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# ON A COMBINATION OF MEET AND JOIN MATRICES

**ISMO KORKEE**

*Department of Mathematics, Statistics and Philosophy*
*University of Tampere*, *FIN*-33014, *Finland*
e-mail: ismo.korkee@uta.fi

## Abstract

Let $S = \{x_1, x_2, ..., x_n\}$ be a subset of a lattice $(P, \leq) = (P, \wedge, \vee)$ and let $f : P \to \mathbf{C}$ be a function. We present two structure theorems for the $n \times n$ matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta} = (m_{ij})$, where

$$m_{ij} = \frac{f(x_i \wedge x_j)^{\alpha} f(x_i \vee x_j)^{\beta}}{f(x_i)^{\gamma} f(x_j)^{\delta}}$$

and $\alpha, \beta, \gamma, \delta$ are appropriate real numbers. We also present formulae for the determinant and the inverse of $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ on meet-closed sets $S$ (i.e., $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$) and join-closed sets $S$ (i.e., $x_i, x_j \in S \Rightarrow x_i \vee x_j \in S$). These formulae are generalizations of those obtained for meet matrices $(S)_f = M_{S,f}^{1,0,0,0}$ and join matrices $[S]_f = M_{S,f}^{0,1,0,0}$ in the literature. We also present our results in a number-theoretic setting, i.e., in the lattice $(\mathbf{Z}_+, |) = (\mathbf{Z}_+, \gcd, \operatorname{lcm})$, where $|$ is the usual divisibility relation.

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## 1. Introduction

Let $(P, \leq) = (P, \wedge, \vee)$ be a locally finite lattice, let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of $P$ and let $f : P \to \mathbf{C}$ be a function. The meet matrix $(S)_f$ and the join matrix $[S]_f$ on $S$ with respect to $f$ are defined as $((S)_f)_{ij} = f(x_i \wedge x_j)$ and $([S]_f)_{ij} = f(x_i \vee x_j)$.

Haukkanen [6] introduced meet matrices and obtained formulae for the determinant and the inverse of $(S)_f$ (see also [16] and [20]). Korkee and Haukkanen [10] used incidence functions (see Section 2) in the study of meet matrices. There we obtained new upper and lower bounds for $\det(S)_f$ and a new formula for $(S)_f^{-1}$ on meet-closed sets $S$ (i.e., $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$). Recently we introduced join matrices [11] and presented formulae for $\det[S]_f$, new upper and lower bounds for $\det[S]_f$ and a new formula for $[S]_f^{-1}$ on join-closed sets $S$ (i.e., $x_i, x_j \in S \Rightarrow x_i \vee x_j \in S$). By assuming the semi-multiplicativity of $f$, formulae for $\det(S)_f$ and $(S)_f^{-1}$ on join-closed sets and formulae for $\det[S]_f$ and $[S]_f^{-1}$ on meet-closed sets are also presented in [11].

Define the $n \times n$ matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta} = (m_{ij})$, where

$$m_{ij} = \frac{f(x_i \wedge x_j)^{\alpha} f(x_i \vee x_j)^{\beta}}{f(x_i)^{\gamma} f(x_j)^{\delta}} \tag{1.1}$$

and $\alpha$, $\beta$, $\gamma$, $\delta$ are appropriate real numbers. Since $(S)_f = M_{S,f}^{1,0,0,0}$ and $[S]_f = M_{S,f}^{0,1,0,0}$, the matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ is a generalization of meet and join matrices. We present two structure theorems for $M_{S,f}^{\alpha,\beta,\gamma,\delta}$. Under certain conditions we obtain formulae for the determinant and the inverse of $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ on meet-closed and join-closed sets $S$.

It is well known that $(\mathbf{Z}_+, |) = (\mathbf{Z}_+, \gcd, \operatorname{lcm})$ is a locally finite lattice, where $|$ is the usual divisibility relation and gcd and lcm stand for the greatest common divisor and the least common multiple of integers. Thus meet and join matrices are generalizations of GCD matrices $((S)_f)_{ij} = f(\gcd(x_i, x_j))$ and LCM matrices $([S]_f)_{ij} = f(\operatorname{lcm}(x_i, x_j))$, see [11, Section 6]. The study of GCD and LCM matrices is considered to begun in 1875/76 when H. J. S. Smith presented his famous determinant formulae, see [18]. For general accounts of GCD and related matrices, see [7], [8] and [11].

We also present our results in a number-theoretic setting, i.e., for combinations of GCD and LCM matrices. Our results are generalizations of those obtained for power GCD matrices and power LCM matrices in [1], [3], [4], [18], see also [9]. Our results also generalize results for GCD-reciprocal LCM matrices and LCM-reciprocal GCD matrices presented in [13], [14] and [19], see also [15]. Note that in the literature these results are mostly obtained for the function $N : \mathbf{Z}_+ \to \mathbf{C}, N(n) = n$.

## 2. Definitions

Let $S$ be a subset of a lattice $(P, \leq) = (P, \wedge, \vee)$. We say that $S$ is *lower-closed* if $(x \in S,\ y \in P,\ y \leq x) \Rightarrow y \in S$. We say that $S$ is *meet-closed* if $x, y \in S \Rightarrow x \wedge y \in S$. We define the dual concepts upper-closed and join-closed analogously. It is clear that a lower-closed set is always meet-closed but the converse does not hold, and dually, an upper-closed set is always join-closed but the converse does not hold. The principal order ideal of $x \in P$ is defined by $\downarrow x = \{z \in P \mid z \leq x\}$.

Let $f$ always be a complex-valued function on $P$ and let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of $P$ such that $x_i < x_j \Rightarrow i < j$.

**Definition 2.1.** The $n \times n$ matrix $(S)_f = (s_{ij})$, where

$$s_{ij} = f(x_i \wedge x_j), \tag{2.1}$$

is called the *meet matrix* on $S$ with respect to $f$. Similarly, the $n \times n$

matrix $[S]_f = (s_{ij})$, where

$$s_{ij} = f(x_i \vee x_j), \tag{2.2}$$

is called the *join matrix* on $S$ with respect to $f$.

**Definition 2.2** [21, p. 190]**.** Let $A$ and $B$ be two matrices of the same size. Then their *Hadamard product* (or *Schur product*) $A \circ B$ is defined by

$$(A \circ B)_{ij} = a_{ij}b_{ij}. \tag{2.3}$$

**Definition 2.3.** We say that $f$ is a *semi-multiplicative function* on $P$ if

$$f(x)f(y) = f(x \wedge y)f(x \vee y) \tag{2.4}$$

for all $x, y \in P$.

We adapt the previous concept from number theory, see [17, p. 49]. Note that for arithmetical functions (i.e., for functions $\mathbf{Z}_+ \to \mathbf{C}$) multiplicativity implies semi-multiplicativity.

Let $\alpha$, $\beta$, $\gamma$, $\delta$ denote real numbers. If $[f(x)]^\alpha$ exists for all $x \in P$, then we define the function $f^\alpha$ on $P$ as $f^\alpha(x) = [f(x)]^\alpha$. Thus by $f^{-1}$ we mean $f^{-1}(x) = 1/f(x)$. Note that $f^\alpha$ is semi-multiplicative if and only if $f$ is semi-multiplicative.

Let $g$ be a complex-valued function on $P \times P$ such that $g(x, y) = 0$ whenever $x \not\leq y$. Then we say that $g$ is an incidence function of $P$. If $g$ and $h$ are incidence functions of $P$, their sum $g + h$ is defined by $(g + h)(x, y) = g(x, y) + h(x, y)$ and their convolution $g * h$ is defined by $(g * h)(x, y) = \sum_{x \leq z \leq y} g(x, z)h(z, y)$. The set of all incidence functions of $P$ under addition and convolution forms a ring with unity, where the unity $\delta$ is defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. The zeta function $\zeta$ of $P$ is defined by $\zeta(x, y) = 1$ if $x \leq y$, and $\zeta(x, y) = 0$ otherwise. The Möbius function $\mu$ of $P$ is the inverse of $\zeta$ under convolution. In this paper the inverse of $f$ (if it exists) is denoted by $f'$. We denote the restriction of an incidence function $f$ on $S \times S$ by $f_S$

and we write $f'_S = (f_S)'$ if it exists. We denote the zeta function of $S$ by $\zeta_S$ and let $\mu_S = \zeta'_S = (\zeta_S)'$.

**Remark 2.1.** In this paper let $(P, \leq)$ always be a finite lattice. However, this is not a proper requirement, since we can always restrict our examination, e.g., to the minimal sublattice of $(P, \leq)$ generated by $S$, see [2].

The method used in [11] requires the assumption of finiteness. Thus we have the least and the greatest element of $P$, which we denote by $0 = \min P$ and $1 = \max P$. We associate $f$ with a "restricted" incidence function $f_{\mathrm{d}}$ of $P$ by the formula $f(z) = f_{\mathrm{d}}(0, z)$, where d means "down". The function $f_{\mathrm{d}}$ can be used in the convolution of usual incidence functions when the first argument is equal to 0 and $f_{\mathrm{d}}$ is the left member in the convolution. Similarly, we associate $f$ with a "restricted" incidence function $f_{\mathrm{u}}$ of $P$ by the formula $f(z) = f_{\mathrm{u}}(z, 1)$, where u means "up". The function $f_{\mathrm{u}}$ can be used in the convolution of usual incidence functions when the second argument is equal to 1 and $f_{\mathrm{u}}$ is the right member in the convolution.



## 3. Two Structure Theorems

The first of two structure theorems is based on expressing the "join part" of $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ in terms of a certain meet matrix. In the second structure theorem we express the "meet part" of $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ in terms of a certain join matrix. We prove only Theorem 3.1, since the proof of Theorem 3.2 is similar. Note that we provide the factorizations with the Hadamard product and also with the ordinary sum of matrices.

**Remark 3.1.** Finding the conditions for the existence of $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ (and $G$ and $H$) in the following structure theorems is not a hard but a laborious task. So for the sake of brevity we prefer not to write the conditions down explicitly.

**Theorem 3.1** (Meet-oriented structure theorem)**.** *Let* $\alpha, \beta, \gamma, \delta$ *be real numbers such that* $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ *exists. Then*

$$M_{S,f}^{\alpha,\beta,\gamma,\delta} = D^{\beta-\gamma}((S)_{f^{\alpha-\beta}} \circ G)D^{\beta-\delta} = D^{\beta-\gamma}((S)_{f^{\alpha-\beta}} + H)D^{\beta-\delta}, \tag{3.1}$$

*where* $D = \operatorname{diag}(f(x_1), f(x_2), \ldots, f(x_n))$,

$$(G)_{ij} = \begin{cases} 1 & \text{if } x_i \le x_j \text{ or } x_j \le x_i, \\ \dfrac{f^{\beta}(x_i \wedge x_j) f^{\beta}(x_i \vee x_j)}{f^{\beta}(x_i) f^{\beta}(x_j)} & \text{otherwise} \end{cases} \tag{3.2}$$

*and*

$$(H)_{ij} = \begin{cases} 0 & \text{if } x_i \le x_j \text{ or } x_j \le x_i, \\ f^{\alpha-\beta}(x_i \wedge x_j)((G)_{ij} - 1) & \text{otherwise,} \end{cases} \tag{3.3}$$

*provided that the denominator in* (3.2) *and* $f^{\alpha-\beta}(x_i \wedge x_j)$ *in* (3.3) *are nonzero.*

**Proof.** First note that

$$(G)_{ij} = [f^{\beta}(x_i \wedge x_j) f^{\beta}(x_i \vee x_j)]/[f^{\beta}(x_i) f^{\beta}(x_j)] \tag{3.4}$$

for all $x_i, x_j \in S$. Thus if $f^{\beta}(x_i \wedge x_j) \neq 0$, then $f^{\beta}(x_i \vee x_j) = (G)_{ij} f^{\beta}(x_i) f^{\beta}(x_j)/f^{\beta}(x_i \wedge x_j)$ and

$$\begin{aligned} (M_{S,f}^{\alpha,\beta,\gamma,\delta})_{ij} &= [f^{\alpha}(x_i \wedge x_j) f^{\beta}(x_i \vee x_j)]/[f^{\gamma}(x_i) f^{\delta}(x_j)] \\ &= f^{\beta-\gamma}(x_i)(f^{\alpha-\beta}(x_i \wedge x_j)(G)_{ij}) f^{\beta-\delta}(x_j) \end{aligned} \tag{3.5}$$

(which obviously holds when $f^{\beta}(x_i \wedge x_j) = 0$). Furthermore,

$$(H)_{ij} = f^{\alpha-\beta}(x_i \wedge x_j)((G)_{ij} - 1) \tag{3.6}$$

for all $x_i, x_j \in S$. Therefore $(G)_{ij} = 1 + (H)_{ij}/f^{\alpha-\beta}(x_i \wedge x_j)$ and

$$(M_{S,f}^{\alpha,\beta,\gamma,\delta})_{ij} = f^{\beta-\gamma}(x_i)(f^{\alpha-\beta}(x_i \wedge x_j) + (H)_{ij}) f^{\beta-\delta}(x_j). \tag{3.7}$$

This completes the proof.

**Theorem 3.2** (Join-oriented structure theorem)**.** *Let* $\alpha$, $\beta$, $\gamma$, $\delta$ *be real numbers such that* $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ *exists. Then*

$$M_{S,f}^{\alpha,\beta,\gamma,\delta} = D^{\alpha-\gamma}([S]_{f^{\beta-\alpha}} \circ G)D^{\alpha-\delta} = D^{\alpha-\gamma}([S]_{f^{\beta-\alpha}} + H)D^{\alpha-\delta}, \tag{3.8}$$

*where* $D = \operatorname{diag}(f(x_1), f(x_2), \ldots, f(x_n))$,

$$(G)_{ij} = \begin{cases} 1 & \text{if } x_i \le x_j \text{ or } x_j \le x_i, \\ \dfrac{f^{\alpha}(x_i \wedge x_j) f^{\alpha}(x_i \vee x_j)}{f^{\alpha}(x_i) f^{\alpha}(x_j)} & \text{otherwise} \end{cases} \tag{3.9}$$

*and*

$$(H)_{ij} = \begin{cases} 0 & \text{if } x_i \le x_j \text{ or } x_j \le x_i, \\ f^{\beta-\alpha}(x_i \vee x_j)((G)_{ij} - 1) & \text{otherwise,} \end{cases} \tag{3.10}$$

*provided that the denominator in* (3.9) *and* $f^{\beta-\alpha}(x_i \vee x_j)$ *in* (3.10) *are nonzero.*

The structure of $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ is simple because $D$ is a diagonal matrix and $G$ and $H$ only affect those elements of $(S)_{f^{\alpha-\beta}}$ and $[S]_{f^{\beta-\alpha}}$, where $x_i$ and $x_j$ are incomparable. The Hadamard product and the matrix sum in (3.1) and (3.8) do not support calculating $\det M_{S,f}^{\alpha,\beta,\gamma,\delta}$ and $(M_{S,f}^{\alpha,\beta,\gamma,\delta})^{-1}$, so we make some further assumptions. Consider the following conditions.

(1) Let $f$ be a semi-multiplicative function on $P$ (or on the set $\{x_i \wedge x_j \mid x_i, x_j \in S\} \cup \{x_i \vee x_j \mid x_i, x_j \in S\}$).

(2) Let $S$ be a chain.

(3) Let $\beta = 0$.

It is obvious that whenever at least one of the conditions (1), (2), (3) holds, then $G$ and $H$ vanish in (3.1) (i.e., $(G)_{ij} = 1$ and $(H)_{ij} = 0$ for all $1 \le i, \ j \le n$). If we replace the condition (3) with

(4) let $\alpha = 0$.

Then the similar arguments also hold for (3.8).

## 4. The Determinant and the Inverse of $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ under certain Conditions

If we replace $f$ with $f^{\alpha-\beta}$ in [11, Propositions 3.2, 3.3, 3.8 and 3.9], then we obtain formulae for the determinant and the inverse of power meet matrix $(S)_{f^{\alpha-\beta}}$ on meet-closed and lower-closed sets. Similar arguments also hold for power join matrix $[S]_{f^{\beta-\alpha}}$ on join-closed and upper-closed sets, see [11, Theorems 4.1 and 4.5, Corollaries 4.1 and 4.2]. Applying these formulae and Theorems 3.1 and 3.2 we obtain Theorems 4.1-4.4 presented below.

In these four theorems, let $f$ be a function and let $\alpha$, $\beta$, $\gamma$, $\delta$ be real numbers such that the matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ exists.

**Theorem 4.1.** *Let at least one of the conditions* (1), (2), (3) *hold. If S is a meet-closed set, then*

$$\det M_{S,f}^{\alpha,\beta,\gamma,\delta} = \det(D^{\beta-\gamma}(S)_{f^{\alpha-\beta}}D^{\beta-\delta})$$

$$= \prod_{k=1}^{n} f^{2\beta-\gamma-\delta}(x_k) \sum_{\substack{z\le x_k \\ z\not\le x_1,\ldots,x_{k-1}}} (f_{\mathrm{d}}^{\alpha-\beta} * \mu)(0, z). \tag{4.1}$$

*Furthermore, if* $\det M_{S,f}^{\alpha,\beta,\gamma,\delta} \neq 0$, *then*

$$((M_{S,f}^{\alpha,\beta,\gamma,\delta})^{-1})_{ij} = f^{\delta-\beta}(x_i)f^{\gamma-\beta}(x_j) \sum_{\substack{x_i\le x_k \\ x_j\le x_k}} \frac{\mu_S(x_i, x_k)\mu_S(x_j, x_k)}{\sum_{\substack{z\le x_k \\ z\not\le x_1,\ldots,x_{k-1}}} (f_{\mathrm{d}}^{\alpha-\beta} * \mu)(0, z)}. \tag{4.2}$$

**Theorem 4.2.** *Let at least one of the conditions* (1), (2), (3) *hold. If S is a lower-closed set, then*

$$\det M_{S,f}^{\alpha,\beta,\gamma,\delta} = \prod_{k=1}^{n} f^{2\beta-\gamma-\delta}(x_k)(f_{\mathrm{d}}^{\alpha-\beta} * \mu)(0, x_k). \tag{4.3}$$

*Furthermore, if* $\det M_{S,f}^{\alpha,\beta,\gamma,\delta} \neq 0$, *then*

$$((M_{S,f}^{\alpha,\beta,\gamma,\delta})^{-1})_{ij} = f^{\delta-\beta}(x_i) f^{\gamma-\beta}(x_j) \sum_{\substack{x_i \leq x_k \\ x_j \leq x_k}} \frac{\mu(x_i, x_k)\mu(x_j, x_k)}{(f_{\mathrm{d}}^{\alpha-\beta} * \mu)(0, x_k)}. \tag{4.4}$$

**Theorem 4.3.** *Let at least one of the conditions* (1), (2), (4) *hold. If S is a join-closed set, then*

$$\det M_{S,f}^{\alpha,\beta,\gamma,\delta} = \det(D^{\alpha-\gamma}[S]_{f^{\beta-\alpha}} D^{\alpha-\delta})$$

$$= \prod_{k=1}^{n} f^{2\alpha-\gamma-\delta}(x_k) \sum_{\substack{x_k \leq z \\ x_{k+1}, \ldots, x_n \not\leq z}} (\mu * f_{\mathrm{u}}^{\beta-\alpha})(z, 1). \tag{4.5}$$

*Furthermore, if* $\det M_{S,f}^{\alpha,\beta,\gamma,\delta} \neq 0$, *then*

$$((M_{S,f}^{\alpha,\beta,\gamma,\delta})^{-1})_{ij} = f^{\delta-\alpha}(x_i) f^{\gamma-\alpha}(x_j) \sum_{\substack{x_k \leq x_i \\ x_k \leq x_j}} \frac{\mu_S(x_k, x_i)\mu_S(x_k, x_j)}{\sum_{\substack{x_k \leq z \\ x_{k+1}, \ldots, x_n \not\leq z}} (\mu * f_{\mathrm{u}}^{\beta-\alpha})(z, 1)}. \tag{4.6}$$

**Theorem 4.4.** *Let at least one of the conditions* (1), (2), (4) *hold. If S is an upper-closed set, then*

$$\det M_{S,f}^{\alpha,\beta,\gamma,\delta} = \prod_{k=1}^{n} f^{2\alpha-\gamma-\delta}(x_k)(\mu * f_{\mathrm{u}}^{\beta-\alpha})(x_k, 1). \tag{4.7}$$

*Furthermore, if* $\det M_{S,f}^{\alpha,\beta,\gamma,\delta} \neq 0$, *then*

$$((M_{S,f}^{\alpha,\beta,\gamma,\delta})^{-1})_{ij} = f^{\delta-\alpha}(x_i) f^{\gamma-\alpha}(x_j) \sum_{\substack{x_k \leq x_i \\ x_k \leq x_j}} \frac{\mu(x_k, x_i)\mu(x_k, x_j)}{(\mu * f_{\mathrm{u}}^{\beta-\alpha})(x_k, 1)}. \tag{4.8}$$

If $f$ is a semi-multiplicative function, i.e., if (1) holds, then $\det(S)_{f^{\alpha-\beta}}$ and $(S)_{f^{\alpha-\beta}}^{-1}$ are also known on join-closed and upper-closed sets, see [11, Section 5.5]. By applying these results to (3.1) we obtain the formulae presented in (4.5)-(4.8). Similarly, if (1) holds, then $\det[S]_{f^{\beta-\alpha}}$ and $[S]_{f^{\beta-\alpha}}^{-1}$ are also known on meet-closed and lower-closed sets, see [11, Section 5.3]. By applying these results to (3.8) we obtain the formulae presented in (4.1)-(4.4).

As special cases of the results above we obtain formulae for a large number of classes of matrices including, e.g., the Hadamard product of power meet and power join matrices

$$(S)_{f^{\alpha}} \circ [S]_{f^{\beta}} = [f^{\alpha}(x_i \wedge x_j) f^{\beta}(x_i \vee x_j)]. \tag{4.9}$$

For the sake of brevity, we do not present these formulae here.

## 5. Results for Number Theory

We can adapt our results to the lattice $(\mathbf{Z}_+, |) = (\mathbf{Z}_+, \gcd, \operatorname{lcm})$. As noted in Remark 2.1, we can restrict the examination to the finite sublattice $(\downarrow \operatorname{lcm} S, |)$ of $(\mathbf{Z}_+, |)$, where $\operatorname{lcm} S$ is the least common multiple of the elements of $S$, see [11, Section 6]. The concepts of meet-, lower-, join- and upper-closed sets can be replaced with the concepts of gcd-, factor-, lcm- and multiple-closed sets respectively. The following table is based on some observations of [11].

$$\begin{array}{rcl}
\text{Expression in } (P, \leq) & \to & \text{Expression in } (\mathbf{Z}_+, |) \\
x_i \wedge x_j,\ x_i \vee x_j & \to & (x_i, x_j),\ [x_i, x_j] \\
\mu(x, y) & \to & \mu(y/x) \\
\mu_S(x_i, x_j) & \to & \sum_{dx_i | x_j;\, dx_i \nmid x_1, \ldots, x_{j-1}} \mu(d) \quad (\text{gcd-closed } S) \\
\mu_S(x_i, x_j) & \to & \sum_{dx_i | x_j;\, dx_{i+1} \ldots, dx_n \nmid x_j} \mu(d) \quad (\text{lcm-closed } S) \\
(f_{\mathrm{d}} * \mu)(0, z) & \to & (f * \mu)(z) \\
(\mu * f_{\mathrm{u}})(z, 1) & \to & \sum_{z | y | \operatorname{lcm} S} f(y) \mu(y/z).
\end{array}$$

For the number-theoretic Möbius function $\mu(n)$ see [12, p. 300].

In the following, let $f$ be an arithmetical function and let $\alpha, \beta, \gamma, \delta$ be real numbers such that the matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta} = (m_{ij})$ exists, where

$$m_{ij} = \frac{f^{\alpha}((x_i, x_j))f^{\beta}([x_i, x_j])}{f^{\gamma}(x_i)f^{\delta}(x_j)}. \tag{5.1}$$

By Theorems 4.1-4.4 we obtain the following corollaries. Note that the condition (2) for $S$ means $x_1 | x_2 | \cdots | x_n$.

**Corollary 5.1.** *Let at least one of the conditions* (1), (2), (3) *hold. Denote*

$$d_k = \sum_{\substack{d \mid x_k \\ d \nmid x_1, \ldots, x_{k-1}}} (f^{\alpha-\beta} * \mu)(d), \quad c_{rk} = \sum_{\substack{dx_r \mid x_k \\ dx_r \nmid x_1, \ldots, x_{k-1}}} \mu(d) \tag{5.2}$$

*for* $1 \le k, r \le n$. *If $S$ is a gcd-closed set, then*

$$\det M_{S,f}^{\alpha,\beta,\gamma,\delta} = \prod_{k=1}^{n} f^{2\beta-\gamma-\delta}(x_k)d_k. \tag{5.3}$$

*Furthermore, if* $\det M_{S,f}^{\alpha,\beta,\gamma,\delta} \neq 0$, *then*

$$((M_{S,f}^{\alpha,\beta,\gamma,\delta})^{-1})_{ij} = f^{\delta-\beta}(x_i)f^{\gamma-\beta}(x_j)\sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{c_{ik}c_{jk}}{d_k}. \tag{5.4}$$

**Corollary 5.2.** *Let at least one of the conditions* (1), (2), (3) *hold. If $S$ is a factor-closed set, then*

$$\det M_{S,f}^{\alpha,\beta,\gamma,\delta} = \prod_{k=1}^{n} f^{2\beta-\gamma-\delta}(x_k)(f^{\alpha-\beta} * \mu)(x_k). \tag{5.5}$$

*Furthermore, if* $\det M_{S,f}^{\alpha,\beta,\gamma,\delta} \neq 0$, *then*

$$((M_{S,f}^{\alpha,\beta,\gamma,\delta})^{-1})_{ij} = f^{\delta-\beta}(x_i)f^{\gamma-\beta}(x_j)\sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{\mu(x_k/x_i)\mu(x_k/x_j)}{(f^{\alpha-\beta} * \mu)(x_k)}. \tag{5.6}$$

**Corollary 5.3.** *Let at least one of the conditions* (1), (2), (4) *hold. Denote*

$$d_k = \sum_{\substack{x_k \mid z \mid x_n \\ x_{k+1}, \ldots, x_n \nmid z}} \sum_{z \mid y \mid x_n} f^{\beta-\alpha}(y)\mu(y/z), \quad c_{kr} = \sum_{\substack{dx_k \mid x_r \\ dx_{k+1}, \ldots, dx_n \nmid x_r}} \mu(d) \tag{5.7}$$

*for* $1 \le k,\ r \le n$. *If* $S$ *is an lcm-closed set, then*

$$\det M_{S,f}^{\alpha,\beta,\gamma,\delta} = \prod_{k=1}^{n} f^{2\alpha-\gamma-\delta}(x_k) d_k. \tag{5.8}$$

*Furthermore, if* $\det M_{S,f}^{\alpha,\beta,\gamma,\delta} \neq 0$, *then*

$$((M_{S,f}^{\alpha,\beta,\gamma,\delta})^{-1})_{ij} = f^{\delta-\alpha}(x_i) f^{\gamma-\alpha}(x_j) \sum_{\substack{x_k \mid x_i \\ x_k \mid x_j}} \frac{c_{ki} c_{kj}}{d_k}. \tag{5.9}$$

**Corollary 5.4.** *Let at least one of the conditions* (1), (2), (4) *hold. If* $S$ *is a multiple-closed set, then*

$$\det M_{S,f}^{\alpha,\beta,\gamma,\delta} = \prod_{k=1}^{n} f^{2\alpha-\gamma-\delta}(x_k) \sum_{x_k \mid y \mid x_n} f^{\beta-\alpha}(y)\mu(y/x_k). \tag{5.10}$$

*Furthermore, if* $\det M_{S,f}^{\alpha,\beta,\gamma,\delta} \neq 0$, *then*

$$((M_{S,f}^{\alpha,\beta,\gamma,\delta})^{-1})_{ij} = f^{\delta-\alpha}(x_i) f^{\gamma-\alpha}(x_j) \sum_{\substack{x_k \mid x_i \\ x_k \mid x_j}} \frac{\mu(x_i/x_k)\mu(x_j/x_k)}{\sum_{x_k \mid y \mid x_n} f^{\beta-\alpha}(y)\mu(y/x_k)}. \tag{5.11}$$

As examples we next present results that are already known in number theory. In the literature the results mostly concern the arithmetical function $f(n) = N(n) = n$, which is semi-multiplicative and nonzero.

**Example 5.1.** Consider the power GCD matrix $M_{S,N}^{\alpha,0,0,0} = [(x_i, x_j)^{\alpha}]$ and the power LCM matrix $M_{S,N}^{0,\beta,0,0} = [[x_i, x_j]^{\beta}]$. By Corollary 5.1 we obtain the formulae presented in [4, Theorems 12 and 13] (note that the role of $\mu$ in [4, Theorem 13] is unclear). Further, by Corollary 5.2 we obtain the formulae presented in [3, Examples 1 (iii) and 3 (i)]. By Corollary 5.2 we also obtain formulae for $[1/(x_i, x_j)]$ and $[1/[x_i, x_j]]$ presented in [1, Corollaries 1 and 3]. Note that Smith [18, (2)] already presented formula for $\det M_{S,N}^{\alpha,0,0,0}$, where $S = \{1, 2, \ldots, n\}$. He also mentioned the possibility to replace $S$ with any factor-closed set $S$.

**Example 5.2.** Consider the matrix $M_{S,N}^{1,-1,0,0} = [(x_i, x_j)/[x_i, x_j]]$ (so-called GCD-reciprocal LCM matrix). By Corollaries 5.1 and 5.3 we obtain the formulae presented in [14, Corollary 1 and Theorem 3] and [13, Corollary 1 and Theorem 2]. Note that the formulae in [13] can be written in terms of the new expressions due to [11]. Further, if $S = \{1, 2, \ldots, n\}$, then by Corollary 5.2 we obtain formulae for $M_{S,N}^{-1,1,0,0} = [[x_i, x_j]/(x_i, x_j)]$ (so-called LCM-reciprocal GCD matrix) presented by [19, Corollary 1 and Theorem 2].

**Remark 5.1.** The structure $(\mathbf{Z}_+, \|)$ is a meet-semilattice, where $\|$ is the unitary divisibility relation defined by $x \| y \Leftrightarrow (x \mid y$ and $(x, y/x) = 1)$, see [5]. However, $(\mathbf{Z}_+, \|)$ is not a lattice, since the least common unitary multiple of integers $x$ and $y$ does not always exist. Thus Corollaries 5.1-5.4 do not necessarily hold for GCUD and LCUM matrices, see [7].

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