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PARAMETRIC WELL-POSEDNESS FOR QUASIVARIATIONAL-LIKE INEQUALITIES

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Abstract

In this paper, the concepts of parametric well-posedness and parametric well-posedness in generalized sense for quasivariational-like inequalities problems are introduced. Some criteria and characterizations are derived for parametric well-posedness and parametric well-posedness in generalized sense.

1. Introduction

Well-posedness of optimization problems, variational problems and equilibrium problems, etc. is to study the property of approximating solutions. Specifically, it is investigated that whether the approximating solution sequence has a subsequence converging to a solution of the considered problems. The importance of this issue has been widely focused by many researchers in the field of computational theory. Well-posedness of scalar optimization problems was first introduced by Tykhonov [1] in 1966. Since then, people gave various concepts of well-posedness and extensively studied the scalar optimization problems [2-6] and the vector

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optimization problems [7, 8]. Recently, the concepts of well-posedness have been generalized in variational inequalities problems [9-14], quasivariational inequalities problems [15-17], equilibrium problems [18, 19], and optimization problems with variational inequalities constraints [20, 21]. In 2006, Lignola [15] introduced and investigated the concepts of well-posedness and L-well-posedness for quasivariational inequalities, and Ceng et al. [16] extended these concepts for mixed quasivariational-like inequalities. In 2007, Fang and Hu [12] considered the concept of parametric well-posedness for variational inequalities and gave some metric characterizations of parametric well-posedness.

Inspired by the above works, in this paper, we introduce the parametric well-posedness and parametric well-posedness in generalized sense for quasivariational-like inequalities. Some necessary and/or sufficient conditions of parametric well-posedness and parametric well-posedness in generalized sense are obtained. Our results improve and extend some known results in the recent literatures.

2. Preliminaries

Let X, E be real Banach spaces and K be a nonempty subset of E. Let $F : K \to 2^K$ be a set-valued mapping, $\eta : K \times K \to E$ be a single-valued mapping and $f : X \times K \times E \to R$ be a real-valued function. We consider the following parametric quasivariational-like inequality problem:

$(QVLI)_x$ Find $u_0 \in K$ such that $u_0 \in F(u_0)$ and $f(x, u_0, \eta(u_0, v)) \leq 0$, $\forall v \in F(u_0)$.

If $\eta(u, v) = u - v$, then the problem $(QVLI)_x$ reduces to the parametric quasivariational inequality problem, i.e.,

$(QVI)_x$ Find $u_0 \in K$ such that $u_0 \in F(u_0)$ and $f(x, u_0, u_0 - v) \leq 0$, $\forall v \in F(u_0)$.

Further, if $F(u) = K$, for all $u \in K$, then the problem reduces to the parametric variational inequality problem which is discussed in [12].

In this paper, we denote $Q(x)$ as the solution set of $(QVLI)_x$.

Let $\{T_n\}$ be a sequence of subsets of E. We recall that the Painlevé-Kuratowski limits of sequence $\{T_n\}$ are defined by:

$\limsup_n T_n = \{u \in E : \exists n_k \to \infty, n_k \in N, \exists u_{u_k} \in T_{n_k}$ with $\lim_k u_{n_k} = u\}$;

$\liminf_n T_n = \{u \in E : \exists u_n \in T_n, n \in N$ with $\lim_n u_n = u\}$.

In the sequel, we recall some known concepts.

Definition 2.1. Let E be a real Banach space, and K be a nonempty closed subset of E. Then a set-valued mapping $F : K \to 2^k$ is termed:

(i) *closed* if the graph $G_F = \{(u, v) : v \in F(u)\}$ is closed in $K \times K$, i.e., if $u_n \in K$ and $u_n \to u$, then we get

$$\limsup_n F(u_n) \subseteq F(u);$$

(ii) *lower semicontinuous* if for any fixed $u_0 \in K$ and any sequence $\{u_n\} \subseteq K$ converging to u_0, for all $v \in F(u_0)$, there exists a sequence of elements $v_n \in F(u_n)$ converging to v, i.e.,

$$F(u) \subseteq \liminf_n F(u_n);$$

(iii) *upper semicontinuous* if for any fixed $u_0 \in K$ and any open set $V \subseteq K$ such that $F(u_0) \subseteq V$, there exists a neighborhood $N(u_0)$ of u_0 such that $F(v) \subseteq V$ for all $v \in N(u_0)$.

(iv) *continuous* if F is both lower semicontinuous and upper semicontinuous.

Definition 2.2. Let E be a real Banach space, and K be a nonempty subset of E. Let $\eta : K \times K \to E$ be a single-valued mapping. Then a bifunction $f : K \times E \to R$ is said to be

(i) η-*pseudomonotone*, if for any $u, v \in K$,

$$f(u, \eta(u, v)) \leq 0 \Rightarrow f(v, \eta(u, v)) \leq 0.$$

(ii) η-*monotone*, if for any $u, v \in K$,

$$f(u, \eta(u, v)) \geq f(v, \eta(u, v)).$$

It is clear that f is η-monotone $\Rightarrow f$ is η-pseudomonotone.

Now, we give a notion of continuity of bifunction f and introduce some definitions of parametric well-posedness for quasivariational-like inequalities.

Definition 2.3. Let E be a real Banach space, K be a nonempty convex subset of E and $\eta : K \times K \to E$ be a map. Then a bifunction $f : K \times E \to R$ is said to be *lower d-semicontinuous* on K if for every $u, v \in K,$ we have

$$f(u, \eta(u, v)) \leq \liminf_{t \to 0^+} f(u + t(v - u), \eta(u, v)).$$

It is clear that the lower semicontinuity of f implies the lower d-semicontinuity of f, but the converse does not true in general.

Definition 2.4. Let $x \in X$ and $\{x_n\} \subseteq X$ with $x_n \to x.$ Then a sequence $\{u_n\} \subseteq K$ is called an *approximating sequence* for $(QVLI)_x$ corresponding to $\{x_n\},$ if there exists a sequence $\{\varepsilon_n\},$ $\varepsilon_n > 0,$ decreasing to 0 such that $d(u_n F(u_n)) \leq \varepsilon_n$ and $f(x_n, u_n, \eta(u_n, v)) \leq \varepsilon_n$ for any $v \in F(u_n).$

Definition 2.5. The family $\{(QVLI)_x : x \in X\}$ is said to be *parametrically well-posed*, if for every $x \in X,$ $(QVLI)_x$ has a unique solution $u_x,$ and for any $\{x_n\} \subseteq X$ with $x_n \to x,$ every approximating sequence for $(QVLI)_x$ corresponding to $\{x_n\}$ converges to $u_x.$

Definition 2.6. The family $\{(QVLI)_x : x \in X\}$ is said to be *parametrically well-posed* in generalized sense, if for every $x \in X,$ the solution set $Q(x)$ of $(QVLI)_x$ is nonempty, and for any $\{x_n\} \subseteq X$ with $x_n \to x,$ every approximating sequence of $(QVLI)_x$ corresponding to $\{x_n\}$ has a subsequence converging to a point of $Q(X).$

Remark 2.1. It is easy to see that the solution set $Q(x)$ is compact for every $x \in X$ when the family $\{(QVLI)_x : x \in X\}$ is parametrically well-posed in generalized sense. In fact, if $\{u_n\} \subseteq Q(x),$ then we can see that $\{u_n\}$ is an approximating sequence for $(QVLI)_x$ corresponding to $\{x_n\}$ (where $x_n = x,$ $n \in N).$ Then there exists a subsequence $\{u_{n_k}\}$ converging to some point of $Q(x).$

3. Parametric Well-posedness for Quasivariational-like Inequalities

In this section, we give some criteria and characterizations of parametric well-posedness for $\{(QVLI)_x : x \in X\}$. Firstly, we introduce the set of approximating solutions of $(QVLI)_x$ as

$$Q_\varepsilon^\delta(x) = \bigcup_{y \in B(x, \delta)} Q_\varepsilon(y),$$

where

$$B(x, \delta) = \{y \in X : \| y - x \| \leq \delta\}$$

and

$$Q_\varepsilon(y) = \{u \in K : d(u, F(u)) \leq \varepsilon, \ f(y, u, \eta(u, v)) \leq \varepsilon, \ \forall v \in F(u)\}.$$

We introduce *Condition A* motivated by the Mohan-Neogy's work in [22]. We say that function $\eta : K \times K \to E$ satisfies *Condition A*, if for any $u, v \in K$ and $\lambda \in (0, 1)$, $\eta(u, u + \lambda(v - u)) = \lambda\eta(u, v)$. Obviously, η satisfies *Condition A* when $\eta(u, v) = u - v$.

Lemma 3.1. *Let E be a real Banach space, and K be a nonempty convex subset of E. Let further* $F : K \to 2^K$ *be a set-valued mapping with nonempty convex value, and* $\eta : K \times K \to E$ *be a single-valued mapping satisfying Condition A. If the function* $f : K \times E \to R$ *is* η*-pseudomonotone, lower d-semicontinuous and positively homogeneous in the second variable, then the following problems are equivalent:*

(i) *find* $u_0 \in K$ *such that*

$$u_0 \in F(u_0) \ \textit{and} \ f(u_0, \eta(u_0, v)) \leq 0, \ \forall v \in \operatorname{int} F(u_0);$$

(ii) *find* $u_0 \in K$ *such that*

$$u_0 \in F(u_0) \ \textit{and} \ f(v, \eta(u_0, v)) \leq 0, \ \forall v \in \operatorname{int} F(u_0).$$

Proof. If $u_0 \in K$ is a solution of the problem (i), then it is clear that $u_0 \in K$ is the solution of the problem (ii) by using the η-pseudomonotonicity of *f*. Conversely, assume that $u_0 \in K$ is a solution of the problem (ii). We only need to prove the case of $\operatorname{int} F(u_0) \neq \varnothing$. Since $F : K \to 2^K$ is nonempty convex valued and η satisfies *Condition A*, for all $\lambda \in (0, 1)$ and $v \in \operatorname{int} F(u_0)$, we have $u_0 + \lambda(v - u_0) \in \operatorname{int} F(u_0)$, and so

$$\begin{aligned} 0 &\geq f(u_0 + \lambda(v - u_0), \eta(u_0, u_0 + \lambda(v - u_0))) \\ &= f(u_0 + \lambda(v - u_0), \lambda\eta(u_0, v)) \\ &= \lambda f(u_0 + \lambda(v - u_0), \eta(u_0, v)), \end{aligned}$$

which shows $f(u_0 + \lambda(v - u_0), \eta(u_0, v)) \leq 0$ for all $v \in \operatorname{int} F(u_0)$. By letting $\lambda \to 0^+$ and using lower *d*-semicontinuity of *f*, we obtain that $f(u_0 + \eta(u_0, v)) \leq 0$, $\forall v \in \operatorname{int} F(u_0)$. Thus u_0 is a solution of the problem (i) and the proof is completed. □

Lemma 3.2 [23]. *Let* $\{K_n\}$ $(n \in N)$ *be a sequence of nonempty convex subsets of the Banach space E such that*

(i) $K_0 \subseteq \liminf_n K_n$;

(ii) *there exists an* $m \in N$ *such that* $\bigcap_{n \geq m} K_n \neq \varnothing$.

Then, for every $u_0 \in \operatorname{int} K_0$, *there exists a positive real number* δ *such that*

$$B(u_0, \delta) \subseteq K_n, \quad \forall n \geq m.$$

Theorem 3.1. *Let X and E be Banach spaces, K be a nonempty closed convex subset of E. Let further* $F : K \to 2^K$ *be a closed and lower semicontinuous set-valued mapping with nonempty convex values,* $\eta : K \times K \to E$ *be a single-valued continuous mapping satisfying Condition A,* $f : X \times K \times E \to R$ *be a real valued function. Suppose that the following conditions hold*:

(i) *for every converging sequence* $\{u_n\} \subseteq K$, *there exists an* $m \in N$ *such that* $\operatorname{int} \bigcap_{n \geq m} F(u_n) \neq \varnothing$;

(ii) $f(\cdot, u, \cdot)$ *is lower semicontinuous for all* $u \in K$;

(iii) *for every* $x \in X$, *the function* $f(x, \cdot, \cdot)$ *is lower d-semicontinuous and* η-*monotone*;

(iv) *for all* $(x, u) \in X \times K$, $f(x, u, \cdot)$ *is positively homogeneous.*

Then $\{(QVLI)_x : x \in X\}$ *is parametrically well-posed if and only if for every* $x \in X$,

$$Q_\varepsilon^\delta(x) \neq \varnothing, \ \forall \varepsilon, \delta > 0, \ \textit{and} \lim_{(\varepsilon, \delta) \to (0,0)} \operatorname{diam} Q_\varepsilon^\delta(x) = 0. \tag{3.1}$$

Proof. "⇒" Assume that $\{(QVLI)_x : x \in X\}$ is parametrically well-posed. It is clear that for every $x \in X$, the unique solution of $(QVLI)_x$ is in the $Q_\varepsilon^\delta(x)$ for all $\varepsilon, \delta > 0$. We only need to show

$$\lim_{(\varepsilon, \delta) \to (0,0)} \operatorname{diam} Q_\varepsilon^\delta(x) = 0. \tag{3.2}$$

Indeed, suppose that (3.2) is false. Then there exist positive number $\alpha > 0$ and sequences $\{\varepsilon_n\}, \{\delta_n\} \subseteq R_+$, ε_n decreasing to 0 and δ_n converging to 0 such that $\operatorname{diam} Q_{\varepsilon_n}^{\delta_n}(x) > \alpha$, for all $n \in N$. We can take $u_n^{(1)}, u_n^{(2)} \in Q_{\varepsilon_n}^{\delta_n}(x)$ such that

$$\| u_n^{(1)} - u_n^{(2)} \| > \frac{\alpha}{2}. \tag{3.3}$$

Thus there exist $x_n^{(1)}, x_n^{(2)} \in B(x, \delta_n)$ with $u_n^{(1)} \in Q_{\varepsilon_n}(x_n^{(1)})$ and $u_n^{(2)} \in Q_{\varepsilon_n}(x_n^{(2)})$, $n \in N$. It is easy to see that the sequences $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ are approximating sequences of $(QVLI)_x$ corresponding to $\{x_n^{(1)}\}$ and $\{x_n^{(2)}\}$, respectively. It follows from the assumption that both of the sequences $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ converge to the unique solution of $(QVLI)_x$, and so $\lim_{n \to \infty} \| u_n^{(1)} - u_n^{(2)} \| \to 0$. This contradicts (3.3). Thus, $\lim_{(\varepsilon, \delta) \to (0,0)} \operatorname{diam} Q_\varepsilon^\delta(x) = 0$.

"⇐" For every $x \in X$, suppose that $Q_\varepsilon^\delta(x) \neq \varnothing \ (\forall \varepsilon, \delta > 0)$ and $\lim_{(\varepsilon, \delta) \to (0,0)} \operatorname{diam} Q_\varepsilon^\delta(x) = 0$. Since every solution of $(QVLI)_x$ belongs to $Q_\varepsilon^\delta(x)$, there is at most one solution of $(QVLI)_x$.

Let $\{x_n\} \subseteq X$ with $x_n \to x$ and $\{u_n\}$ be an approximating sequence of $(QVLI)_x$ corresponding to $\{x_n\}$. Then there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$, decreasing to 0 such that $d(u_n, F(u_n)) \leq \varepsilon_n$ and $f(x_n, u_n, \eta(u_n, v)) \leq \varepsilon_n$, $\forall v \in F(u_n)$.

We first show that $\{u_n\}$ is a convergent sequence. In fact, let $\delta_n = \| x_n - x \|$. It is obvious that $u_n \in Q_{\varepsilon_n}^{\delta_n}(x)$. Since $\varepsilon_n, \delta_n \to 0$ and $\lim_{(\varepsilon, \delta) \to (0, 0)} \operatorname{diam} Q_\varepsilon^\delta(x) = 0$, it follows that $\{u_n\}$ is a Cauchy sequence, and so converges to some $u_x \in K$.

Next, we show that u_x is a solution of $(QVLI)_x$. For each $n \in N$, we can take $u'_n \in F(u_n)$ such that $\| u_n - u'_n \| \leq 2d(u_n, F(u_n)) \leq 2\varepsilon_n$. Thus $u'_n \to u_x$. Since F is closed, we get $u_x \in F(u_x)$. The lower semicontinuity of F implies that $F(u_x) \subseteq \liminf_n F(u_n)$. Further, we have that $\operatorname{int} F(u_x) \neq \varnothing$ from condition (i) by taking $u_n = u_x$ for all $n \in N$. Thus, from Lemma 3.2, it follows that, if $v \in \operatorname{int} F(u_x)$, then $v \in F(u_n)$, for n sufficiently large. By using conditions (ii) and (iii), we have that

$$\begin{aligned} f(x, v, \eta(u_x, v)) &\leq \liminf_{n \to \infty} f(x_n, v, \eta(u_n, v)) \\ &\leq \liminf_{n \to \infty} f(x_n, u_n, \eta(u_n, v)) \\ &\leq \liminf_{n \to \infty} \varepsilon_n = 0. \end{aligned}$$

By Lemma 3.1, we have that $f(x, u_x, \eta(u_x, v)) \leq 0$ for all $v \in \operatorname{int} F(u_x)$. If $v \in F(u_x) - \inf F(u_x)$, then there exists a sequence $\{v_n\} \subseteq \operatorname{int} F(u_x)$ such that $v_n \to v$, and so $f(x, u_x, \eta(u_x, v)) \leq 0$ from condition (ii) and the continuity of η. Hence, u_x is the solution of the $(QVLI)_x$. Thus, $\{(QVLI)_x : x \in X\}$ is parametrically well-posed and the proof is completed. □

When f is a lower semicontinuous function, the conditions associated with F and η can be weakened and we can obtain the succinct result as follows.

Theorem 3.2. *Let X and E be Banach spaces*, *K be a nonempty closed subset of E. Let further $F : K \to 2^K$ be a closed and lower semicontinuous set-valued* *mapping with nonempty values*, $\eta : K \times K \to E$ *be a single-valued continuous mapping*, *and* $f : X \times K \times E \to R$ *be a lower semicontinuous function*. *Then* $\{(QVLI)_x : x \in X\}$ *is parametrically well-posed if and only if for every* $x \in X$,

$$Q_\varepsilon^\delta(x) \neq \emptyset,\ \forall \varepsilon, \delta > 0, \text{ and } \lim_{(\varepsilon, \delta) \to (0, 0)} \operatorname{diam} Q_\varepsilon^\delta(x) = 0. \tag{3.4}$$

Proof. The necessity can be shown as in the proof of Theorem 3.1. For the sufficiency, let $x \in X$ be fixed, $\{x_n\} \subseteq X$ with $x_n \to x$ and $\{u_n\} \subseteq K$ be an approximating sequence of $(QVLI)_x$ corresponding to $\{x_n\}$. Then there exists a sequence $\{\varepsilon_n\}, \varepsilon_n > 0$, decreasing to 0 such that $d(u_n, F(u_n)) \leq \varepsilon_n$ and $f(x_n, u_n, \eta(u_n, v)) \leq \varepsilon_n, \forall v \in F(u_n)$. From (3.4), we know that the problem $(QVLI)_x$ has at most one solution and u_n converges to some point $u_x \in F(u_x)$. The lower semicontinuity of F implies that for any $v \in F(u_x)$, there exists $v_n \in F(u_n)$ such that v_n converges to v. In light of the lower semicontinuity of f and the continuity of η, we have that

$$f(x, u_x, \eta(u_x, v)) \leq \liminf_{n \to \infty} f(x_n, u_x, \eta(u_n, v_n)) \leq \liminf_{n \to \infty} \varepsilon_n = 0.$$

Consequently, u_x is the unique solution of $(QVLI)_x$. Thus $\{(QVLI)_x : x \in X\}$ is parametrically well-posed. □

Remark 3.1. When $\eta(u, v) = u - v$ and $F(u) = K$ for all $u \in K$, the above result implies Theorem 3.2 in [12]. It also generalized Proposition 2.3 in [11].

Now, we give the following examples to show the applications of Theorem 3.2.

Example 3.1. Let $X = E = R$, $K = [1, +\infty]$, $f(x, u, v) = u^2 v$, $\eta(u, v) = u^2 - v^2$ and consider the set-valued mapping F defined by $F(u) = [1, 2u]$. We observe that the functions f, η and set-valued mapping F are continuous. We can calculate that the problem $(QVLI)_x$ has the unique solution $u = 1$ for all $x \in X$, and the set $Q_\varepsilon^\delta(x) = \left[1, \sqrt{\frac{1 + \sqrt{1 + 4\varepsilon}}{2}}\right]$. It follows that $\lim_{(\varepsilon, \delta) \to (0, 0)} \operatorname{diam} Q_\varepsilon^\delta(x) = 0$. By Theorem 3.2, $\{(QVLI)_x : x \in X\}$ is parametrically well-posed.

Example 3.2. Let $X = E = R$, $K = [1, +\infty]$, $f(x, u, v) = e^{-u}v$, $\eta(u, v) = u^2 - v^2$ and consider the set valued function F defined by $F(u) = [1, 2u]$. We observe that the functions *f*, η and set-valued mapping F are continuous. We can calculate that the problem $(QVLI)_x$ has the unique solution $u = 1$ for all $x \in X$, and the set $Q_\varepsilon^\delta(x) = \{u \in [1, +\infty] : e^{-u}(u^2 - 1) \le \varepsilon\}$. By Theorem 3.2, $(QVLI)_x : x \in X$ is not parametrically well-posed since $\lim_{(\varepsilon, \delta) \to (0,0)} \operatorname{diam} Q_\varepsilon^\delta(x) \ne 0$.

When the subset K is compact, we can obtain the following conclusion that $\{(QVLI)_x : x \in X\}$ is parametrically well-posed if and only if $(QVLI)_x$ has a unique solution for every $x \in X$.

Theorem 3.3. *Let X, E be Banach spaces and K be a nonempty compact subset of E. Let further* $F : K \to 2^K$ *be a closed and lower semicontinuous set-valued mapping with nonempty values,* $\eta : K \times K \to E$ *be a single-valued continuous mapping and* $f : X \times K \times E \to R$ *be a real valued lower semicontinuous function. Then* $\{(QVLI)_x : x \in X\}$ *is parametrically well-posed if and only if* $(QVLI)_x$ *has a unique solution for every* $x \in X$.

Proof. The necessity is trivial by the definition of parametric well-posedness of $\{(QVLI)_x : x \in X\}$. Conversely, let u_x be the unique solution of $(QVLI_x)$ and $\{u_n\}$ be an approximating sequence for $(QVLI_x)$ corresponding to $\{x_n\}$ $(x_n \to x)$. Then there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$, decreasing to 0 such that $d(u_n, F(u_n)) \le \varepsilon_n$ and $f(x_n, u_n, \eta(u_n, v)) \le \varepsilon_n$, $\forall v \in F(u_n)$. Suppose that $\{u_{n_k}\}$ is a convergent subsequence of $\{u_n\}$ with limit u_0. We can take $u'_{u_k} \in F(u_{n_k})$ $(k \in N)$ such that $\| u'_{n_k} - u_{n_k} \| \le 2\varepsilon_{n_k}$, and so $u'_{n_k} \to u_0$. Since F is closed, we have that $u_0 \in F(u_0)$. The lower semicontinuity of F implies that for any $v \in F(u_0)$, there exists a sequence $\{v_{n_k}\}$ with $v_{n_k} \in F(u_{n_k})$ such that v_{n_k} converges to *v*. In light of the lower semicontinuity of *f* and the continuity of η, we have that

$$\begin{aligned} f(x, u_0, \eta(u_0, v)) &\le \liminf_{k \to \infty} f(x_{n_k}, v_{n_k}, \eta(u_{n_k}, v_{n_k})) \\ &\le \liminf_{k \to \infty} \varepsilon_{n_k} = 0, \ \forall v \in F(u_0). \end{aligned}$$

This means that u_0 is a solution of $(QVLI)_x$. Thus $u_0 = u_x$. Since K is compact, we have that the sequence $\{u_n\}$ converges to $u_0 = u_x$ and the proof is completed. □

Remark 3.2. In finite dimensional space, without the condition that K is compact, Fang and Hu in [12] (Theorem 5.1) proved that variational inequalities problem is parametrically well-posed if and only if it has a unique solution for every parameter. For infinite dimensional space and the quasivariational-like inequalities problem, the compactness of K in Theorem 3.3 is essential: if we drop it, the parametric well-posedness of $\{(QVLI)_x : x \in X\}$ cannot be guaranteed, as Example 3.2 shows.

4. Parametric Well-posedness in Generalized Sense for Quasivariational-like Inequalities

In this section, we consider the parametric well-posedness in generalized sense for $\{(QVLI)_x : x \in X\}$.

Let E be a complete metric space. Recall that the Kuratowski measure of noncompactness μ for a subset A of E is defined by

$$\mu(A) = \inf\left\{\varepsilon > 0 : A \subseteq \bigcup_{1 \le i \le n} C_i,\ \operatorname{diam} C_i \le \varepsilon,\ i = 1, \ldots, n\right\},$$

where $\operatorname{diam} C_i$ is the diameter of C_i.

For given two nonempty subsets A and B of E, the Hausdorff distance between A and B is defined as

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

Theorem 4.1. *Let X and E be Banach spaces, K be a nonempty closed convex subset of E. Let further $F : K \to 2^K$ be a closed and lower semicontinuous set-valued mapping with nonempty convex values, $\eta : K \times K \to E$ be a single-valued continuous mapping satisfying Condition A, $f : X \times K \times E \to R$ be a real valued function. Suppose that the following conditions hold*:

(i) *for every converging sequence $\{u_n\} \subseteq K$, there exists an $m \in N$ such that* $\operatorname{int} \bigcap_{n \ge m} F(u_n) \neq \emptyset$;

(ii) $f(\cdot, u, \cdot)$ *is lower semicontinuous for all* $u \in K$;

(iii) *for every* $x \in K$, *the function* $f(x, \cdot, \cdot)$ *is lower d-semicontinuous and* η-*monotone*;

(iv) *for all* $(x, u) \in X \times K$, $f(x, u, \cdot)$ *is positively homogeneous.*

Then $\{(QVLI)_x : x \in X\}$ *is parametrically well-posed in generalized sense if and only if for every* $x \in X$,

$$Q_\varepsilon^\delta(x) \neq \emptyset, \forall \varepsilon, \delta > 0, \text{ and } \lim_{(\varepsilon, \delta)\to(0,0)} \mu(Q_\varepsilon^\delta(x)) = 0. \tag{4.5}$$

Proof. "$\Rightarrow$" Assume that $\{(QVLI)_x : x \in X\}$ is parametrically well-posed in generalized sense. For every $x \in X$, the solution set $Q(x)$ of $(QVLI)_x$ is nonempty and $Q(x) \subseteq Q_\varepsilon^\delta(x)$ for all $\delta > 0$. It follows that the set $Q_\varepsilon^\delta(x)$ is nonempty. Since $Q(x)$ is compact, we have

$$\begin{aligned}\mu(Q_\varepsilon^\delta(x)) &\leq 2H(Q_\varepsilon^\delta(x), Q(x)) + \mu(Q(x)) \\ &= 2H(Q_\varepsilon^\delta(x), Q(x)) = 2 \sup_{u \in Q_\varepsilon^\delta(x)} d(u, Q(x)).\end{aligned}$$

Assume that $\lim_{(\varepsilon, \delta)\to(0,0)} \mu(Q_\varepsilon^\delta)(x) \neq 0$. Then there exist $\alpha > 0$ and the sequences of positive numbers ε_n and δ_n decreasing to 0 such that $\mu(Q_{\varepsilon_n}^{\delta_n}(x)) > \alpha$, and so there is a sequence $\{u_n\} \subseteq K$ with $u_n \in Q_{\varepsilon_n}^{\delta_n}(x)$ such that

$$d(u_n, Q(x)) > \frac{\alpha}{3}. \tag{4.6}$$

On the other hand, there exists $\{x_n\} \subseteq X$ such that $\| x_n - x \| \leq \delta_n$ with $u_n \in Q_{\varepsilon_n}(x_n)$. Then the sequence $\{u_n\}$ is an approximating sequence of $(QVLI)_x$ corresponding to $\{x_n\}$. Since $\{(QVLI)_x : x \in X\}$ is parametrically well-posed in generalized sense, there exists a subsequence $\{u_{n_k}\}$ converging to some point of $Q(x)$, which contradicts (4.6). Thus (4.5) holds.

"$\Leftarrow$" Assume that (4.5) holds. For every $x \in X$, let $\{u_n\}$ be an approximating sequence of $(QVLI)_x$ corresponding to $\{x_n\}$ $(x_n \to x)$. Then there exists a sequence

$\{\varepsilon_n\}$, $\varepsilon_n > 0$, decreasing to 0 such that $u_n \in Q_{\varepsilon_n}^{\delta_n}(x)$, where $\delta_n = \| x_n - x \|$. Without loss of generality, we suppose that δ_n is a decreasing sequence. Then $\{Q_{\varepsilon_n}^{\delta_n}(x)\}$ is a decreasing sequence. By using the similar method in [24, p. 4], we consider the decreasing sequence of sets $Q_n = \{u_k : k \geq n\}$. Then $Q_n \subseteq Q_{\varepsilon_n}^{\delta_n}(x)$, and so $\mu(Q_1) = \mu(Q_n) \leq \mu(Q_{\varepsilon_n}^{\delta_n}(x))$ for all $n \in N$. This shows that $\mu(Q_1) = 0$, and so $Q_1 = \{u_n : n \in N\}$ is precompact. Thus there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging to some point $u_x \in K$. From the proof of Theorem 3.1, we know that $u_x \in F(u_x)$ is a solution of $(QVLI)_x$. Therefore, $\{(QVLI)_x : x \in X\}$ is parametrically well-posed in generalized sense. □

When the continuity of *f* is strengthened, we can obtain the following theorem:

Theorem 4.2. *Let X and E be Banach spaces, K be a nonempty closed subset of E. Let further* $F : K \to 2^k$ *be a closed and lower semicontinuous set-valued mapping with nonempty values,* $\eta : K \times K \to E$ *be a single-valued continuous mapping, and* $f : X \times K \times E \to R$ *be a lower semicontinuous function. Then* $\{(QVLI)_x : x \in X\}$ *is parametrically well-posed in generalized sense if and only if for every* $x \in X$,

$$Q_\varepsilon^\delta(x) \neq \emptyset, \forall \varepsilon, \delta > 0, \text{ and } \lim_{(\varepsilon, \delta) \to (0, 0)} \mu(Q_\varepsilon^\delta(x)) = 0. \tag{4.7}$$

Proof. The necessity follows from Theorem 4.1. Conversely, assume that (4.7) holds. For every $x \in X$, let $\{u_n\}$ be an approximating sequence of $(QVLI)_x$ corresponding to $\{x_n\}$ $(x_n \to x)$. As the same discussion in Theorem 4.1, we have that $\{u_n\}$ has subsequence $\{u_{n_k}\}$ converging to some u_x and $u_x \in F(u_x)$. Since F is lower semicontinuous, we get that for any $v \in F(u_x)$, there exists $v_{n_k} \in F(u_{n_k})$ such that v_{n_k} converges to *v*. In light of the lower semicontinuity of *f* and continuity of η, we have that

$$f(x, u_x, \eta(u_x, v)) \leq \liminf_{k \to \infty} (x_{n_k}, u_{n_k}, \eta(u_{n_k}, v_{n_k})) \leq \liminf_{k \to \infty} \varepsilon_{n_k} = 0.$$

Thus, u_x is a solution of $(QVLI)_x$ and $\{(QVLI)_x : x \in X\}$ is parametrically well-posed in generalized sense. □

Remark 4.1. When $\eta(u, v) = u - v$ and $F(u) = K$, for all $u \in K$, the above result implies the Theorem 4.2 in [12].

Example 4.1. Let $X = E = R$, $K = [0, +\infty]$, $f(x, u, v) = u^2 v$, $\eta(u, v) = u^2 - v^2$ and consider the set valued function F defined by

$$F(u) = \begin{cases} [u, 1], & 0 \leq u \leq 1, \\ [1, 2u], & u > 1. \end{cases}$$

We observe that the functions f and η are continuous, and that the set-valued mapping F is closed and lower semicontinuous with nonempty values. We can calculate that $Q(x) = [0, 1]$ and $Q_\varepsilon^\delta(x) = \left[0, \sqrt{\frac{1 + \sqrt{1 + 4\varepsilon}}{2}}\right]$. It follows that $\lim_{(\varepsilon, \delta) \to (0, 0)} \mu(Q_\varepsilon^\delta(x)) = 0$. By Theorem 4.2, $\{(QVLI)_x : x \in X\}$ is parametrically well-posed in generalized sense.

In the end, we will give some sufficient conditions for parametric well-posedness in generalized sense of $\{(QVLI)_x : x \in X\}$.

Theorem 4.3. *Let X and E be Banach spaces, K be a nonempty compact convex subset of E. Let further $F : K \to 2^K$ be a continuous set-valued mapping with nonempty convex closed values, $\eta : K \times K \to E$ be a single-valued continuous mapping with $\eta(u, u) = 0$ for all $u \in K$ and $f : X \times K \times E \to R$ be a real valued continuous function satisfying that $f(x, u, \eta(u, \cdot))$ is concave and $f(x, u, 0) = 0$ for all $(x, u) \in X \times K$. Then $\{(QVLI)_x : x \in X\}$ is parametrically well-posed in generalized sense.*

Proof. Since K is compact and F is closed-valued, the set $F(u)$ is compact for all $u \in K$. For every fixed $x \in X$, we define the set-valued mapping $T_x : K \to 2^K$ by

$$T_x(u) = \{w \in K : f(x, u, \eta(u, w)) = \max_{v \in F(u)} f(x, u, \eta(u, v))\}.$$

Since the functions f and η are continuous and the set-valued mapping F is continuous with nonempty compact values, we have that the set-valued mapping T_x is upper semicontinuous with nonempty compact values (see [25, p. 120]). Since $f(x, u, \eta(u, \cdot))$ is concave and F has convex values, we deduce that $T_x(u)$ is convex set for all $u \in K$. The Kakutani fixed-point theorem implies that there exists a point $u \in K$ such that $u \in T_x(u)$, i.e.,

$$u \in F(u) \text{ and } f(x, u, \eta(u, v)) \leq f(x, u, \eta(u, u)) = 0, \forall v \in F(u).$$

This shows that the solution set $Q(x)$ of $(QVLI)_x$ is nonempty.

Now, let $\{u_n\} \subseteq K$ be an approximating sequence corresponding to $\{x_n\}$ $(x_n \to x)$ for $(QVLI)_x$. Then there exists a sequence $\{\varepsilon_n\}, \varepsilon_n > 0$, decreasing to 0 such that

$$d(u_n, F(u_n)) \leq \varepsilon_n \text{ and } f(x_n, u_n, \eta(u_n, v)) \leq \varepsilon_n, \forall v \in F(u_n).$$

Since K is compact, there exists a subsequence $\{u_{n_k}\}$ converging to some point $u_0 \in K$. We can take a sequence of elements $u'_{n_k} \in F(u_{n_k})$ such that $\| u'_{n_k} - u_{n_k} \| \leq 2d(u_{n_k}, F(u_{n_k})) \leq 2\varepsilon_{n_k}$. It follows that $u'_{n_k} \to u_0$. Since every upper semicontinuous set-valued mapping with closed values is closed, we have that $u_0 \in F(u_0)$. The lower semicontinuity of F implies that for any $v \in F(u_0)$, there exist $v_{n_k} \in F(u_{n_k})$ $(k = 1, 2, ...)$ such that the sequence $\{v_{n_k}\}$ converges to v. By using the continuity of f and η, we have that

$$f(x, u_0, \eta(u_0, v)) = \lim_{k \to \infty} f(x_{n_k}, u_{n_k}, \eta(u_{n_k}, v_{n_k})) \leq \lim_{k \to \infty} \varepsilon_{n_k} = 0.$$

Thus, $u_0 \in Q(x)$ and the problem $\{(QVLI)_x : x \in X\}$ is parametrically well-posed in generalized sense. □

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