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ON CHEBYSHEV VARIETIES, I: INTEGRAL POINTS

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Abstract

Two families of determinantal varieties $V_n, W_n, n \geq 1,$ defined over $\mathbf{Q}$, are introduced and their integral points $V_n(\mathbf{Z}), W_n(\mathbf{Z})$ are determined completely.

1. Introduction

The main purpose of this paper is to introduce two families of determinantal varieties $V_n, W_n, n \geq 1,$ defined over $\mathbf{Q}$ and to determine the sets $V_n(\mathbf{Z}), W_n(\mathbf{Z})$ of their integral points completely. As a result we will see that for any n, the whole $V_n(\mathbf{Z})$ (resp. $W_n(\mathbf{Z})$) is contained in a finite union of linear subvarieties of V_n (resp. W_n). Our result also enables us to obtain a complete list of the n-tuples $(a_1, ..., a_n)$ of integers such that the continued fraction $a_1 - \cfrac{1}{a_2 - \cdots \cfrac{1}{a_{n-1} - \cfrac{1}{a_n}}}$ vanishes.

Originally, these varieties arise in the course of our study of generalized Chebyshev polynomials [1]. Moreover the diagonal specialization of the defining equation of V_n (resp. W_n) is essentially equal to the Chebyshev polynomial of the second (resp. the first) kind (see

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Proposition 2.7 and Proposition 4.4). For this reason we may be entitled to call them *Chebyshev varieties*. In view of the fact that there are not so many examples of higher-dimensional varieties defined over $\mathbf{Q}$ for which the set of integral points is determined completely, the study in this paper may be of some interest.

The plan of this paper is as follows. In Section 2 we give a definition of Chebyshev variety of the second kind. The defining equation satisfies quite a few identities. We restrict our attention on, and prove a part of those identities which play some roles in our determination of their integral points. In Section 3, we introduce three families of morphisms between Chebyshev varieties of the second kind. They enable us to construct any integral points on a Chebyshev variety from those on lower-dimensional ones. Thus they provide us with an inductive procedure which produces all integral points on the Chebyshev varieties (Theorem 3.5), as well as a recursive procedure which decide whether a point of $\mathbf{Z}^n$ lies on V_n or not (Theorem 3.7). Section 4 is devoted to the investigation of the Chebyshev varieties of the first kind W_n, and gives a complete description of the sets of the integral points on them.

2. Polynomial Identities for the Defining Equations

For any n independent variables $x_1, x_2, ..., x_n$, let

$$U(x_1, x_2, \ldots, x_n) = \begin{pmatrix} x_1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & x_2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & x_3 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & x_{n-1} & -1 \\ 0 & 0 & 0 & \cdots & -1 & x_n \end{pmatrix},$$

and let $u(x_1, x_2, ..., x_n) = \det U(x_1, x_2, ..., x_n)$. One of the main purposes of this article is to determine the set of integral points of the variety $V_n = \{u(x_1, x_2, ..., x_n) = 0\} \subset \mathbf{A}^n$, which we call *Chebyshev variety of the second kind*. In this preliminary section we derive some polynomial identities which play important roles in the next section.

Proposition 2.1.

(i) $u(x_1, x_2, \ldots, x_n) = u(x_n, x_{n-1}, \ldots, x_1)$.

(ii) $u(x_1, x_2, \ldots, x_n) = x_1 u(x_2, \ldots, x_n) - u(x_3, \ldots, x_n)$.

(iii) $u(x_1, x_2, \ldots, x_n) = x_n u(x_1, \ldots, x_{n-1}) - u(x_1, \ldots, x_{n-2})$.

Proof. (i) Exchange the rows of $U(x_1, x_2, \ldots, x_n)$ according to the permutation $\rho = (1\ n)(2\ n-1)\cdots$, and apply it to the columns, then the resulted matrix is $U(x_n, x_{n-1}, \ldots, x_1)$.

(ii) Expanding the determinant $u(x_1, x_2, \ldots, x_n)$ along the first row, we have

$$u(x_1, \ldots, x_n) = x_1 \det \begin{pmatrix} x_2 & -1 & \cdots & 0 & 0 \\ -1 & x_3 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & x_{n-1} & -1 \\ 0 & 0 & \cdots & -1 & x_n \end{pmatrix} + \det \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ -1 & x_3 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & x_{n-1} & -1 \\ 0 & 0 & \cdots & -1 & x_n \end{pmatrix}$$

$$= x_1 u(x_2, x_3, \ldots, x_n) - u(x_3, \ldots, x_n).$$

(iii) Combine (i) and (ii).

Remark 2.2. We will use the simplified notation $u[i, j]$ to express $u(x_i, x_{i+1}, \ldots, x_j)$ when $i < j$. Moreover we employ the convention that

$$u[i, i] = x_i, \quad u[i, i-1] = 1, \quad u[i, i-2] = 0,$$

for any $i \geq 1$. This will enable one to describe various identities in a unified way. The reader might be convinced of its usefulness by setting $n = 1$ in (iii) above.

The following proposition is entitled to be called a *mother identity*, since this produces several other remarkable identities.

Proposition 2.3. *Let* $A(x)$ *denote the matrix* $\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$. *Then we have*

$$A(x_n)A(x_{n-1})\cdots A(x_1) = \begin{pmatrix} -u[2,\ n-1] & -u[1,\ n-1] \\ u[2,\ n] & u[1,\ n] \end{pmatrix} \tag{2.1}$$

for any $n \geq 1$.

(Note that we are in the convention explained in Remark 2.2.)

Proof. When $n = 1$, (2.1) holds by Remark 2.2. When $n \geq 2$, we assume (2.1) holds for smaller n. Then we have

$$\begin{aligned} A(x_n)A(x_{n-1})\cdots A(x_1) &= A(x_n)\begin{pmatrix} -u[2,\ n-2] & -u[1,\ n-2] \\ u[2,\ n-1] & u[1,\ n-1] \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & x_n \end{pmatrix}\begin{pmatrix} -u[2,\ n-2] & -u[1,\ n-2] \\ u[2,\ n-1] & u[1,\ n-1] \end{pmatrix} \\ &= \begin{pmatrix} -u[2, n-1] & -u[1, n-1] \\ x_n u[2, n-1] - u[2, n-2] & x_n u[1, n-1] - u[1, n-2] \end{pmatrix} \\ &= \begin{pmatrix} -u[2,\ n-1] & -u[1,\ n-1] \\ u[2,\ n] & u[1,\ n] \end{pmatrix}, \end{aligned}$$

which shows that (2.1) holds for n too.

As an application, we derive the following.

Proposition 2.4. *For any k with* $1 \leq k \leq n-1$, *we have*

$$u[1,\ n] = u[1,\ k]u[k+1,\ n] - u[1,\ k-1]u[k+2,\ n]. \tag{2.2}$$

Proof. Divide the product on the left hand side of (2.1) into two:

$$A(x_n)A(x_{n-1})\cdots A(x_1) = (A(x_n)\cdots A(x_{k+1}))(A(x_k)\cdots A(x_1)). \tag{2.3}$$

Then each product on the right hand side is expressed as follows by Proposition 2.3:

$$A(x_n)\cdots A(x_{k+1}) = \begin{pmatrix} -u[k+2,\ n-1] & -u[k+1,\ n-1] \\ u[k+2,\ n] & u[k+1,\ n] \end{pmatrix},$$

$$A(x_k)\cdots A(x_1) = \begin{pmatrix} -u[2,\ k-1] & -u[1,\ k-1] \\ u[2,\ k] & u[1,\ k] \end{pmatrix}.$$

Comparing the lower right entries on the both sides of (2.3), we obtain the desired equality.

Remark 2.5. When $k = 1$ (resp. $k = n - 1$), the equality (2.2) specializes to Proposition 2.1 (ii) (resp. (iii)). When $k = 0$ or $k = n$, (2.2) still holds by the convention in Remark 2.2.

Remark 2.6. The equality (2.2) as well as its generalization for the determinant of band matrices has been known for a long time (see [2, Formula (1), p. 518], for example). Several other identities described in [2] can be, however, proved more easily if one uses the mother equality (2.1) and its generalization

$$\begin{pmatrix} 0 & -1 \\ -c_{n-1} & a_n \end{pmatrix}\begin{pmatrix} 0 & b_{n-1} \\ -c_{n-2} & a_{n-1} \end{pmatrix}\cdots\begin{pmatrix} 0 & b_2 \\ -c_1 & a_2 \end{pmatrix}\begin{pmatrix} 0 & b_1 \\ 1 & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} -f[2, n-1] & -f[1, n-1] \\ f[2, n] & f[1, n] \end{pmatrix}, \tag{2.1'}$$

where

$$f[1, n] = \det\begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & c_{n-1} & a_n \end{pmatrix}.$$

The following proposition reveals an intimate connection of our function u and the Chebyshev polynomial of the second kind.

Proposition 2.7. *Let $U_n(z)$ be the Chebyshev polynomial of the second kind defined by*

$$U_n(z) = \frac{\sin(n+1)\theta}{\sin\theta} \text{ with } z = \cos\theta.$$

Then we have

$$u(\underbrace{x, x, \ldots, x}_{n}) = U_n(x/2).$$

Proof. Let $u_n(x) = u(\underbrace{x, x, \ldots, x}_{n})$, and let $x_1 = \cdots = x_n = x$ in Proposition 2.1 (ii). Then we have

$$u_n(x) = xu_{n-1}(x) - u_{n-2}(x)$$

and $u_0(x) = 1$, $u_1(x) = x$. On the other hand, if we put $\alpha = e^{i\theta}$, then we have $z = \cos\theta = (\alpha + \alpha^{-1})/2$, and hence

$$U_n(z) = \frac{\alpha^{n+1} - \alpha^{-(n+1)}}{\alpha - \alpha^{-1}}.$$

Therefore

$$\begin{aligned}
&2zU_{n-1}(z) - U_{n-2}(z) \\
&= \frac{1}{(\alpha - \alpha^{-1})}((\alpha + \alpha^{-1})(\alpha^n - \alpha^{-n}) - (\alpha^{n-1} - \alpha^{-(n-1)})) \\
&= \frac{\alpha^{n+1} - \alpha^{-(n+1)}}{\alpha - \alpha^{-1}} \\
&= U_n(z).
\end{aligned}$$

Moreover we have $U_0(z) = 1$, $U_1(z) = 2z$. Hence the two sequences $\{u_n(x)\}$ and $\{U_n(z)\}$ satisfy one and the same three-term recurrence formula with common initial values if $x = 2z$. This completes the proof of Proposition 2.7.

Remark 2.8. This proposition may justify our naming the variety $V_n = \{u(x_1, x_2, \ldots, x_n) = 0\}$ Chebyshev variety of the second kind.

Next we introduce three maps, called *blow-up*, *splitting*, and *pasting*. These will play a crucial role to generate the whole integral solutions of the Chebyshev varieties.

Definition 2.9. For any $n \geq 1$, we define three maps

(Blow-up) $\quad \mathrm{blup}^n_{(i;\pm)} : \mathbf{A}^n \to \mathbf{A}^{n+1}, \ 1 \leq i \leq n+1,$

(Splitting) $\quad \mathrm{split}^n_{(i;c)} : \mathbf{A}^n \to \mathbf{A}^{n+2}, \ 1 \leq i \leq n, \ c \in \mathbf{Q},$

(Pasting) $\quad \mathrm{paste}^n_{(\pm;c)} : \mathbf{A}^n \to \mathbf{A}^{n+2}, \ c \in \mathbf{Q},$

by the following rules:

(i) $\mathrm{blup}^n_{(i;\pm)}(x_1, \ldots, x_n)$

$$= (x_1, \ldots, x_{i-2}, x_{i-1} \pm 1, \pm 1, x_i \pm 1, x_{i+1}, \ldots, x_n), \quad 2 \le i \le n,$$

$$\mathrm{blup}^n_{(1;\pm)}(x_1, \ldots, x_n) = (\pm 1, x_1 \pm 1, x_2, \ldots, x_n),$$

$$\mathrm{blup}^n_{(n+1;\pm)}(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, x_n \pm 1, \pm 1),$$

(ii) $\mathrm{split}^n_{(i;c)}(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i - c, 0, c, x_{i+1}, \ldots, x_n)$,

(iii) $\mathrm{paste}^n_{(-;c)}(x_1, \ldots, x_n) = (0, c, x_1, \ldots, x_n)$,

$$\mathrm{paste}^n_{(+;c)}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, c, 0).$$

Remark 2.10. The map "blow-up" is related to the corresponding notion in the geometry of algebraic surfaces in the following sense. Let X be a projective surface and let D_i, $1 \le i \le n$, be curves on X isomorphic to $\mathbf{P}^1$ with self-intersection numbers $(D_i^2) = -a_i$, $1 \le i \le n$. Assume furthermore that $(D_i . D_j) = \delta_{i+1, j}$ for any i, j, and let $D_{i-1} \cap D_i = \{p_i\}$, $2 \le i \le n$. Let $\pi_i : \widetilde{X} \to X$ be the blow-up of X at the point p_i. Let $\widetilde{D}_i$, $1 \le i \le n$, denote the proper transforms of D_i, and E denote the exceptional curve. Then the self-intersection numbers of them are given by

$$(\widetilde{D}_1^2) = -a_1, \ldots, (\widetilde{D}_{i-2}^2) = -a_{i-2},$$

$$(\widetilde{D}_{i-1}^2) = -(a_{i-1} + 1), \quad (E^2) = -1, \quad (\widetilde{D}_i^2) = -(a_i + 1),$$

$$(\widetilde{D}_{i+1}^2) = -a_{i+1}, \ldots, (\widetilde{D}_n^2) = -a_n.$$

Thus the map $\mathrm{blup}^n_{(i;\pm)}$, $2 \le i \le n$, defined above corresponds to the transition of the self-intersection numbers under the blow-up.

A remarkable fact is that all of these maps restrict to maps between the Chebyshev varieties as follows.

Theorem 2.11. *These three maps reserve (or reverse the sign) of the value of the function u. More precisely, the following three formulas hold*:

(i) $u(\mathrm{blup}^n_{(i;\,\pm)}(x_1, \ldots, x_n)) = \pm u(x_1, \ldots, x_n),\ 1 \le i \le n+1,$

(ii) $u(\mathrm{split}^n_{(i;\,c)}(x_1, \ldots, x_n)) = -u(x_1, \ldots, x_n),\ 1 \le i \le n,\ c \in \mathbf{Q},$

(iii) $u(\mathrm{paste}^n_{(\pm;\,c)}(x_1, \ldots, x_n)) = -u(x_1, \ldots, x_n),\ c \in \mathbf{Q}.$

In particular, these maps restrict to morphisms between the Chebyshev varieties: $\mathrm{blup}^n_{(i;\pm)} : V_n \to V_{n+1}, \mathrm{split}^n_{(i;c)} : V_n \to V_{n+2}, \mathrm{paste}^n_{(\pm;c)} : V_n \to V_{n+2}.$

Proof. (i) We will give a proof of the statement for $\mathrm{blup}^n_{(i;\,+)}$, since the other is proved similarly. When $2 \le i \le n,$ we compute as follows:

$$\begin{aligned}
&u(\mathrm{blup}^n_{(i;\,+)}(x_1, \ldots, x_n))\\
&= u(x_1, \ldots, x_{i-2}, x_{i-1}+1, 1, x_i+1, x_{i+1}, \ldots, x_n)\\
&= u(x_1, \ldots, x_{i-2}, x_{i-1}+1, 1)u(x_i+1, x_{i+1}, \ldots, x_n)\\
&\quad - u(x_1, \ldots, x_{i-2}, x_{i-1}+1)u(x_{i+1}, \ldots, x_n) \quad \text{(by Proposition 2.4)}\\
&= (u(x_1, \ldots, x_{i-2}, x_{i-1}+1) - u(x_1, \ldots, x_{i-2}))u(x_i+1, x_{i+1}, \ldots, x_n)\\
&\quad -u(x_1, \ldots, x_{i-2}, x_{i-1}+1)u(x_{i+1}, \ldots, x_n). \quad \text{(by Proposition 2.1 (iii))} \qquad (2.4)
\end{aligned}$$

Here we need the following lemma, which can be easily derived from Proposition 2.1 (ii), (iii):

Lemma 2.11.1. *For any* $d \in \mathbf{Q}$, *we have*

(i) $$\begin{aligned} u(x_1, \ldots, x_{n-1}, x_n+d) &= u(x_1, \ldots, x_{n-1}, x_n) + du(x_1, \ldots, x_{n-1})\\ &= u[1, n] + du[1, n-1], \end{aligned}$$

(ii) $$\begin{aligned} u(x_1+d, x_2, \ldots, x_n) &= u(x_1, x_2, \ldots, x_n) + du(x_2, \ldots, x_n)\\ &= u[1, n] + du[2, n]. \end{aligned}$$

By using these formulas, we can continue the computation in (2.4) as follows:

$$
\begin{aligned}
& u(\mathrm{blup}^n_{(i;+)}(x_1, \ldots, x_n)) \\
&= (u[1, i-1] + u[1, i-2] - u[1, i-2])(u[i, n] + u[i+1, n]) \\
&\quad - (u[1, i-1] + u[1, i-2])u[i+1, n] \\
&= u[1, i-1]u[i, n] - u[1, i-2]u[i+1, n] \\
&= u[1, n],
\end{aligned}
$$

the last equality coming from Proposition 2.4. When $i = 1$, we can compute as follows:

$$
\begin{aligned}
& u(\mathrm{blup}^n_{(1;+)}(x_1, \ldots, x_n)) \\
&= u(1, x_1 + 1, x_2, \ldots, x_n) \\
&= u(x_1 + 1, x_2, \ldots, x_n) - u(x_2, \ldots, x_n) \quad \text{(by Proposition 2.1 (ii))} \\
&= u(x_1, x_2, \ldots, x_n). \quad \text{(by Lemma 2.11.1 (ii))}
\end{aligned}
$$

Since we can compute similarly for $i = n + 1$, the proof of (i) is completed.

(ii) This is seen to hold by the following computation:

$$
\begin{aligned}
& u(\mathrm{split}^n_{(i;c)}(x_1, \ldots, x_n)) \\
&= u(x_1, \ldots, x_{i-1}, x_i - c, 0, c, x_{i+1}, \ldots, x_n) \\
&= u(x_1, \ldots, x_{i-1}, x_i - c, 0)u(c, x_{i+1}, \ldots, x_n) \\
&\quad - u(x_1, \ldots, x_{i-1}, x_i - c)u(x_{i+1}, \ldots, x_n) \quad \text{(by Proposition 2.4)} \\
&= -u[1, i-1](cu[i+1, n] - u[i+2, n]) - (u[1, i] - cu[1, i-1])u[i+1, n] \\
&\qquad \text{(by Lemma 2.11.1)} \\
&= u[1, i-1]u[i+2, n] - u[1, i]u[i+1, n] \\
&= -u[1, n].
\end{aligned}
$$

(iii) These are easiest of all. We have only to apply Proposition 2.1 (ii) or (iii):

$$u(\mathrm{paste}^n_{(-;c)}(x_1, \ldots, x_n)) = u(0, c, x_1, \ldots, x_n) = -u(x_1, \ldots, x_n),$$

$$u(\mathrm{paste}^n_{(+;c)}(x_1, ..., x_n)) = u(x_1, ..., x_n, c, 0) = -u(x_1, ..., x_n).$$

Thus the proof of Theorem 2.11 is finished.

3. Integral Points on the Chebyshev Varieties

In this section we see that any integral points on the Chebyshev varieties V_n are constructed from those points on lower dimensional V_k, $k < n$ through the three maps introduced in the previous section.

The following proposition plays a crucial role.

Lemma 3.1. $\dfrac{u(x_1, ..., x_n)}{u(x_2, ..., x_n)} = x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3 - \cdots \cfrac{1}{x_{n-1} - \cfrac{1}{x_n}}}}.$

Proof. This is a direct consequence of Proposition 2.1 (ii). For, dividing the both sides of $u(x_1, x_2, ..., x_n) = x_1 u(x_2, ..., x_n) - u(x_3, ..., x_n)$ by $u(x_2, ..., x_n)$, we have the equality

$$\frac{u(x_1, ..., x_n)}{u(x_2, ..., x_n)} = x_1 - \frac{u(x_3, ..., x_n)}{u(x_2, ..., x_n)} = x_1 - \frac{1}{\dfrac{u(x_2, ..., x_n)}{u(x_3, ..., x_n)}},$$

which implies the desired continued fraction expansion by induction.

By using the lemma, we have the following non-vanishing result.

Proposition 3.2. *Suppose that* $|x_i| \geq 2, 1 \leq i \leq n.$ *Then we have*

$$u(x_1, ..., x_n) \neq 0. \tag{3.1}$$

Proof. We prove this by induction on n. When $n = 1$, this holds trivially, since $u(x_1) = x_1$. We want to prove (3.1), assuming that $(*)$ $u(x_k, ..., x_n) \neq 0$ with $|x_i| \geq 2$, $k \leq i \leq n$, for any $k > 1$. Let us use the continued fraction expansion in Lemma 3.1. Note that

$$\left| \frac{u(x_i, ..., x_n)}{u(x_{i+1}, ..., x_n)} \right| > 1$$

holds for any $i \geq 1$. For, we have

$$\left| \frac{u(x_i, \ldots, x_n)}{u(x_{i+1}, \ldots, x_n)} \right| = \left| x_i - \frac{1}{\dfrac{u(x_{i+1}, \ldots, x_n)}{u(x_{i+2}, \ldots, x_n)}} \right|$$

$$\geq | x_i | - \left| \frac{1}{\dfrac{u(x_{i+1}, \ldots, x_n)}{u(x_{i+2}, \ldots, x_n)}} \right|$$

$$\geq 2 - \left| \frac{1}{\dfrac{u(x_{i+1}, \ldots, x_n)}{u(x_{i+2}, \ldots, x_n)}} \right|$$

by downward induction. (Here we have used $(*)$ implicitly.) In particular, we have $u(x_1, \ldots, x_n) \neq 0$, which completes the proof of Proposition 3.2.

The above proof also shows that the following estimate holds.

Corollary 3.2.1. *If* $| x_i | \geq 2$, $1 \leq i \leq n$, *then* $| u(x_1, \ldots, x_n) | \geq \max\{| x_1 |, | x_n |\}$.

Now we can prove one of the main theorems of this paper. For any variety V defined over $\mathbf{Q}$, we denote the set of integral points on V by $V(\mathbf{Z})$.

Theorem 3.3. *The set of integral points on the Chebyshev variety* V_n *is given by the following*:

(i) $V_1(\mathbf{Z}) = \{0\}$,

(ii) $V_2(\mathbf{Z}) = \{(1, 1), (-1, -1)\}$,

(iii) $$V_n(\mathbf{Z}) = \left(\bigcup_{1 \leq i \leq n} \mathrm{blup}^{n-1}_{(i;\pm)}(V_{n-1}(\mathbf{Z})) \right) \cup \left(\bigcup_{\substack{1 \leq i \leq n-2 \\ c \in \mathbf{Z}}} \mathrm{split}^{n-2}_{(i;c)}(V_{n-2}(\mathbf{Z})) \right) \cup \left(\bigcup_{c \in \mathbf{Z}} \mathrm{paste}^{n-2}_{(\pm;c)}(V_{n-2}(\mathbf{Z})) \right),$$

for any $n \geq 3$.

Proof. The statements (i) and (ii) are direct consequence of the definitions $u(x_1) = x_1, u(x_1, x_2) = x_1x_2 - 1.$ The inclusion (LHS) $\supset$ (RHS) of (iii) is already proved in Theorem 2.11. To prove the converse, let $(x_1, ..., x_n) \in V_n(\mathbf{Z})$. Then it follows from Proposition 3.2 that there exists an i with $1 \le i \le n$ such that $|x_i| \le 1$, namely $x_i = 0, \pm 1$. When $x_i = \pm 1$, Theorem 2.11 (i) implies that $(x_1, ..., x_n) \in \mathrm{blup}_{(i;\pm)}^{n-1}(V_{n-1}(\mathbf{Z}))$. When $x_i = 0$ and $2 \le i \le n-1$, we see that

$$\begin{aligned}(x_1, ..., x_n) &= (x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) \\ &= \mathrm{split}_{(i-1;\, x_{i+1})}^{n-2}(x_1, ..., x_{i-1} + x_{i+1}, ..., x_n) \\ &\in \mathrm{split}_{(i-1;\, x_{i+1})}^{n-2}(V_{n-2}(\mathbf{Z}))\end{aligned}$$

by Theorem 2.11 (ii). When $x_1 = 0$, we have

$$(x_1, ..., x_n) = (0, x_2, ..., x_n) = \mathrm{paste}_{(-;\, x_2)}^{n-2}(x_3, ..., x_n) \in \mathrm{paste}_{(-;\, x_2)}^{n-2}(V_{n-2}(\mathbf{Z})),$$

furthermore when $x_n = 0$, we have

$$\begin{aligned}(x_1, ..., x_n) = (x_1, ..., x_{n-1}, 0) &= \mathrm{paste}_{(+;\, x_{n-1})}^{n-2}(x_1, ..., x_{n-2}) \\ &\in \mathrm{paste}_{(+;\, x_{n-1})}^{n-2}(V_{n-2}(\mathbf{Z})).\end{aligned}$$

Thus we complete the proof of Theorem 3.3.

Remark 3.4. This theorem tells us that all of integral points on V_n lie on a finite union of linear subvarieties.

The theorem provides us with an *inductive* procedure which produces all the integral points on the whole family V_n, $n \ge 1$, of the Chebyshev varieties as follows:

Theorem 3.5. *For any* $n \ge 3$, *we obtain every integral point on* V_n *by applying the maps* $\mathrm{blup}_{(i;\pm)}^k$, $\mathrm{split}_{(i;c)}^k$, $\mathrm{paste}_{(\pm;c)}^k$ $(c \in \mathbf{Z})$ *successively starting from the set* $\{0, (1, 1), (-1, -1)\}$.

This together with Lemma 3.1 provides us with an interesting by-product.

Corollary 3.5.1. *For any n-tuple* $(a_1, \ldots, a_n)$ *of integers, the continued fraction* $a_1 - \cfrac{1}{a_2 - \cdots \cfrac{1}{a_{n-1} - \cfrac{1}{a_n}}}$ *vanishes if and only if* $(a_1, \ldots, a_n)$ *is obtained from the inductive procedure stated in Theorem* 3.5.

Example 3.6 (Determination of $V_3(\mathbf{Z})$)**.** By Theorem 3.3, we have

$$\begin{aligned} V_3(\mathbf{Z}) &= \left(\bigcup_{1 \le i \le 3} \mathrm{blup}^2_{(i;\, \pm)}(V_2(\mathbf{Z}))\right) \cup \left(\bigcup_{c \in \mathbf{Z}} \mathrm{split}^1_{(1;\, c)}(V_1(\mathbf{Z}))\right) \\ &\qquad \cup \left(\bigcup_{c \in \mathbf{Z}} \mathrm{paste}^1_{(\pm;\, c)}(V_1(\mathbf{Z}))\right) \\ &= \left(\bigcup_{1 \le i \le 3} \mathrm{blup}^2_{(i;\, \pm)}(\{(1,\, 1),\, (-1,\, -1)\})\right) \cup \left(\bigcup_{c \in \mathbf{Z}} \mathrm{split}^1_{(1;\, c)}(\{0\})\right) \\ &\qquad \cup \left(\bigcup_{c \in \mathbf{Z}} \mathrm{paste}^1_{(\pm;\, c)}(\{0\})\right) \\ &= \{(1,\, 2,\, 1),\, (2,\, 1,\, 2),\, (-1,\, -2,\, -1),\, (-2,\, -1,\, -2)\} \\ &\qquad \cup \{(-c,\, 0,\, c);\, c \in \mathbf{Z}\} \cup \{(0,\, c,\, 0);\, c \in \mathbf{Z}\}. \end{aligned}$$

On the other hand, Theorem 3.3 provides us with a *recursive* procedure which decides whether a given point $(x_1, \ldots, x_n) \in \mathbf{Z}^n$ lies in $V_n(\mathbf{Z})$ or not.

Theorem 3.7. *For any* $(x_1, \ldots, x_n) \in \mathbf{Z}^n$, *the algorithm below decides whether it lies in* $V_n(\mathbf{Z})$ *or not*:

(1) *if* $|x_i| \ge 2$ *for any i, then* $(x_1, \ldots, x_n) \notin V_n(\mathbf{Z})$,

(2.1) *if* $x_i = 1$ *for some* i, *then let* $(x'_1, \ldots, x'_{n-1}) = (x_1, \ldots, x_{i-2}, x_{i-1} - 1,\ x_{i+1} - 1, x_{i+2}, \ldots, x_n)$ *and go to* (1), (2.2) *if* $x_i = -1$ *for some* i, *then let* $(x'_1, ..., x'_{n-1}) = (x_1, ..., x_{i-2}, x_{i-1} + 1, x_{i+1} + 1, x_{i+2}, ..., x_n)$ *and go to* (1),

(2.3) *if* $x_i = 0$ *with* $2 \le i \le n-1$, *then let* $(x'_1, ..., x'_{n-2}) = (x_1, ..., x_{i-2}, x_{i-1} + x_{i+1}x_{i+2}, ..., x_n)$ *and go to* (1),

(2.4) *if* $x_1 = 0$, *then let* $(x'_1, ..., x'_{n-2}) = (x_3, ..., x_n)$ *and go to* (1),

(2.5) *if* $x_n = 0$, *then let* $(x'_1, ..., x'_{n-2}) = (x_1, ..., x_{n-2})$ *and go to* (1).

If the algorithm ends with (0), (1, 1) or $(-1, -1)$, *then* $(x_1, ..., x_n) \in V_n(\mathbf{Z})$, *otherwise* $(x_1, ..., x_n) \notin V_n(\mathbf{Z})$.

Remark 3.8. The proof for Theorem 3.3 tells us another interesting fact that for any integer m, there is an inductive procedure to create infinitely many solutions to the Diophantine equation $u(x_1, ..., x_n) = m$, $n \ge 3$. Indeed, one has only to start with the trivial solution $x_1 = m$ for $u(x_1) = m$ and apply the three families of maps to it as much as one likes.

4. Chebyshev Varieties of the First Kind

In this section we introduce another family of Chebyshev varieties, and investigate what integral points lie on them.

For any n independent variables $x_1, x_2, ..., x_n$, let

$$T(x_1, x_2, ..., x_n) = \begin{pmatrix} x_1 & -1 & 0 & \cdots & 0 & 1 \\ -1 & x_2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & x_3 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & x_{n-1} & -1 \\ 1 & 0 & 0 & \cdots & -1 & x_n \end{pmatrix}, n \ge 3,$$

$$T(x_1, x_2) = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix},$$

$$T(x_1) = (x_1 + 2), \tag{4.1}$$

and let $t(x_1, x_2, ..., x_n) = \det T(x_1, x_2, \cdots, x_n)$. As is the case for u, we write $t[1, n]$ for $t(x_1, ..., x_n)$. We call the zero locus $W_n = \{t(x_1, x_2, ..., x_n) = 0\} \subset \mathbf{A}^n$ *Chebyshev variety of the first kind* (see Proposition 4.4 for the origin of the name). The defining polynomial $t(x_1, x_2, ..., x_n)$ is related to $u(x_1, x_2, ..., x_n)$ as follows.

Proposition 4.1. *For any n, we have*

$$t(x_1, x_2, ..., x_n) = u(x_1, x_2, ..., x_n) - u(x_2, ..., x_{n-1}) + 2.$$

Proof. This is proved by appropriate expansion of the determinant as follows:

$$t(x_1, x_2, ..., x_n) = \det\begin{pmatrix} x_1 & -1 & 0 & \cdots & 0 & 1 \\ -1 & x_2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & x_3 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & x_{n-1} & -1 \\ 1 & 0 & 0 & \cdots & -1 & x_n \end{pmatrix}$$

$$= x_1 \det\begin{pmatrix} x_2 & -1 & \cdots & 0 & 0 \\ -1 & x_3 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & x_{n-1} & -1 \\ 0 & 0 & \cdots & -1 & x_n \end{pmatrix}$$

$$+ \det\begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \\ -1 & x_3 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & x_{n-1} & -1 \\ 0 & 0 & \cdots & -1 & x_n \end{pmatrix}$$

$$+ (-1)^{n-1} \det\begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \\ x_2 & -1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -1 & 0 \\ 0 & 0 & \cdots & x_{n-1} & -1 \end{pmatrix}$$

(by the expansion along the first column)

$= x_1 u[2, n] + (-u[3, n] + (-1)^{n-2} \cdot (-1)^{n-2}) + (-1)^{n-1}((-1)^{n-1} + (-1)^n u[2, n-1])$

(by the expansion along the first rows of the last two determinants)

$= u[1, n] - u[2, n-1] + 2.$ (by Proposition 2.1 (ii))

This completes the proof.

Corollary 4.1.1. *Let* $A(x)$ *denote the matrix* $\begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$, *introduced in Proposition* 2.3. *Then we have*

$$t(x_1, x_2, .., x_n) = tr(A(x_n)A(x_{n-1}) \cdots A(x_1)) + 2. \tag{4.2}$$

Proof. Combine Proposition 4.1 with Proposition 2.3.

Corollary 4.1.2. *The polynomial* $t(x_1, ..., x_n)$ *remains invariant under the cyclic permutation* $x_1 \mapsto x_2, x_2 \mapsto x_3, ..., x_{n-1} \mapsto x_n, x_n \mapsto x_1$.

Proof. This is a direct consequence of Corollary 4.1.1.

Remark 4.2. The formula (4.2) was first pointed out to the author by S. Sato. Actually his discovery of (4.2) was a great stimulus to the author to begin the study of this paper.

Remark 4.3. The equality in Proposition 4.1 justifies the convention we have employed in (4.1).

The following proposition reveals an intimate connection of our polynomial t and the Chebyshev polynomial of the first kind.

Proposition 4.4. *Let* $T_n(z)$ *be the Chebyshev polynomial of the first kind defined by*

$$T_n(z) = \cos n\theta \text{ with } z = \cos\theta.$$

Then we have

$$t(\underbrace{x, x, ..., x}_{n}) = 2T_n(x/2) + 2.$$

Proof. Let $t_n(x) = t(\underbrace{x, x, ..., x}_{n})$, and let $x_1 = \cdots = x_n = x$ in Proposition 4.1. Let $\alpha = e^{i\theta}$, $x = 2z = \alpha + \alpha^{-1}$, as in the proof of Proposition 2.7. Then we have

$$\begin{aligned}
t_n(x) &= u_n(x) - u_{n-2}(x) + 2 \\
&= \frac{\alpha^{n+1} - \alpha^{-(n+1)}}{\alpha - \alpha^{-1}} - \frac{\alpha^{n-1} - \alpha^{-(n-1)}}{\alpha - \alpha^{-1}} + 2 \\
&= (\alpha^n + \alpha^{n-2} + \cdots + \alpha^{-n}) - (\alpha^{n-2} + \alpha^{n-4} + \cdots + \alpha^{-(n-2)}) + 2 \\
&= \alpha^n + \alpha^{-n} + 2 \\
&= 2\cos n\theta + 2 \\
&= 2T_n(x/2) + 2.
\end{aligned}$$

This completes the proof.

Remark 4.5. This proposition may justify our naming the variety $W_n = \{t(x_1, x_2, \ldots, x_n) = 0\}$ "Chebyshev variety of the first kind."

As opposed to the case for the Chebyshev varieties of the second kind, we need only use two types of maps, blow-up's, splitting, to deal with the inductive structure of the sets of the integral points on the Chebyshev varieties of the first kind. This is due to the cyclicity of the polynominal t. The definitions of the maps are modified slightly accordingly. (Actually only $\mathrm{blup}^n_{(1;\pm)}$ and $\mathrm{blup}^n_{(n+1;\pm)}$ are modified.)

Definition 4.6. For any $n \geq 1$, we define two families of maps

(Blow-up) $\mathrm{blup}^n_{(i;\pm)} : \mathbf{A}^n \to \mathbf{A}^{n+1}, 1 \leq i \leq n+1,$

(Splitting) $\mathrm{split}_{(i;c)} : \mathbf{A}^n \to \mathbf{A}^{n+2}, 1 \leq i \leq n, c \in \mathbf{Q},$

by the following rules:

(i) $\mathrm{blup}^n_{(i;\pm)}(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-2}, x_{i-1} \pm 1, \pm 1, x_i \pm 1, x_{i+1}, \ldots, x_n),$

$$2 \leq i \leq n,$$

$\mathrm{blup}^n_{(1;\pm)}(x_1, \ldots, x_n) = (\pm 1, x_1 \pm 1, x_2, \ldots, x_n \pm 1),$

$\mathrm{blup}^n_{(n+1;\pm)}(x_1, \ldots, x_n) = (x_1 \pm 1, x_2, \ldots, x_{n-1}, x_n \pm 1, \pm 1),$

(ii) $\mathrm{split}^n_{(i;c)}(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i - c, 0, c, x_{i+1}, \ldots, x_n).$

This time the polynomial t changes their values under these maps through the following rules:

Theorem 4.7.

(i) $t(\mathrm{blup}^n_{(i;+)}(x_1, \ldots, x_n)) = t(x_1, \ldots, x_n),\ 1 \le i \le n+1,$

$t(\mathrm{blup}^n_{(i;-)}(x_1, \ldots, x_n)) = -t(x_1, \ldots, x_n) + 4,\ 1 \le i \le n+1,$

(ii) $t(\mathrm{split}^n_{(i;c)}(x_1, \ldots, x_n)) = -t(x_1, \ldots, x_n) + 4,\ 1 \le i \le n,\ c \in \mathbf{Q}.$

Proof. (i) By cyclicity we have only to give a proof for the case when $i = 1$. By using Proposition 4.1, we can compute as follows:

$$\begin{aligned}
& t(\mathrm{blup}^n_{(1;+)}(x_1, \ldots, x_n)) \\
&= u(1, x_1 + 1, x_2, \ldots, x_n + 1) - u(x_1 + 1, x_2, \ldots, x_{n-1}) + 2 \\
&= (u(x_1 + 1, x_2, \ldots, x_n + 1) - u(x_2, \ldots, x_n + 1)) \\
&\quad - u(x_1 + 1, x_2, \ldots, x_{n-1}) + 2 \ \text{(by Proposition 2.1 (ii))} \\
&= (u[1, n] + u[1, n-1] + u[2, n] + u[2, n-1] - u[2, n] - u[2, n-1]) \\
&\quad - (u[1, n-1] + u[2, n-1]) + 2 \ \text{(by Lemma 2.11.1)} \\
&= u[1, n] - u[2, n-1] + 2 \\
&= t[1, n]
\end{aligned}$$

which proves the first equality in (i). The second one can be proved similarly. (The extra “+4” comes from the sign-change of u.)

(ii) By cyclicity, we may assume that $i = 1$. Then we have

$$\begin{aligned}
t(\mathrm{split}^n_{(1;c)}(x_1, \ldots, x_n)) &= u(x_1 - c, 0, c, x_2, \ldots, x_n) \\
&\quad - u(0, c, x_2, \ldots, x_{n-1}) + 2 \\
&= -u(x_1, x_2, \ldots, x_n) + u(x_2, \ldots, x_{n-1}) + 2 \\
&\quad \text{(by Theorem 2.11 (ii) and Proposition 2.1 (ii))}
\end{aligned}$$

$$= -t(x_1, \dots, x_n) + 4.$$

Thus we complete the proof.

Remark 4.8. At this point the reader might feel uneasy with "4" appearing in the formulae above. If one modify t through $t'(x_1, \dots, x_n) = t(x_1, \dots, x_n) - 2$, then the corresponding formulae just look like those in Theorem 2.11, and the determination of the set of integral points goes completely parallel to the argument given in Section 3. The reason why we have chosen t as the main object of our concern is that it is expressed as the determinant of a good-looking matrix, and that the very "4" makes the structure of the set of integral points a little bit more fascinating than that of the modified one.

The following proposition plays a crucial role to describe $W_n(\mathbf{Z})$.

Proposition 4.9. *Suppose that $|x_i| \geq 2, 1 \leq i \leq n,$ and there exists an index i such that $|x_i| \geq 3$. Then we have*

$$|t(x_1, \dots, x_n) - 2| > 2. \tag{4.3}$$

Proof. By cyclicity, we may assume that the first coordinate has the absolute value greater than or equal to 3. We prove (4.3) by induction on n. When $n = 1$, this holds trivially, since $t(x_1) = x_1 + 2$. When $n \geq 2$, it follows from Proposition 4.1 that

$$\frac{t[1, n] - 2}{u[2, n]} = \frac{u[1, n]}{u[2, n]} - \frac{u[2, n-1]}{u[2, n]}.$$

(Note that $u[2, n] \neq 0$ by Proposition 3.2.) This implies by Lemma 3.1 that

$$\frac{t[1, n] - 2}{u[2, n]} = x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3 - \cdots \cfrac{1}{x_{n-1} - \cfrac{1}{x_n}}}} - \cfrac{1}{x_n - \cfrac{1}{x_{n-1} - \cdots \cfrac{1}{x_3 - \cfrac{1}{x_2}}}}. \tag{4.4}$$

Recall that we have assumed $|x_1| \geq 3$, and note that the last two denominators on the right hand side have absolute values greater than one by the proof for Proposition 2.3. Therefore we see that $|(t[1, n]-2)/u[2, n]| > 1$ and hence $|t[1, n]-2| > |u[2, n]|$. Since $|u[2, n]| \geq \max\{|x_2|, |x_n|\} \geq 2$ by Corollary 3.2.1, Proposition 4.9 is proved.

Thus in order to have a result like Theorem 2.11 for t, we must deal with the case when $|x_i| = 2, 1 \leq i \leq n$.

Proposition 4.10. *Suppose that* $|x_i| = 2, 1 \leq i \leq n$, *and not all the coordinates have the same sign. Then* $|t(x_1, \ldots, x_n) - 2| > 2$.

Proof. It follows from the assumption that there is an index i such that $x_i > 0, x_{i+1} < 0$ or $x_i < 0, x_{i+1} > 0$. Hence by cyclicity we may assume that $x_1 > 0, x_2 < 0$ or $x_1 > 0, x_n < 0$. In the first case, the sum of the first two terms on the right hand side of (4.4) is greater than two, hence $|(t[1, n]-2)/u[2, n]| > 1$ by the same argument as in the proof of Proposition 4.9. In the second case, the sum of the first and the third terms on the right hand side is greater than two. Hence we finish the proof.

Thus we are left with the case when

$$x_i = 2, 1 \leq i \leq n, \tag{4.5}$$

$$x_i = -2, 1 \leq i \leq n. \tag{4.6}$$

Proposition 4.11. (i) *If* (4.5) *holds, then* $t(x_1, \ldots, x_n) = 4$.

(ii) *If* (4.6) *holds, then* $t(x_1, \ldots, x_n) = 2(1 + (-1)^n)$.

Proof. It follows from Proposition 4.4 that

$$t\underbrace{(2, 2, \ldots, 2)}_{n} = 2T_n(1) + 2 = 2\cos(n \cdot 0) + 2 = 4.$$

Moreover the same proposition implies that

$$t\underbrace{(-2, -2, \ldots, -2)}_{n} = 2T_n(-1) + 2 = 2\cos(n\pi) + 2 = 2(1 + (-1)^n).$$

This completes the proof.

Summing up the results in Propositions 4.9-4.11, we obtain the following.

Proposition 4.12. *When* $|x_i| \geq 2,\ 1 \leq i \leq n,$ *we have* $t(x_1, \ldots, x_n) \in \{0, 4\}$ *if and only if* $(x_1, \ldots, x_n) \in \{(2, \ldots, 2), (-2, \ldots, -2)\}$.

Now we can determine the set $W_n(\mathbf{Z})$ of the integral points on W_n completely. For ease of its description, we introduce some notation. For any $k \in \mathbf{Z}$, let $W_n(k) = \{(x_1, \ldots, x_n) \in \mathbf{Z}^n;\ t(x_1, \ldots, x_n) = k\}$ and $W(k) = \bigcup_{n \geq 1} W_n(k)$. For any pair (i, j) of integers, let $W_n(i, j) = W_n(i) \cup W_n(j)$. Furthermore let

$$\mathrm{Op}^+ = \bigcup_{n \geq 2} \{\mathrm{blup}^n_{(i;\,+)};\ 1 \leq i \leq n+1\},$$

$$\mathrm{Op}^- = \bigcup_{n \geq 2} \{\mathrm{blup}^n_{(i;\,-)};\ 1 \leq i \leq n+1\} \cup \bigcup_{n \geq 1} \{\mathrm{split}^n_{(i;\,c)};\ 1 \leq i \leq n,\ c \in \mathbf{Z}\},$$

$$\mathrm{Op} = \mathrm{Op}^+ \cup \mathrm{Op}^-,$$

$$\mathrm{Composite}_{\mathrm{even}} = \bigcup_{k \geq 1} \{f_1 \circ \cdots \circ f_k;\ f_i \in \mathrm{Op}, 1 \leq i \leq k, \text{ the composite is well-defined and } \#(\{f_1, \ldots, f_k\} \cap \mathrm{Op}^-) \equiv 0 \pmod 2\},$$

$$\mathrm{Composite}_{\mathrm{odd}} = \bigcup_{k \geq 1} \{f_1 \circ \cdots \circ f_k;\ f_i \in \mathrm{Op}, 1 \leq i \leq k, \text{ the composite is well-defined and } \#(\{f_1, \ldots, f_k\} \cap \mathrm{Op}^-) \equiv 1 \pmod 2\}.$$

Theorem 4.13. *The set* $W_n(0)$ *of integral points on the Chebyshev variety of the first kind and the set* $W_n(4)$ *are given by the following*:

(i) $W_1(0) = \{-2\}, W_1(4) = \{2\}$,

(ii) $W_2(0) = \{(d, 0);\ d \in \mathbf{Z}\} \cup \{(0, d);\ d \in \mathbf{Z}\}$,

$W_2(4) = \{(4, 1), (2, 2), (1, 4), (-4, -1), (-2, -2), (-1, -4)\}$,

$$\text{(iii)}\ \ W_n(0,4)=\left(\bigcup_{1\le i\le n}\mathrm{blup}^{n-1}_{(i;\pm)}(W_{n-1}(0,4))\right)\cup\left(\bigcup_{\substack{1\le i\le n-2\\ c\in\mathbf{Z}}}\mathrm{split}^{n-2}_{(i;c)}(W_{n-2}(0,4))\right)$$

$$\cup W_n(0)_{\text{exceptional}}\cup W_n(4)_{\text{exceptional}},$$

for any $n \ge 3$,

where

$$W_n(0)_{\text{exceptional}} = \begin{cases}\{\underbrace{(-2,\ \ldots,\ -2)}_{n}\}, & \textit{if } n \textit{ is odd},\\ \phi, & \textit{if } n \textit{ is even},\end{cases}$$

$$W_n(4)_{\text{exceptional}} = \begin{cases}\{\underbrace{(2,\ \ldots,\ 2)}_{n}\}, & \textit{if } n \textit{ is odd},\\ \{\underbrace{(2,\ \ldots,\ 2)}_{n},\ \underbrace{(-2,\ \ldots,\ -2)}_{n}\}, & \textit{if } n \textit{ is even}.\end{cases}$$

Moreover we have

$$\begin{aligned} W(0) = {} & W_1(0)\cup W_2(0)\cup W(0)_{\text{exceptional}}\\ & \cup\left(\bigcup_{\Phi\in\mathrm{Composite}_{\text{even}}}\Phi(W_1(0)\cup W_2(0)\cup W(0)_{\text{exceptional}})\right)\\ & \cup\left(\bigcup_{\Phi\in\mathrm{Composite}_{\text{odd}}}\Phi(W_1(4)\cup W_2(4)\cup W(4)_{\text{exceptional}})\right),\end{aligned}$$

$$\begin{aligned} W(4) = {} & W_1(4)\cup W_2(4)\cup W(4)_{\text{exceptional}}\\ & \cup\left(\bigcup_{\Phi\in\mathrm{Composite}_{\text{even}}}\Phi(W_1(4)\cup W_2(4)\cup W(4)_{\text{exceptional}})\right)\\ & \cup\left(\bigcup_{\Phi\in\mathrm{Composite}_{\text{odd}}}\Phi(W_1(0)\cup W_2(0)\cup W(0)_{\text{exceptional}})\right),\end{aligned}$$

where

$$W(0)_{\text{exceptional}} = \bigcup_{n \geq 1} W_n(0)_{\text{exceptional}},$$

$$W(4)_{\text{exceptional}} = \bigcup_{n \geq 1} W_n(4)_{\text{exceptional}}.$$

Proof. The statements (i) and (ii) are direct consequence of the definitions $t(x_1) = x_1 + 2$, $t(x_1, x_2) = x_1 x_2$. The other statements can be proved by a similar argument to that for Theorem 3.3, in view of Propositions 4.6 and 4.12.

Remark 4.14. One can see that the sets $W(1)$, $W(2)$, and $W(3)$ have similar descriptions to the theorem, and details are left to the reader. Moreover the above proof shows that for any $k \geq 3$, there is an inductive procedure to produce infinitely many elements of $W(2 + k)$ together with $W(2 - k)$.

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