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AN APPLICATION OF ENERGY BALANCE METHOD FOR A CONSERVATIVE $X^{1/3}$ FORCE NONLINEAR OSCILLATOR AND OSCILLATOR AND THE DUFFING EQUATIONS

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Abstract

In this paper, we use Energy Balance Method (EBM) to nonlinear vibration and oscillation equations to obtain the periodic solutions of a conservative nonlinear oscillator for which the elastic force term is proportional to $X^{1/3}$. The results show that the EBM is very effective and simple so that do not require linearization or small perturbation.

1. Introduction

The nonlinear vibration and oscillation equations have been considered in several papers [1-3]. Main model was introduced by Mickens [16] and has been studied by many investigators [17, 26, 29, 39]. Various methods have been used to solve them, for example, Homotopy Perturbation Method (HPM) and Harmonic Balance Method (HBM). Our purpose is to solve conservative $X^{1/3}$ force nonlinear

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oscillator and oscillator and the Duffing equations by EBM, and the results of each of them will be compared with exact, HPM, HBM solutions, respectively.

2. Solution Procedure

We consider the following nonlinear oscillator:

$$\frac{d^2X}{dt^2} + X^{\frac{1}{3}} = 0 \tag{1}$$

with initial conditions:

$$x(0) = A, \quad \frac{dX}{dt}(0) = 0. \tag{2}$$

Equation (1) is a conservative nonlinear oscillator with a fractional power restoring force. We denote the angular frequency of these oscillations by ω and note that one of our major tasks is to determine $\omega(A)$, the functional behavior of ω as a function of the initial amplitude.

In order to assess the advantages and the accuracy of EBM, we will apply this method to (1) with initial conditions (2).

Its formulation can easily be established by:

$$J = \int_0^t \left(\frac{1}{2}\left(\frac{dX}{dt}\right)^2 + \frac{3}{4} X^{\frac{4}{3}} \right) dt. \tag{3}$$

Its Hamiltonian can be written in the form:

$$H = \left(\frac{1}{2}\left(\frac{dX}{dt}\right)^2 + \frac{3}{4} X^{\frac{4}{3}} \right) \tag{4}$$

with initial conditions:

$$X(0) = A, \quad \frac{dX(0)}{dt} = 0. \tag{5}$$

Therefore,

$$H_{t=0} = \frac{3}{4} A^{\frac{4}{3}}, \tag{6}$$

$$H_t - H_{t=0} = \left(\frac{1}{2}\left(\frac{dX}{dt}\right)^2 + \frac{3}{4} X^{\frac{4}{3}} \right) - \frac{3}{4} A^{\frac{4}{3}}. \tag{7}$$

We will use the trial function to determine the angular frequency ω, that is,

$$X(t) = A\cos(\omega t). \tag{8}$$

If we substitute (7) into (6), then we get the following residual equation:

$$\frac{1}{2}A^2\omega^2\sin^2(\omega t) + \frac{3}{4}(A\cos(\omega t))^{\frac{4}{3}} - \frac{3}{4}A^{\frac{4}{3}} = 0. \tag{9}$$

If we use $\omega t = \frac{\pi}{4}$, then we obtain:

$$\frac{1}{4}A^2\omega^2 + \frac{3}{8}(2A)^{\frac{1}{3}} - \frac{3}{4}A^{\frac{4}{3}} = 0 \tag{10}$$

hence

$$\omega = \frac{1}{2}\frac{\sqrt{6}}{A^{\frac{1}{3}}}\sqrt{-2^{\frac{1}{3}} + 2}. \tag{11}$$

If we compare our result with the exact solution given in [40], then we get:

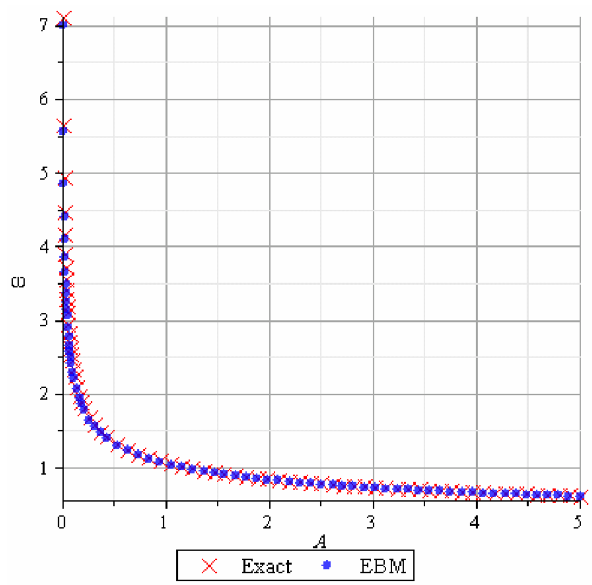

Figure (1-1)

Now, for second example, we consider the following nonlinear oscillator:

$$\frac{d^2}{dt^2}X(t) + X(t)^{\frac{1}{3}} + X(t)^{\frac{1}{3}} = 0 \tag{12}$$

with initial conditions:

$$X(0) = A, \quad \frac{dX(0)}{dt} = 0. \tag{13}$$

We will apply EBM to solve (12). First, we establish its formulation by:

$$J = \int_0^t \left(\frac{1}{2}\left(\frac{dX}{dt}\right)^2 + \frac{3}{4}X^{\frac{4}{3}} + \frac{1}{4}X^4 \right) dt. \tag{14}$$

Thus, its Hamiltonian can be written in the form:

$$H = \left(\frac{1}{2}\left(\frac{dX}{dt}\right)^2 + \frac{3}{4}X^{\frac{4}{3}} \right) \tag{15}$$

with

$$H_{t=0} = \frac{3}{4}A^{\frac{4}{3}} + \frac{1}{4}A^4 \tag{16}$$

and

$$H_t - H_{t=0} = \left(\frac{1}{2}\left(\frac{dX}{dt}\right)^2 + \frac{3}{4}X^{\frac{4}{3}} + \frac{1}{4}X^4 \right) - \frac{3}{4}A^{\frac{4}{3}} - \frac{1}{4}A^4. \tag{17}$$

We will use the trial function

$$X(t) = A\cos(\omega t) \tag{18}$$

in (17) to determine the angular frequency ω. This leads to

$$\frac{1}{2}A^2\omega^2\sin^2(\omega t) + \frac{3}{4}(A\cos(\omega t))^{\frac{4}{3}} + \frac{1}{4}((A\cos(\omega t))^4) - \frac{3}{4}A^{\frac{4}{3}} - \frac{1}{4}A^4 = 0. \tag{19}$$

If we put $\omega t = \frac{\pi}{4}$, then we obtain:

$$\frac{1}{4}A^2\omega^2 + \frac{3}{16}2^{\frac{2}{3}}(A\sqrt{2})^{\frac{4}{3}} - \frac{3}{16}A^4 - \frac{3}{4}A^{\frac{4}{3}} = 0. \tag{20}$$

Hence

$$\omega = -\frac{1}{2}\frac{\sqrt{-6 2^{\frac{1}{6}} A (A\sqrt{2})^{\frac{1}{3}} + 3A^4 + 12A^{\frac{4}{3}}}}{A}. \tag{21}$$

So, if we compare our result with the result of HPM given in [2], then we get:

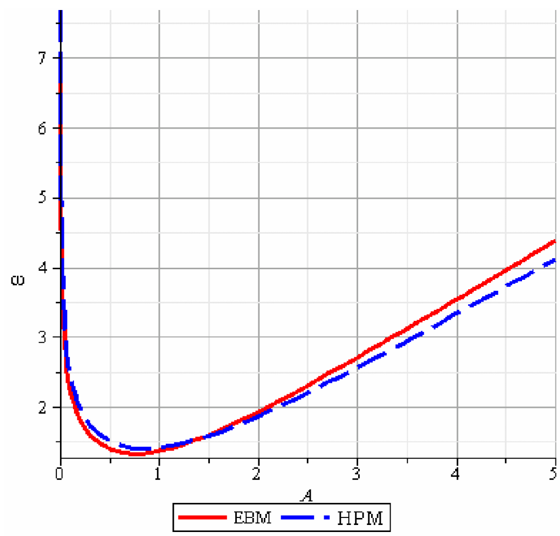

Figure (1-2)

Our third example is the Duffing oscillator:

Consider

$$\frac{d^2}{dt^2}X(t) + X(t) + \varepsilon X(t)^3 \tag{22}$$

with initial conditions:

$$X(0) = A, \quad \frac{dX(0)}{dt} = 0. \tag{23}$$

We will apply EBM to solve (22). We establish again its formulation by:

$$J = \int_0^t \left(\frac{1}{2}\left(\frac{d}{dt}X(t)\right)^2 + \frac{1}{2}X(t)^2 + \frac{1}{4}\varepsilon X(t)^4\right)dt. \tag{24}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \left(\frac{1}{2}\left(\frac{dX(t)}{dt}\right)^2 + \frac{1}{2}X(t)^2 + \frac{1}{4}\varepsilon X(t)^4\right) \tag{25}$$

with

$$H_{t=0} = \frac{1}{2}A^2 + \frac{1}{4}\varepsilon A^4 \tag{26}$$

and

$$H_t - H_{t=0} = \left(\frac{1}{2}\left(\frac{dX(t)}{dt}\right)^2 + \frac{1}{2}X(t)^2 + \frac{1}{4}\varepsilon X(t)^4\right) - \frac{1}{2}A^2 - \frac{1}{4}\varepsilon A^4. \tag{27}$$

We will use the trial function to determine the angular frequency ω, that is,

$$X(t) = A\cos(\omega t). \tag{28}$$

In (27), the result will be

$$\frac{1}{2}A^2\omega^2\sin^2(\omega t) + \frac{1}{2}(A\cos(\omega t))^2 + \frac{1}{4}(\varepsilon(A\cos(\omega t))^4) - \frac{1}{2}A^2 - \frac{1}{4}\varepsilon A^4 = 0. \tag{29}$$

If we put $\omega t = \frac{\pi}{4}$, then we get:

$$\frac{1}{4}A^2\omega^2 - \frac{1}{4}A^2 - \frac{3}{16}\varepsilon A^4 = 0, \tag{30}$$

hence

$$\omega = \frac{1}{2}\sqrt{4 + 3\varepsilon A^2}. \tag{31}$$

Once more, we compare our result with the result given in [3] and get:

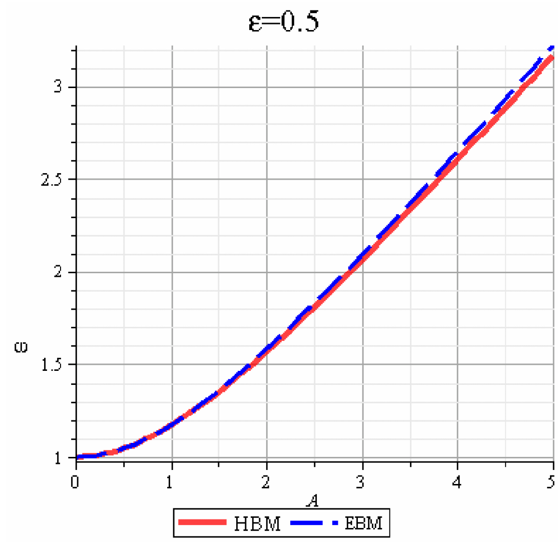

Figure (1-3)

3. Conclusion

Results in this paper show the accuracy and efficiency of EBM in studying of nonlinear vibrating equations, and is a powerful mathematical tool to investigate them and can easily be extended to any such nonlinear equations.

Results show that the solutions obtained by this method, are in a good agreement, with other results.

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