). *Then there exists an $l(x) \geq 0$ such that $J(u_0, x, x_0) = V(x_0)$ for every $x_0 \in \Re^n$ and every admissible $h(x)$.*

Proof. For all admissible $h(x)$, consider $V(x)$ of (3.15) as a Lyapunov function candidate. Clearly, we obtain $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$. We next show that $\dot{V}(x) < 0$ for $x \neq 0$,

$$\begin{aligned}
\dot{V} &= L_f V(x) + L_g V(x) u + L_h V(x) \\
&\leq \psi(x) + \|C(x)\| N(x) - \beta'(x)\left(\eta\Xi + \frac{(P(x) + \sqrt{P^2(x) + \|\beta(x)\|^2})}{\|\beta(x)\|}\Xi\right) \\
&\leq \left[\frac{\psi(x) + \|C(x)\| N(x)}{\|\beta(x)\|} - \beta'(x)\Xi\left(\frac{(P(x) + \sqrt{P^2(x) + \|\beta(x)\|^2})}{\|\beta(x)\|^2} + \frac{\eta}{\|\beta(x)\|}\right)\right]\|\beta(x)\| \\
&\leq \left[\frac{\psi(x) + \|C(x)\| N(x)}{\|\beta(x)\|} - \|\beta(x)\|^2\left(\frac{(P(x) + \sqrt{P^2(x) + \|\beta(x)\|^2})}{\|\beta(x)\|^2} + \frac{\eta}{\|\beta(x)\|}\right)\right]\|\beta(x)\| \\
&\leq [P(x) - (P(x) + \sqrt{P^2(x) + \|\beta(x)\|^2})]\|\beta(x)\| - \eta\|\beta(x)\|^2 \\
&\leq -(\sqrt{P^2(x) + \|\beta(x)\|^2})\|\beta(x)\| - \eta\|\beta(x)\|^2 < 0.
\end{aligned} \tag{4.2}$$

Thus, u_0 globally asymptotically stabilizes the equilibrium point $x = 0$ of the system (2.1).

Letting $u_1 = 0.5u_0$ and following the procedure to find $\dot{V}$ similar to (4.2), it yields the same result $(\dot{V} \leq 0)$.

Choose:

$$l(x) = -\psi(x) - \| C(x) \| N(x) - \beta'(x)u_1. \tag{4.3}$$

Then $l(x) > 0$ since

$$l(x) = -\psi(x) - \| C(x) \| N(x) + \frac{1}{4} R^{-1}(x) \| \beta(x) \|^2 \geq 0.$$

Substituting $u = v - \frac{1}{2} R^{-1}(x)\Xi$ into $\int_0^\infty (l(x) + R(x)\| u \|^2)dt$, yields

$$\begin{aligned}
&\int_0^\infty (l(x) + R(x)\| u \|^2)dt \\
&= \int_0^\infty \left(-\psi(x) - \| C(x) \| N(x) + \frac{1}{4} R^{-1}(x)\| \beta(x) \|^2 + R(x)\| u \|^2\right)dt \\
&= \int_0^\infty \left(-\psi(x) - \| C(x) \| N(x) + \frac{1}{2} R^{-1}(x)\| \beta(x) \|^2 - \Xi^T v + R(x)\| v \|^2\right)dt \\
&= \int_0^\infty \left(-\psi(x) - \| C(x) \| N(x) + \frac{1}{2} R^{-1}(x)\Xi^T\Xi - \Xi^T v + R(x)\| v \|^2\right)dt \\
&= \int_0^\infty \left(-\psi(x) - \| C(x) \| N(x) + \Xi^T\left(\frac{1}{2} R^{-1}(x)\Xi - v\right) + R(x)\| v \|^2\right)dt \\
&\leq -\int_0^\infty \frac{\partial V}{\partial x}(f(x) + h(x) + \beta'(x)u)dt + \int_0^\infty (R(x)\| v \|^2)dt \\
&= V(x_0) - \lim_{t\to\infty} V(x) + \int_0^\infty R(x)\| v \|^2 dt.
\end{aligned} \tag{4.4}$$

Since u is chosen to stabilize the closed systems, it follows that $\lim_{t\to\infty}(x(t)) = 0$ and then $\lim_{t\to\infty} V(x(t)) = 0$. Moreover, we obtain

$$\int_0^\infty (l(x) + R(x)\| u \|^2)dt \leq V(x_0) + \int_0^\infty R(x)\| v \|^2 dt.$$

Taking $v = 0$ then $u = u_0$. This implies that

$$J(u_0, x, x_0) \le V(x_0) \tag{4.5}$$

for every $x_0 \in \Re^n$ and admissible $h(x)$. Let

$$\delta_0(x) = \begin{cases} \dfrac{C^T(x)N(x)}{\| C(x) \|} & \text{for } C(x) \neq 0, \\ \dfrac{1}{\sqrt{m}}[N(x) \cdots N(x)]^T & \text{for } C(x) = 0. \end{cases}$$

Then $\delta_0(x) \in \Gamma$ and $C(x)\delta_0(x) = \| C(x) \| N(x)$. For every $x_0 \in \Re^n$ and $\delta_0(x)$, $\dot{V}$ can be integrated along the solution $x(t)$ of the systems (2.1) and (3.18). Thus, for all $T \ge 0$:

$$V(x_0) = V(x(T)) - \int_0^T \frac{dV}{dt}\, dt \tag{4.6}$$

$$= V(x(T)) - \int_0^T (\psi(x) + C(x)\delta_0(x) + \beta'(x)u_0)\, dt \tag{4.7}$$

$$= V(x(T)) - \int_0^T (\psi(x) + \| C(x) \| N(x) + \beta'(x)u_0)\, dt. \tag{4.8}$$

Therefore, we obtain

$$V(x_0) = V(x(T)) - \int_0^T (l(x) + R(x)\| u_0 \|^2)\, dt. \tag{4.9}$$

Because $V(x(T)) \to 0$ as $T \to \infty$, (4.9) becomes

$$V(x_0) = J(u_0, x, x_0). \tag{4.10}$$

Thus, using (4.5) and (4.10), we can conclude that u_0 is exactly the optimal solution to (3.14) for the $R(x)$ and $l(x)$ chosen as (4.1) and (4.3), respectively. This completes the proof. □

5. Relations to Sliding Mode Control

For general control designs, the way to obtain high accuracy is usually connected with increasing gains in the feedback systems. Sliding mode control is an effective tool for achieving high accuracy without using any high gain approach [20]. Our proposed control law is in the class of first-order SMC, since it has a form

$$u_i = z_i(x) f(s_i), \tag{5.1}$$

where $f(s_i) = \operatorname{sign}(s_i)$ and $z_i(x) = -\frac{1}{2} R^{-1}(x) | \beta_i' |$.

From (5.1), each component of the sliding vector s_i is defined as

$$s_i = \beta_i'(x), \quad i = 1, 2, \ldots, m. \tag{5.2}$$

The control law above was shown to make the system global asymptotically stable in Section 4, so this implies that the reaching and sliding on the sliding manifold are also ensured.

Based on the sliding mode control concept, instead of the formula of the control law (5.1), we can use other formulas, i.e.,

$$u_i = -\frac{1}{\overline{\rho}} s_i - z_i(x) f(s_i), \tag{5.3}$$

where $\overline{\rho} > 1$ is a positive constant.

Using the same function as (3.15), the stability of the control law above can be easily investigated by considering

$$\begin{aligned}\dot{V} &= L_f V(x) + L_g V(x) u + L_h V(x) \\ &= L_f V(x) + \beta(x) \begin{bmatrix} -\frac{1}{\overline{\rho}} s_1 - \frac{1}{2} R^{-1}(x) | \beta_1' | \operatorname{sign}(s_1) \\ -\frac{1}{\overline{\rho}} s_2 - \frac{1}{2} R^{-1}(x) | \beta_2' | \operatorname{sign}(s_2) \\ \vdots \\ -\frac{1}{\overline{\rho}} s_m - \frac{1}{2} R^{-1}(x) | \beta_m' | \operatorname{sign}(s_m) \end{bmatrix} + \| C(x) \| N(x),\end{aligned} \tag{5.4}$$

which can be further written as

$$\dot{V} = L_f V(x) - \frac{1}{\overline{\rho}} \sum_{i=1}^{3} s_i^2 - \frac{1}{2} R^{-1}(x) \sum_{i=1}^{3} s_i^2 + \| C(x) \| N(x). \tag{5.5}$$

Using (2.5), we obtain $\dot{V} < 0$ and this guarantees the reaching and sliding on the sliding manifold. However, since this control is designed using the first-order SMC concept similar to the controller (5.1), both (5.1) and (5.3) usually provide the same behaviors of system responses.

Due to the chattering in the sliding mode controller design, the sign function of the control law (5.1) is replaced by

$$f(s_i) = \tanh\left(\frac{s_i}{\varepsilon}\right), \quad i = 1, 2, \cdots m, \tag{5.6}$$

where $\tanh(u)$ is the hyperbolic tangent function.

6. Simulation Example

Here an example of a rigid-body satellite [24] is presented with numerical simulations to verify the performance of our proposed controller.

The dynamic equations of the rotational motion of a spacecraft [24] are described by

$$J\dot{\omega} = -[\omega \times] J\omega + u + G\xi(t), \tag{6.1}$$

where $\omega = [\omega_1 \quad \omega_2 \quad \omega_3]^T$ is the angular velocity vector of the spacecraft, $u = [u_1 \quad u_2 \quad u_3]^T$ represents the control vector, $\xi = [\xi_1 \quad \xi_2 \quad \xi_3]^T$ is bounded disturbance, J is the inertia matrix, and G is a 3×3 input matrix for $\xi(t)$. The skew-symmetric matrix $[\omega \times]$ is

$$[\omega \times] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{6.2}$$

The orientation of a rigid body with respect to the inertia frame using quaternion is considered. We define here the quaternion $\overline{Q} = [q^T \quad q_4]^T$ with

$$q = [q_1 \quad q_2 \quad q_3]^T.$$

Then the kinematic equations of the rigid body motion described in terms of the attitude quaternion [25] is given by

$$\dot{q} = \frac{1}{2}([q\times] + q_4 I_{3\times 3})\omega,$$

$$\dot{q}_4 = -\frac{1}{2} q^T \omega, \tag{6.3}$$

where $I_{3\times 3}$ is a 3×3 identity matrix and the elements of $\overline{Q}$ are restricted by

$$\| \overline{Q} \| = 1 \text{ or } q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1. \tag{6.4}$$

In this example, the spacecraft is assumed to have the inertia matrix $J = \mathrm{diag}(10, 15, 20)$. Then the control law (5.1) is considered with the presence of the external disturbance signal given by [24]

$$\xi(t) = W^{-1} G^T \omega(t), \tag{6.5}$$

where $W > 0$ is a 3×3 positive definite Hermitian matrix. We assumed that $G = I_{3\times 3}$. $\xi(t)$ is used as the external disturbance and its gain matrix is $W = \mathrm{diag}(1, 2, 3)$. In the simulation, it is also assumed that

$$e(x) = g(x) = [J^{-1} \quad 0_{3\times 3}]^T.$$

For the performance index (3.14), $l(x)$ is chosen using (4.3). The initial conditions for the attitude quaternion are

$$\overline{Q}(0) = [0.4618 \quad 0.1915 \quad 0.7999 \quad 0.3320]^T.$$

A rest-to-rest manoeuvre is considered and thus, the initial condition for $\omega(t)$ is given as $\omega(0) = [0 \quad 0 \quad 0]^T$ rad/sec.

Let a proper smooth function be chosen as [26]

$$V(x) = \frac{1}{2}[\omega^T \quad q^T]\begin{bmatrix} aJ & bJ \\ bJ & c \end{bmatrix}\begin{bmatrix} \omega \\ q \end{bmatrix}. \tag{6.6}$$

Clearly, the conditions for $V(x)$ to be positive defined are given as [26]

$$c > 0, \quad acJ > b^2 J^2, \tag{6.7}$$

where a, b and c are positive constants. Then it is straightforward to verify that

$$\beta'(x) = L_g V(x) = a\omega^T + bq^T,$$

$$\psi(x) = -(a\omega^T + bq^T)[\omega \times]\omega + \frac{1}{2}([q \times] + q_4 I_{3\times 3}) \tag{6.8}$$

and

$$C(x) = L_g V(x). \tag{6.9}$$

Figures 1 and 2 clearly show the performance of the developed controller. The responses of quaternion and angular velocity components reach zero after 60 seconds. Obviously, the effect of external disturbances on quaternion and angular velocity is totally removed. As shown in Figure 3, the sliding vector remains on the sliding manifold in about 10 seconds. From Figure 4, it can be seen that the developed controller stabilizes the closed-loop system and provides smooth control torque responses. As shown in Figure 5, the external disturbances converge to the neighborhood of zero after 60 seconds.

The developed control law seems to be a useful approach for general nonlinear models. The difficulty of using the inverse approach is how to find a suitable CLF. Once this function is already known, the inverse optimal control approach can be used to design the optimal controller that yields global asymptotic stability.

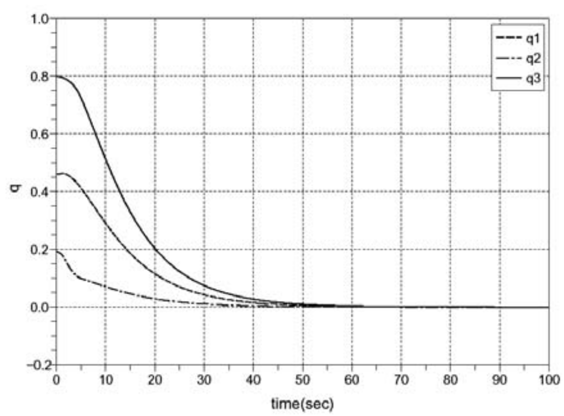

Figure 1. Components of quaternion vector with external disturbances.

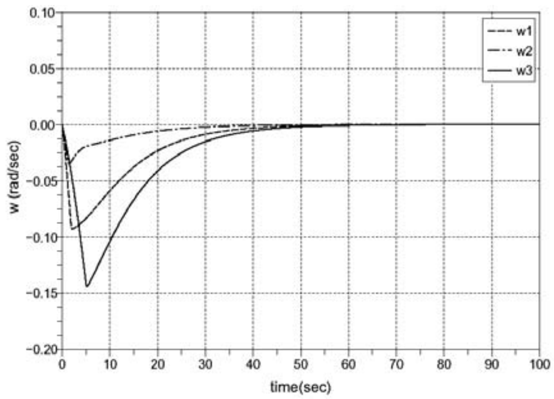

Figure 2. Components of angular velocity vector with external disturbances.

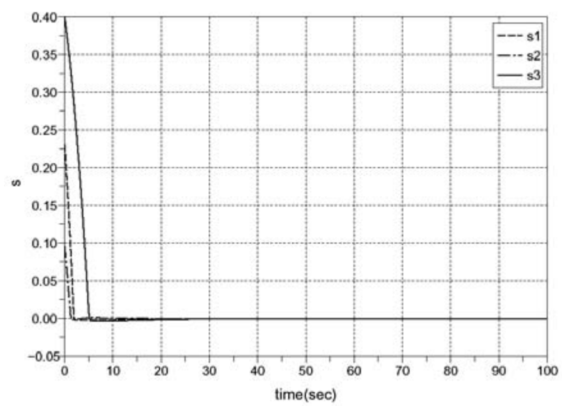

Figure 3. Components of sliding vector with external disturbances.

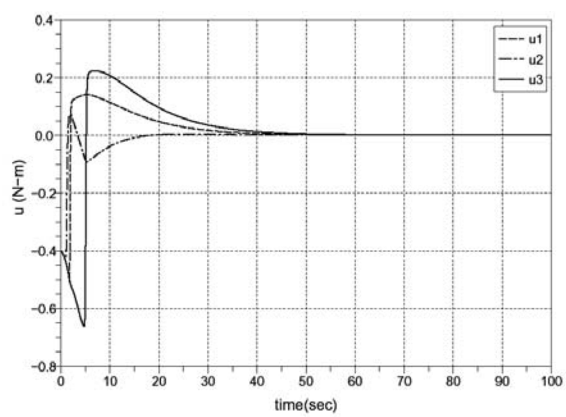

Figure 4. Control torques with external disturbances.

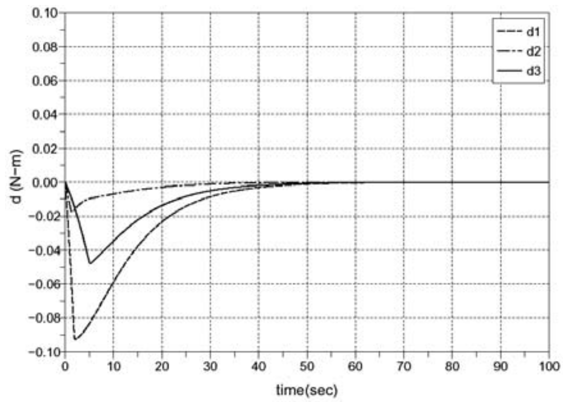

Figure 5. Disturbance responses.

7. Conclusion

We have studied the bounded controller design for a class of nonlinear systems with structural uncertainty. Based on the CLF, with the SCP, the developed control law is a smooth feedback stabilizer and makes the system globally asymptotically stable. Furthermore, the bounded control based on the CLF and inverse optimal control approach has been successfully applied to nonlinear systems with a structural uncertainty. This control law has proven to actually minimize the derived performance index and has a formula similar to the first-order sliding mode control. An example of multiaxial attitude manoeuvres is presented and simulation results are given to verify the usefulness of the developed controller.

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