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MODIFIED REDUCTION AND ALTERNATIVE AXIOMATIZATIONS OF THE CORE OF FUZZY TU GAMES

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Abstract

Different from the results of Hwang [4], we adopt the modified reduction proposed by Hwang [4] to characterize the core on the domain of all balanced fuzzy transferable-utility (TU) games and the domain of all fuzzy TU games, respectively.

1. Introduction

The theory of fuzzy games started with the work of Aubin [1, 2] where the notions of a fuzzy game and the core of a fuzzy game are introduced. In the meantime, many solution concepts have been developed. In the framework of fuzzy transferable-utility (TU) games, Hwang [4] first extended the max-reduced games to fuzzy TU games and offered axiomatizations of the core on the domain of all fuzzy TU games.

Inspired by Peleg [5], we adopt the modified reduction proposed by Hwang [4] to show that the core is the only solution satisfying non-emptiness, individually rationality, consistency and superadditivity on the domain of all balanced fuzzy TU games. Also, we introduce a weakening of non-emptiness by restricting its application to balanced fuzzy TU games. With its help, we extend the results of Peleg [5] to the domain of all fuzzy TU games.

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2. Preliminaries

Let U be the universe of players. If $N \subseteq U$ is a set of players, then a *fuzzy coalition* is a vector $\alpha \in [0, 1]^N$. The ith coordinate α_i of α is called the *participation level* of player i in the fuzzy coalition α. For all $T \subseteq N$, let $|T|$ be the number of elements in T. Instead of $[0, 1]^T$, we will write F^T for the set of fuzzy coalitions. A player-coalition $T \subseteq N$ corresponds in a canonical way to the fuzzy coalition $e^T(N) \in F^N$, which is the vector with $e_i^T(N) = 1$ if $i \in T$, and $e_i^T(N) = 0$ if $i \in N \backslash T$. The fuzzy coalition $e^T(N)$ corresponds to the situation where the players in T fully cooperate (i.e., with participation level 1) and the players outside T are not involved at all (i.e., they have participation level 0). Denote the zero vector in $\mathbb{R}^N$ by 0_N. The fuzzy coalition 0_N corresponds to the empty player-coalition. Note that $e^T(N)$ will be denoted by e^T if no confusion arises.

A fuzzy TU game is a pair (N, v), where N is a non-empty and finite set of players and $v : F^N \to \mathbb{R}$ is a characteristic function with $v(0_N) = 0$. The map v assigns to each fuzzy coalition $\alpha = (\alpha_i)_{i \in N} \in F^N$ a number, telling what such a coalition can achieve in cooperation. Denote the class of all fuzzy TU games by $\mathcal{FG}$.

Let $(N, v) \in \mathcal{FG}$. A *payoff vector* of (N, v) is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$. Then a payoff vector x of $(N, v) \in \mathcal{FG}$ is

- *efficient* (*EFF*) if $\sum_{i \in N} x_i = v(e^N)$,
- *individually rational* (*IR*) if for all $i \in N$ and for all $j \in [0, 1]$, $jx_i \geq v(je^{\{i\}})$.

Moreover, x is an *imputation* of (N, v) if it is EFF and IR. The set of *feasible payoff vectors* of (N, v) is denoted by

$$X^*(N, v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq v(e^N) \right\},$$

whereas

$$X(N, v) = \{x \in \mathbb{R}^N \mid x \text{ is EFF}\}$$

is the set of *preimputations* of (N, v) and the set of imputations of (N, v) is denoted by $I(N, v)$.

Given $(N, v) \in \mathcal{FG}$, $x \in \mathbb{R}^N$, $\alpha \in F^N$ and $S \subseteq N$, we denote $x_S \in \mathbb{R}^S$ to be the restriction of x to S, and $x(\alpha) = \sum_{i \in N} \alpha_i x_i$.

A *solution on* $\mathcal{FG}$ is a function σ which associates with each $(N, v) \in \mathcal{FG}$ a subset $\sigma(N, v)$ of $X^*(N, v)$. The core of a fuzzy TU game (N, v) (Aubin [1, 2]) is as follows.

Definition 1. The *core* $C(N, v)$ of $(N, v) \in \mathcal{FG}$ consists of all $x \in X(N, v)$ that satisfy for all $\alpha \in F^N$, $x(\alpha) \geq v(\alpha)$.

3. Reduced Games and Axiomatizations

We say that the fuzzy TU game (N, v) is *balanced*[1] if $C(N, v) \neq \emptyset$. Let $\mathcal{FG}_c$ denote the set of all balanced fuzzy TU games. Let $\mathcal{FG}' \subseteq \mathcal{FG}$ and σ be a solution on $\mathcal{FG}'$. σ satisfies *non-emptiness* (*NE*) if for all $(N, v) \in \mathcal{FG}'$, $\sigma(N, v) \neq \emptyset$. σ satisfies *non-emptiness for balanced games* (*NEB*) if for all $(N, v) \in \mathcal{FG}_c$, $\sigma(N, v) \neq \emptyset$. σ satisfies *individually rationality* (*IR*) if for all $(N, v) \in \mathcal{FG}'$, $\sigma(N, v) \subseteq I(N, v)$. σ satisfies *one-person rationality* (*OPR*) if for all $(N, v) \in \mathcal{FG}$ with $|N| = 1$, $\sigma(N, v) = I(N, v)$. σ satisfies *efficience* (*EFF*) if for all $(N, v) \in \mathcal{FG}'$, $\sigma(N, v) \subseteq X(N, v)$. σ satisfies *superadditivity* (*SUPA*)[2] if for all $(N, v) \in \mathcal{FG}'$, $\sigma(N, v + w) \supseteq \sigma(N, v) + \sigma(N, w)$, where for all $\alpha \in M^N$, $(v + w)(\alpha) = v(\alpha) + w(\alpha)$.

[1]A characterization of balanced fuzzy games was given by Sharkey and Telser [7].

[2]If $N \subseteq U$ and $A, B \subset \mathbb{R}^N$, then $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$.

Hwang [4] extended the reduction introduced by Davis and Maschler [3] to fuzzy TU games as follows.

Definition 2 (Hwang [4]).[3] Let $(N, v) \in \mathcal{FG}$, $x \in \mathbb{R}^N$ and $S \subseteq N$, $S \neq \emptyset$. The *DM-reduced game with respect to S and x* is the game $(S, v_{S,x}^{DM})$, where

$$v_{S,x}^{DM}(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0^S, \\ v(e^N) - x(e_{N\setminus S}^N), & \text{if } \alpha = e_S^N, \\ \sup\{v(\alpha, \beta) - x(\beta, 0_S) \mid \beta \in F^{N\setminus S}\}, & \text{otherwise.} \end{cases}$$

Consistency requires that if x is prescribed by σ for a game (N, v), then the projection of x to S should be prescribed by σ for the reduced game with respect to S and x for all S. Thus, the projection of x to S should be consistent with the expectations of the members of S as reflected by their reduced game. Let σ be a solution on $\mathcal{FG}$.

- *DM-consistency* (*DMCON*): If $(N, v) \in \mathcal{FG}$, $S \subseteq N$, $S \neq \emptyset$, and $x \in \sigma(N, v)$, then $(S, v_{S,x}^{DM}) \in \mathcal{FG}$ and $x_S \in \sigma(S, v_{S,x}^{DM})$.

Converse consistency requires that if the projection of an efficient payoff vector x to every proper S is consistent with the expectations of the members of S as reflected by their reduced game, then x itself should be recommended for whole game.

- *Converse DM-consistency* (*CDMCON*): If $(N, v) \in \mathcal{FG}$ with $|N| \geq 2$, $x \in X(N, v)$, and for all $S \subset N$, $0 < |S| < |N|$, $(S, v_{S,x}^{DM}) \in \mathcal{FG}$ and $x_S \in \sigma(S, v_{S,x}^{DM})$, then $x \in \sigma(N, v)$.

The following axiom is a weakening of the previous axiom, since it requires that x to be individually rational as well.

- *Weak converse DM-consistency* (*WCDMCON*): If $(N, v) \in \mathcal{FG}$ with $|N|$

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[3]From now on, we restrict our attention to bounded fuzzy TU games, defined as those games (N, v) such that, there exists a real number M_v such that for all $\alpha \in F^N$, $v(\alpha) \leq M_v$. We use it here in order to guarantee that, in Definition 2, $v_{S,x}^{DM}$ is well-defined.

≥ 2, $x \in I(N, v)$, and for all $S \subset N$, $0 < |S| < |N|$, $(S, v^{DM}_{S,x}) \in \mathcal{FG}$ and $x_S \in \sigma(S, v^{DM}_{S,x})$, then $x \in \sigma(N, v)$.

Inspired by Serrano and Volij [6], Hwang [4] also proposed an extended reduced game on fuzzy TU games as follows.

Definition 3. Let $(N, v) \in \mathcal{FG}$, $x \in \mathbb{R}^N$ and $S \subseteq N$, $S \neq \emptyset$. The *modified reduced game with respect to S and x* is the game $(S, v^M_{S,x})$, where

$$v^M_{S,x}(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0_S, \\ \sup\{v(\alpha, \beta) - x(\beta, 0_S) \mid \beta \in F^{N \setminus S}\}, & \text{otherwise.} \end{cases}$$

The idea of the modified reduced game was first introduced by Serrano and Volij [6]. The only difference between this and the reduced game in Definition 1 is the fact that the coalition S is also allowed to imagine potential interaction with any of the subsets of $N \setminus S$. Informally, in order to reach the maximal benefit, all the coalitions comprised by the members of $N \setminus S$ should be considered to cooperate with S.

"Modified-reduction" instead of "DM-reduction", Hwang [4] introduced the *M-consistency* (*MCON*), *converse M-consistency* (*CMCON*) and *weak converse M-consistency* (*WCMCON*). Hwang [4] showed that the core is the only solution satisfying OPR, IR, DMCON (MCON) and WCDMCON (WCMCON). Next, we provide alternative axiomatizations by means of MCON.

Lemma 1 (Hwang [4])**.** *Both on* $\mathcal{FG}$ *and* $\mathcal{FG}_c$, *the core satisfies DMCON, WCDMCON, MCON and WCMCON.*

Lemma 2. *Let* σ *be a solution on* $\mathcal{FG}$. *If* σ *satisfies IR and MCON, then it also satisfies EFF.*

Proof. Assume that the solution σ satisfies IR and MCON. Let $(N, v) \in \mathcal{FG}$ with $|N| \geq 2$ and let $x \in \sigma(N, v)$,

$$v^M_{\{i\},x}(e^N_{\{i\}}) = \sup\{v(e^N_{\{i\}}, \beta) - x(\beta, 0) \mid \beta \in F^{N \setminus \{i\}}\}$$

$$\geq v(e^N) - x(e^N_{N \setminus \{i\}}, 0) \text{ (Take } \beta = e^N_{N \setminus \{i\}}).$$

By MCON of σ, $x_i \in \sigma(\{i\}, v^M_{\{i\},x})$. By IR of σ, $x_i \geq v^M_{\{i\},x}(e^N_{\{i\}}) = v(e^N) - x(e^N_{N\setminus\{i\}}, 0)$. Hence, $x(e^N) \geq v(e^N)$. Since σ is a solution, $\sigma(N, v) \subseteq X^*(N, v)$. Hence, $x(e^N) \leq v(e^N)$. Therefore, $x(e^N) = v(e^N)$. □

Lemma 3. *Let* $\mathcal{FG}' \subset \mathcal{FG}$ *and let* σ *be a solution on* $\mathcal{FG}'$. *If* σ *satisfies IR and MCON, then for all* $(N, v) \in \mathcal{FG}'$, $\sigma(N, v) \subseteq C(N, v)$.

Proof. Let $(N, v) \in \mathcal{FG}'$. The proof proceeds by induction on the number $|N|$. If $|N| = 1$, then by IR of σ and C, $\sigma(N, v) \subseteq C(N, v)$. Assume that $\sigma(N, v) \subseteq C(N, v)$ if for $|N| \leq k - 1$, where $k \geq 2$.

The case $|N| = k$:

Since σ satisfies IR and MCON, by Lemma 2, σ satisfies EFF. Hence, $\sigma(N, v) \subseteq I(N, v)$. Let $x \in \sigma(N, v)$. Since σ satisfies MCON, for all $S \subseteq N$ with $0 < |S| < |N|$, $x_S \in \sigma(S, v^M_{S,x})$. Hence, by the induction hypotheses, $x_S \in \sigma(S, v^M_{S,x}) \subseteq C(S, v^M_{S,x})$. Since C satisfies WCMCON, $x \in C(N, v)$. □

Theorem 1. (1) *On* $\mathcal{FG}_c$, *the core is the only solution satisfying NE, IR, SUPA and MCON.*

(2) *Let* $\mathcal{FG}' \subset \mathcal{FG}$. *On* $\mathcal{FG}'$, *the core is the only solution satisfying NEB, IR, SUPA and MCON.*

Proof. By Lemma 1, the core satisfies MCON. By definition of the core, it is easy to check that it satisfies IR and SUPA. Finally, for all $(N, v) \in \mathcal{FG}_c$, $C(N, v) \neq \emptyset$.

To prove the uniqueness of (1), assume that a solution σ satisfies NE, IR, SUPA and MCON on $\mathcal{FG}_c$. Let $(N, v) \in \mathcal{FG}_c$. Two cases may be distinguished:

Case 1. Assume that $|N| \geq 3$. Let $x \in C(N, v)$. Define $(N, w) \in \mathcal{FG}$ by the following rule:

$$w(t) = \begin{cases} v(je^{\{i\}}), & t = je^{\{i\}} \text{ for all } i \in N \text{ and for all } j \in [0, 1], \\ x(t), & \text{otherwise.} \end{cases}$$

As the reader can easily verify that $C(N, w) = \{x\}$. Thus, by NE of σ and Lemma 3, $\sigma(N, w) = \{x\}$. Now let $u = v - w$. Clearly, for all $i \in N$ and for all $j \in [0, 1]$, $u(je^{\{i\}}) = 0$. And for all $t \in F^N$, $u(e^N) = 0$ and $u(t) \leq 0$. Hence, $C(N, u) = \{0_N\}$. By NE of σ and Lemma 3, $\sigma(N, u) = \{0_N\}$. Since $v = u + w$ and σ satisfies SUPA,

$$\sigma(N, v) \supseteq \sigma(N, u) + \sigma(N, w) = \{x\}.$$

Thus, $\sigma(N, v) \supseteq C(N, v)$. By Lemma 3, $\sigma(N, v) \subseteq C(N, v)$. Hence, $\sigma(N, v) = C(N, v)$.

Case 2. Assume that $|N| \leq 2$. If $|N| = 1$, then by NE and IR, $\sigma(N, v) = C(N, v)$. Thus, let $|N| = 2$. Denote that $N = \{i, k\}$ and let $p \in U \setminus N$. Define $(H, u) \in \mathcal{FG}$ with $H = \{i, k, p\}$ in the following rule. For all $t \in F^H$, $u(t) = v(t_N)$. Let $x \in C(N, v)$. By definitions of u and $u^M_{N,x}$, it is easy to verify that $(x, 0_p) \in C(H, u)$ and $(N, u^M_{N,x}) = (N, v)$, where $0_p = 0$. Since $|H| = 3$, by Case 1, $\sigma(H, u) = C(H, u)$. Thus, $(x, 0_p) \in \sigma(H, u)$. By MCON of σ, $x = (x, 0_p)_N \in \sigma(N, u^M_{N,x}) = \sigma(N, v)$. Hence, $C(N, v) \subseteq \sigma(N, v)$. By Lemma 3, $\sigma(N, v) \subseteq C(N, v)$. Hence, $\sigma(N, v) = C(N, v)$.

This proof of uniqueness of (2) is a copy of (1) except "NEB and $\mathcal{FG}'$" instead of "NE and $\mathcal{FG}_c$"; hence, we omit it. □

The following examples show that each of the axioms used in Theorem 1 is logically independent of the others.[4]

Example 1. Let $\sigma(N, v) = \emptyset$ for all $(N, v) \in \mathcal{FG}_c$. Then σ satisfies IR, SUPA and MCON, but it violates NE (NEB).

Example 2. Let $\sigma(N, v) = X(N, v)$ for all $(N, v) \in \mathcal{FG}_c$. Then σ satisfies NE (NEB), SUPA and MCON, but it violates IR.

Example 3. Let $\sigma(N, v) = I(N, v)$ for all $(N, v) \in \mathcal{FG}_c$. Then σ satisfies NE (NEB), IR and SUPA, but it violates MCON.

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[4]In order to show the logical independence of the used axioms, $|U| \geq 2$ is needed.

Example 4. For all $(N, v) \in \mathcal{FG}_c$, we define a solution σ on $\mathcal{FG}_c$ to be

$$\sigma(N, v) = \begin{cases} C(N, v), & \text{if for all } x \in C(N, v) \text{ with } x_i \neq 0, \\ \{x \in C(N, v) \mid x_i = 0\}, & \text{otherwise.} \end{cases}$$

Then σ satisfies NE (NEB), IR and MCON, but it violates SUPA.

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