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ON SEMIESSENTIAL SUBSEMIMODULES

LANDING FALL, JEAN RAOUL TSIBA and DJIBY SOW

Departement de Mathématiques et Informatique
Laboratoire d'Algèbre, de Cryptologie, de Géométrie Algébrique et Applications
Université Cheikh Anta Diop
BP 5005 Dakar Fann
Sénégal
e-mail: sowdjibab@ucad.sn
sowdjibab@yahoo.fr
landingfall@yahoo.fr
jrtsiba@hotmail.com

Abstract

In semimodules over semirings theory, essential subsemimodules and essential monomorphisms of semimodules are studied differently by some authors.

This paper deals with the extension of essential subsemimodules to semiessential subsemimodules following the pattern on essential submodules in module theory.

We have defined and characterized many new notions such as semiessential subsemimodules, semiessential monomorphisms, quasiessential monomorphisms, pseudoessential monomorphisms and *M*-complement subsemimodules.

We compare the class of semiessential semimodules and the class of essential semimodules.

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0. Introduction

In semimodules over semirings theory, we do not meet document dealing with exhaustive manner on essential subsemimodules. We propose to study in this paper, a new class of subsemimodules similar to essential subsemimodules with new variants.

We follow the process proposed in [2] on essential submodules.

New notions as semiessential monomorphisms, quasiessential monomorphisms and pseudoessential monomorphisms are introduced in order to characterize the notion of semiessential monomorphisms relatively to the image $Im f$, to the proper image $f(M)$ (of a morphism $f : M \to N$) restricted to substractive subsemimodules.

We prove that the class of semiessential semimodules is not contained in the class of essential semimodules.

Moreover, other results connecting semiessential subsemimodules and pseudo direct sum are studied.

Throughout this article, for basics notions and notations in modules theory and semimodules theory, we will follow [1, 2, 5, 13-15].

In the following, we recall some definitions and notations that will be retained in this note.

All semirings are associative with identity 1 (if R is a semiring, then we assume that $1 \neq 0$), all semimodules are unital and all semiring extensions contain the common identity.

- Let M be a left R-semimodule. An equivalence relation ρ on M is an R-congruence relation if and only if: $m\rho m'$ and $n\rho n' \Rightarrow (m+n)\rho(m'+n')$ and $(rm)\rho(rm')$, $\forall m, m', n, n' \in M$ and $r \in R$. The set of all R-congruence relations on M, $R\text{-}cong(M)$, is partially-ordered by the relation $\leq$ defined by $\rho \leq \rho'$ if and only if $m\rho m' \Rightarrow m\rho' m'$, $\forall m, m' \in M$. For $m, m' \in M$, $\rho_{(m, m')}$ is the unique smallest element ρ of $R\text{-}cong(M)$ satisfying $m\rho m'$.

- Let N be a subsemimodule of a left R-semimodule M. N induces on M an R-congruence relation $\equiv_N$, known as the Bourne relation: $\forall m, m' \in M$; $m \equiv_N m'$ $\Leftrightarrow \exists n, n' \in N$ such that $m + n = m' + n'$.

- M/N denotes the factor R-semimodule $M/\equiv_N$, and $\overline{m} = [m] = m + N$ denotes an element of M/N.

- Let M_1 and M_2 be subsemimodules of a left R-semimodule M.

*** If M_1 and M_2 are independent and span M (i.e., $M_1 \cap M_2 = 0$ and $M = M_1 + M_2$), then M is the weak direct sum of its subsemimodules M_1 and M_2. Also, we write $M = M_1 \overline{\oplus} M_2$. For each $m \in M$, there exist $m_1 \in M_1$ and $m_2 \in M_2$ such that $m = m_1 + m_2$ but this decomposition of m is not unique, hence we cannot define projection with the weak summand.

*** If M_1 and M_2 span M (i.e., $M = M_1 + M_2$), and the restriction of $\equiv_{M_2}$ to M_1 and the restriction of $\equiv_{M_1}$ to M_2 are trivial, then M is the direct sum of its subsemimodules M_1 and M_2. Also, we write $M = M_1 \oplus M_2$. For each $m \in M$, there exists a unique pair $(m_1, m_2) \in M_1 \times M_2$ such that $m = m_1 + m_2$. With this direct sum, we can define projection.

- A subsemimodule N of a left R-semimodule M is called *substractive* in M if: $\forall (m, m') \in M^2$ $(m + m' \in N$ and $m \in N) \Rightarrow m' \in N$.

- N is strongly substractive if for all $(m, m') \in M^2$: $m + m' \in N \Rightarrow m \in N$ and $m' \in N$.

- A left R-semimodule M is called *cancellative* if for all elements m, m' and x of M: $m + x = m' + x \Rightarrow m = m'$.

Example 0.1 (See [10], [8] and [9])**.** Let $R = \{0, 1\}$ be the Boole semiring and the set $M = \{0, 1, a, b\}$. Define on M the operations as the following:

$0_R = 0_M = 0$; $1_R = 1_M = 1$,

$1 + 1 = 1 + a = 1 + b = a + b = 1$; $a + 0 = a + a = a$; $0 + b = b + b = b$,

$0 \times a = 0 \times b = a \times b = 0$; $1 \times 1 = 1$; $1 \times a = a \times a = a$; $1 \times b = b \times b = b$.

Then $(M, +, \times, 0, 1)$ is a commutative left R-semimodule.

- $\{0, a\}$ is a substractive subsemimodule of M.

- $\{0\}$ is a strongly substractive subsemimodule of M.

- $\{0, 1, a\}$ is a nonsubstractive subsemimodule of M because $1 + b = 1$ and b is not in $\{0, 1, a\}$.

- $M = \{0, a\} + \{0, 1, b\}$ and $\{0, a\} \cap \{0, 1, b\} = \{0\}$. But $1 = 0 + 1 = a + b$ with $0 \neq a$ and $b \neq 1$, so the decomposition of 1 is not unique. Therefore, $M = \{0; a\} \overline{\oplus} \{0, 1, b\}$.

- $M = \{0, a\} + \{0; b\}$ and there does not exist $x, y \in \{0, a\}/0 + x = b + y$, therefore $m \equiv_{\{0, a\}} m' \Leftrightarrow m = m'$, $\forall m, m' \in \{0, b\}$. So the restriction of $\equiv_{\{0, a\}}$ to $\{0, b\}$ is trivial. Similarly, the restriction of $\equiv_{\{0, b\}}$ to $\{0, a\}$ is trivial. Thus $M = \{0, a\} \oplus \{0, b\}$.

This paper is organized as follows:

- In Section 1, we study some basics notions relatively to semimodules.

- In Section 2, we study the notions of semiessential subsemimodules and semiessential monomorphisms.

1. Basics Notions

In this section, some important results are given on basics notions on semimodules. We give the definitions of essential subsemimodules and essential monomorphisms according to [5].

- Throughout this paper, we consider the left R-semimodules. Let $f : M \to N$ be a homomorphism of left R-semimodules. Denote by

(1) $Ker f = \{m \in M / f(m) = 0\}$, $Ker f$ is the kernel of f.

(2) $f(M) = \{f(m); m \in M\}$, $f(M)$ is the proper image of f.

(3) $Im f = \{n \in N / n + f(m) = f(m'); m, m' \in M\}$, $Im f$ is the image of f.

$Ker f$ is a subsemimodule of M; $f(M)$ and $Im f$ are subsemimodules of N.

Moreover, in general $f(M) \subsetneq Im f$.

Definition 1.1. *Let* $f: M \to N$ be a homomorphism of left R-semimodules. Then f is

(1) *monomorphism* if, $\forall g;\ h \in Hom_R(L, M),\ f \circ g = f \circ h \Rightarrow g = h$,

(2) *semimonomorphism* if, $Ker\, f = 0$,

(3) *isomorphism* if f is both injective and surjective,

(4) *semi-isomorphism* if, $Ker\, f = 0$ and f is surjective,

(5) *i-regular* (*image-regular*), if $f(M) = Im\, f$,

(6) *k-regular* (*kernel-regular*), if $f(m) = f(m') \Rightarrow m + k = m' + k'$ for some $k, k' \in Ker\, f$,

(7) *regular*, if f is both *i*-regular and *k*-regular.

Proposition 1.2. *Let* $f: M \to N$ *be a homomorphism of left R-semimodules. Then the following conditions are equivalent*:

(1) *f is injective*,

(2) *f is k-regular and* $Ker\, f = 0$,

(3) *f is k-regular* (*semi*)*monomorphism*,

(4) *f is a monomorphism*.

Proof. See [1] and [11]. □

In [5], essential subsemimodules and essential monomorphisms are defined as follows.

Definition 1.3. (1) An R-monomorphism $f: M \to N$ of left R-semimodules is essential if for any R-homomorphism $g: N \to N'$, $g \circ f$ is a monomorphism implying that g is a monomorphism.

(2) A subsemimodule M' of a left R-semimodule M is essential (or large) in M if the inclusion map $i_{M'}: M' \to M$ is an essential R-homomorphism. The class of subsemimodules essential for an R-semimodule is denoted by $\mathfrak{C}_{_RM}$.

(3) $f: M \to N$ is an essential R-homomorphism if and only if $f(M)$ is a large subsemimodule of N.

Proposition 1.4. *If N is a subsemimodule of a left R-semimodule M, then the following conditions are equivalent*:

(1) *N is large in M.*

(2) *If ρ is a nontrivial R-congruence relation on M, then the restriction of ρ to N is also nontrivial.*

(3) *If m and m' are distinct elements of M, then there exist distinct elements n and n' of N satisfying $n\rho_{(m, m')}n'$.*

Proof. See [5]. □

Example 1.5 (See [5])**.** Let R be the semiring $(\mathbb{I}, \max, \min)$ in which $\mathbb{I} = [0, 1]$ is the unit interval on the real line with addition (respectively, multiplication) of x and y and is defined by $x \oplus y = \max(x, y)$ (respectively, $x \otimes y = \min(x, y)$). R is a left R-semimodule. Consider $N = R \setminus \{1\}$, N is a subsemimodule of R.

Let ρ be a nontrivial R-congruence relation on R. Suppose that the restriction of ρ to N is trivial. 1 is not in N and ρ is nontrivial in R, then there exists $n \in N$ such that $n\rho 1$. Let $n' \in N$ be such that $n < n' < 1$, we have $n\rho 1$ and $n'\rho n'$. Then $\min(n, n')\rho \min(1, n')$ because R is an R-semimodule, thus $n\rho n' \Rightarrow n = n'$, which is a contradiction.

Then by Proposition 1.4, N is essential in R.

2. On Semiessential Subsemimodules and Monomorphisms

In this section, we define and characterize semiessential subsemimodules, semiessential, quasiessential and pseudoessential monomorphisms of semimodules.

Definition 2.1. (1) Let M be a left R-semimodule. Then a left R-subsemimodule K of M is said to be *semiessential* in M, written as $K \trianglelefteq_s M$, if for every R-subsemimodule L of M: $L \cap K = 0 \Rightarrow L = 0$. The class of subsemimodules semiessential for an R-semimodule M is denoted by $\overline{\mathfrak{C}}_{R}M$.

We say also that M is a *semiessential extension* of K.

(2) A monomorphism (respectively, semimonomorphism) $f : M \to N$ of left R-semimodules is said to be *semiessential* if $f(M) \trianglelefteq_s N$.

(3) A monomorphism (respectively, semimonomorphism) $f : M \to N$ of left R-semimodules is said to be *quasiessential* if $Im f \trianglelefteq_s N$.

Lemma 2.2. *Let $f : M \to N$ be an R-monomorphism* (*respectively*, *R-semimonomorphism*) *of left semimodules*. *If f is semiessential*, *then f is quasiessential*.

Proof. Suppose that f is semiessential and let L be a subsemimodule of N such that $Im f \bigcap L = 0$. Since $f(M) \subset Im f \subset N$, $Im f \bigcap L = 0 \Rightarrow f(M) \bigcap L = 0$. Since f is semiessential, $f(M) \bigcap L = 0 \Rightarrow L = 0$, thus $Im f \trianglelefteq_s N$. Therefore, f is quasiessential. □

Remark 2.3. If f is i-regular, then f quasiessential is equivalent to f semiessential.

Indeed, if f is i-regular, then $f(M) = Im f$. Thus, by Definition 2.1, f quasiessential $\Leftrightarrow$ f semiessential.

Definition 2.4. (1) Let M be a left R-semimodule. Then an R-subsemimodule K of M is called *pseudoessential* in M, written as $K \trianglelefteq_p M$, if for all substractive R-subsemimodule S of M: $K \bigcap S = 0 \Rightarrow S = 0$.

(2) A monomorphism (respectively, semimonomorphism) $f : M \to N$ of left R-semimodules is said to be *pseudoessential* if $f(M) \trianglelefteq_p N$.

(3) A monomorphism (respectively, semimonomorphism) $f : M \to N$ of left R-semimodules is said to be *i-pseudoessential* if $Im f \trianglelefteq_p N$.

Remark 2.5. (1) If N is semiessential in M, then N is pseudoessential in M.

(2) If f is a monomorphism (respectively, semimonomorphism) semiessential, then f is pseudoessential.

(3) If f is a monomorphism (respectively, semimonomorphism) quasiessential, then f is i-pseudoessential.

(4) If f is i-regular, then f pseudoessential is equivalent to f i-pseudoessential.

Indeed, if N is semiessential in M, then N is semiessential for all subsemimodules of M, in particular, for its substractive subsemimodules, thus N is pseudoessential in M. Similarly, f semiessential $\Rightarrow$ f pseudoessential.

If f is a monomorphism quasiessential, then $Im f$ is semiessential for all subsemimodules of N, in particular, for its substractive subsemimodules, thus f is i-pseudoessential.

If f is i-regular, then $f(M) = Im f$, thus f pseudoessential $\Leftrightarrow$ f i-pseudoessential.

Proposition 2.6. *Let M be a left R-semimodule and K be a subsemimodule of M. Then the following conditions are equivalent*:

(1) $K \trianglelefteq_s M$.

(2) *The canonical injection* $i_K : K \to M$ *is a semiessential monomorphism*.

Proof. We have $i_K(K) = K$, then $K \trianglelefteq_s M \Leftrightarrow i_K(K) \trianglelefteq_s M$. Therefore (1) $\Leftrightarrow$ (2). □

Proposition 2.7. *Let M be a left R-semimodule and K be a substractive subsemimodule of M. Then the following conditions are equivalent*:

(1) $K \trianglelefteq_s M$.

(2) *The canonical injection* $i_K : K \to M$ *is a quasiessential monomorphism*.

Proof. If K is substractive, then $Im\, i_K = i_K(K) = K$ by [11], thus (1) $\Leftrightarrow$ (2). □

Proposition 2.8. *Let M be a left R-semimodule and K be a subsemimodule of M.*

(1) *If* $K \trianglelefteq_s M$, *then for all left semimodules N and for all* $h \in Hom_R(M; N)$, $(Ker h) \bigcap K = 0 \Rightarrow Ker h = 0$.

(2) *If for all homomorphisms* $h : M \to N$ *of left R-semimodules*, $(Ker h) \bigcap K = 0 \Rightarrow Ker h = 0$, *then* $K \trianglelefteq_p M$.

Proof. (1) $K \trianglelefteq_s M \Rightarrow \forall L \leq M$, if $L \bigcap K = 0$, then $L = 0$. $(Ker h) \leq M$, therefore $(Ker h) \bigcap K = 0 \Rightarrow Ker h = 0$.

(2) Consider the canonical epimorphism $n_S : M \to M/S$, where S is a substractive subsemimodule of M. Prove that $Ker n_S = S$.

If $x \in Ker n_S$, then $[x] = [0]$. $[x] = [0] \Rightarrow \exists (s, s') \in S^2$ such that $x + s = s'$. Also, as S is substractive, $(x + s \in S, s \in S) \Rightarrow x \in S$.

Conversely, letting $x \in S$, $x = x + 0 = 0 + x$, we have $\langle 0, x \rangle \in S^2$, therefore $[x] = [0] \Rightarrow x \in Kern_S$. Thus $Kern_S = S$, $S \bigcap K = 0 \Rightarrow (Kern_S) \bigcap K = 0 \Rightarrow Kern_S = 0$, therefore $S = 0$. Thus $K \trianglelefteq_p M$. □

Proposition 2.9. (1) *Let* $f : M \to N$ *be a semiessential homomorphism of R-semimodule and let* $h \in Hom_R(M; N)$. *If* $h \circ f$ *is a semimonomorphism, then h is a semimonomorphism.*

(2) *Let M be a cancellative left R-semimodule,* $f : M \to N$ *be a quasiessential homomorphism of R-semimodules and let* $h \in Hom_R(M; N)$. *If* $h \circ f$ *is a monomorphism, then h is a semimonomorphism.*

Proof. (1) Prove that $Ker h \bigcap f(L) = 0$. Let $x \in Ker h \bigcap f(L)$, therefore $h(x) = 0$ and $x = f(l)$ with $l \in L$, $h(x) = 0 \Rightarrow h \circ f(l) = 0$, therefore $l \in Ker(h \circ f)$. We have $h \circ f$ is a semimonomorphism, then $l = 0$ consequently, $x = f(0) = 0$. Therefore, $Ker h \bigcap f(L) = 0$. We know that *f* is semiessential so $Ker h = 0$. Thus *h* is a semimonomorphism.

(2) *f* is quasiessential so $Im f \trianglelefteq_s M$. Let $h \in Hom_R(M; N)$ be such that $h \circ f$ is a monomorphism. Prove that $Ker h \bigcap Im f = 0$. Let $x \in Ker h \bigcap Im f$. Then $x \in Ker h \bigcap Im f \Rightarrow (x \in Ker h$ and $x \in Im f)$, therefore $h(x) = 0$ and $x + f(l) = f(l')$ with $(l, l') \in L^2$, $x + f(l) = f(l') \Rightarrow h(x) + h[f(l)] = h[f(l')]$.

So $h \circ f(l) = h \circ f(l')$, consequently, $l = l'$ because $h \circ f$ is a monomorphism. Therefore, $x + f(l) = 0 + f(l)$ and since M is cancellative, we have $x = 0$. So $Ker h \bigcap Im f = 0$. We have also $Im f \trianglelefteq_s M$ and $Ker h \leq M$, therefore $Ker h = 0$. Thus *h* is a semimonomorphism. □

Corollary 2.10. *Let* $f : L \to M$ *be a monomorphism of left R-semimodules and let* $h \in Hom_R(M; N)$. *If* $h \circ f$ *semimonomorphism* $\Rightarrow$ *h semimonomorphism, then* $Im f \trianglelefteq_p M$.

Proof. Let $K = Im f$ and suppose that $Ker h \bigcap K = 0$. Then $Ker h \bigcap K = 0 \Rightarrow Ker(h \circ i_K) = 0 \Rightarrow Ker h = 0$. Therefore $K \trianglelefteq_p M$ from Proposition 2.8. Thus $Im f \trianglelefteq_p M$. □

Proposition 2.11. *Let M be a left R-semimodule, K, N and H be subsemimodules of M such that* $K \leq N \leq M$ *and* $H \leq M$. *Then we have*

(1) $K \trianglelefteq_s M \Leftrightarrow (K \trianglelefteq_s N$ *and* $N \trianglelefteq_s M)$.

(2) $(H \cap K) \trianglelefteq_s M \Leftrightarrow (H \trianglelefteq_s M$ *and* $K \trianglelefteq_s M)$.

Proof.

(1) • $(\Rightarrow)$ Let $L \leq N$ be such that $L \cap K = 0$. Since $L \leq M$ and $K \trianglelefteq_s M$, $L = 0$ thus $K \trianglelefteq_s N$. Let $L' \leq M$ be such that $L' \cap N = 0$, $L' \cap N = 0 \Rightarrow L' \cap K = 0 \Rightarrow L' = 0$ because $K \trianglelefteq_s M$.

• $(\Leftarrow)$ Let $L \leq M$ be such that $L \cap K = 0$, $L \cap K = 0 \Rightarrow L \cap K \cap N = 0 \Rightarrow (L \cap N) \cap K = 0$. We have $(L \cap N) \leq N$ and $K \trianglelefteq_s N$, then $L \cap N = 0$ consequently, $L = 0$ because $(L \cap N) \leq M$ and $N \trianglelefteq_s M$. Thus $K \trianglelefteq_s M$.

(2) • $(\Rightarrow)$ $(H \cap K) \leq H \leq M$, therefore from (1), we have $(H \cap K) \trianglelefteq_s M \Rightarrow H \trianglelefteq_s M$. $(H \cap K) \leq K \leq M$, therefore from (1), we have $(H \cap K) \trianglelefteq_s M \Rightarrow K \trianglelefteq_s M$.

• $(\Leftarrow)$ Let $L \leq M$ be such that $L \cap H \cap K = 0$, $L \cap H \cap K = 0 \Rightarrow (L \cap K) \cap H = 0$. Therefore, $L \cap K = 0$ because $(L \cap K) \leq M$ and $H \trianglelefteq_s M$. Consequently, $L = 0$ because $(L \cap K) \leq M$ and $K \trianglelefteq_s M$. Thus $(H \cap K) \trianglelefteq_s M$.

□

Lemma 2.12. *A subsemimodule K of a left R-semimodule M is semiessential if and only if for all* $x \neq 0$, *elements of M, there exists* $r \in R$ *such that* $0 \neq rx \in K$.

Proof.

• $(\Rightarrow)$ Suppose that $K \trianglelefteq_s M$ and let $0 \neq x \in M$. We know $(Rx) \leq M$ and $Rx \neq 0$ therefore $(Rx) \cap K \neq 0$ because $K \trianglelefteq_s M$. Consequently, there exists $r \in R$ such that $rx \in K$.

• $(\Leftarrow)$ Let $L \leq M$. Suppose that $L \neq 0$, so there exists $0 \neq x \in L$ and for assumption there exists $r \in R$ such that $rx \in K$. We have $rx \in L$ and $rx \neq 0$ therefore $rx \in (L \cap K)$ and $rx \neq 0$. Consequently, $L \cap K \neq 0$. Thus $K \trianglelefteq_s M$. □

Example 2.13. Consider the left R-semimodule $R = (\mathbb{I}, \max, \min)$. $N = R\backslash\{1\}$ is an essential subsemimodule of R (see Example 2.14). Prove that N is semiessential.

• Let $x \in \,]0; 1[$, $\min(x; 1) = x \in \,]0; 1[\, \subset N \Rightarrow 0 \neq x \otimes 1 \in N$.

• Let $x = 1$, $\min(0, 5; 1) = 0, 5 \in \,]0; 1[\, \subset N \Rightarrow 0 \neq 0, 5 \otimes 1 \in N$.

This proves that N is semiessential in R. □

Example 2.14. In this example, as above, we build a subsemimodule (of a semimodule) which is simultaneously semiessential and essential.

Set $M = \{0, 1, a, b\}$ and define on M the two commutative operations $(+, \times)$ as follows:

(1) $0_M = 0; 1_M = 1$.

(2) $1 + 1 = 1 + a = 1 + b = 1$; $a + 0 = a + a = a$; $0 + b = b + b = b$; $a + b = 0$.

(3) $0 \times a = 0 \times b = a \times b = b \times b = a \times a = 0$; $1 \times 1 = 1$; $1 \times a = a$; $1 \times b = b$.

Then $(M, +, 0)$ and $(M, \times, 1)$ are commutative semigroups.

Multiplying (2) successively by a and b and using (3), we can easily see that multiplications are distributive relatively to addition.

Hence $(M, +, \times, 0, 1)$ is a commutative semiring.

Now, put $N = \{0; a; b\}$, then N is an ideal of M. Thus ${}_M N$ is a subsemimodule of ${}_M M$.

• Let us prove that $N = \{0; a; b\}$ is semiessential. Let $0 \neq x \in M$. Then we can easily find $r \in M = \{0; 1; a; b\}$ such that $0 \neq rx \in N$. This proves that N is semiessential in M.

• Let us prove that $N = \{0, a; b\}$ is essential in ${}_M M$. Let ρ be a nontrivial R-congruence relation on M. Suppose that the restriction of ρ to N is trivial. Since ρ is not trivial on M, there exist $x_0, y_0 \in M$, $x_0 \neq y_0$ such that $x_0 \rho y_0$. ρ trivial on N implies that $x_0 \notin N$ or $y_0 \notin N$ hence $x_0 = 1$ or $y_0 = 1$. Therefore, there exists $c \neq 1$ such that $1\rho c$ thus $1\rho 0$ or $1\rho b$ or $1\rho a$.

Now, we are going to prove that these three last cases are impossible by using the fact that the restriction of ρ to N is trivial.

Case 1: $1\rho 0$ and $a\rho a$ imply that $1 \times a\rho 0 \times a$ hence $a\rho 0$ which contradicts the fact that the restriction of ρ to N is trivial.

Case 2: $1\rho b$ and $a\rho a$ imply that $1 \times a\rho b \times a$ hence $a\rho 0$ which contradicts the fact that the restriction of ρ to N is trivial.

Case 3: $1\rho a$ and $b\rho b$ imply that $1 \times b\rho a \times b$ hence $b\rho 0$ which contradicts the fact that the restriction of ρ to N is trivial.

This proves that the restriction of ρ to N is nontrivial and so N is essential in M.

Remark 2.15. Let $\overline{\mathfrak{C}}_{_RM}$ be the class of subsemimodules semiessential in an R-semimodule M introduced in this paper and let $\mathfrak{C}_{_RM}$ be the class of essential subsemimodules in an R-semimodule M given in [5]. Then in the two following examples, we are going to prove that neither of these classes is contained in other.

Example 2.16. Here we prove that $\overline{\mathfrak{C}}_{_RM} \not\subseteq \mathfrak{C}_{_RM}$ (see [5]).

Let $n \geq 1$ be a positive integer. Consider the set $R = \{r \in \mathbb{Q}^+ / r \leq n\} \cup \{-\infty\}$ in which $\mathbb{Q}^+$ is the set of all nonnegative rational numbers, $-\infty$ is assumed to satisfy the conditions that $-\infty \leq i$ and $-\infty + i = -\infty$, $\forall i \in R$. Define on R the operations $\oplus$ and $\otimes$ as follows: $\forall i; j \in R$; $i \oplus j = \max(i; j)$ and $i \otimes j = \min(i + j; n)$.

We can easily verify that $(R; \oplus; \otimes)$ is a commutative semiring having $-\infty$ as additive identity. R is also a left R-semimodule. Put $R^* = R \backslash \{0\}$, then R^* is an ideal of R. R^* is a subsemimodule of R.

- Let us prove that R^* is semiessential. Let $-\infty \neq x \in R$; $\exists r \in \mathbb{Q}_+^*$ with $r \leq n$ such that $r + x \leq n$ and so $r \otimes x = \min(r + x; n) = r + x$, now $r + x \neq -\infty$ and $r + x \neq 0$, this implies that $r \otimes x \in R^*$, therefore $R^* \trianglelefteq_s R$.

- Let us prove that R^* is not essential. It will suffice to find an R-congruence relation which is not trivial on R but trivial on R^*.

Let ρ be an R-congruence relation on R such that $-\infty\rho 0$ (ρ is the smallest R-congruence relation for which the elements $-\infty$ and 0 are in relation). Suppose there exists $r \neq r' \in R^*$ such that $r\rho_{(-\infty,\, 0)}r'$. Then $r\rho_{(-\infty,\, 0)}r' \Rightarrow r \equiv_{\{-\infty;\, 0\}} r'$. $r \equiv_{\{-\infty;\, 0\}} r' \Rightarrow r = 0$ or $r' = 0$ or $r = r'$ which is a contradiction. Thus ρ is trivial on R^*. This is proved by Proposition 1.4 that R^* is nonessential in R.

Example 2.17. In this example, we prove that $\mathfrak{C}_{R^M} \not\subseteq \overline{\mathfrak{C}}_{R^M}$.

Set $R = \{0, 1, a\}$ and define on R the two commutative operations $(+, \times)$ as follows:

(1) $0_R = 0;\ 1_R = 1$.

(2) $1 + 1 = 1 + a = 1;\ a + 0 = a + a = a$.

(3) $0 \times 0 = 0 \times 1 = 0 \times a = 0;\ 1 \times 1 = 1;\ 1 \times a = a \times a = a$.

Then $(R, +, \times, 0, 1)$ is a commutative semiring.

Let $M = \{0, 1, a, b\}$ with the same operations defined in R and $1_M = 1_R = 1$, $0_M = 0_R = 0$, $b + 0 = b + b = b$, $b + 1 = b + a = a$, $0 \times b = b \times a = 0$, $b \times 1 = b \times b = b$. Then it is easy to see that $(M, +, \times, 0, 1)$ is a commutative R-semimodule.

Now, put $N = R = \{0; 1; a\}$, then N is a subsemimodule of M.

- Let us prove that $N = \{0; 1; a\}$ is essential in M. Let ρ be a nontrivial R-congruence relation on M. Suppose that the restriction of ρ to N is trivial. If ρ is not trivial on M, then there exist $x_0, y_0 \in M$, $x_0 \neq y_0$ such that $x_0\rho y_0$. ρ trivial on N implies that $x_0 \notin N$ or $y_0 \notin N$ hence $x_0 = b$ or $y_0 = b$. Therefore there exists $c \neq b$ such that $b\rho c$ thus $b\rho 0$ or $b\rho 1$ or $b\rho a$.

Now, we are going to prove that these three last cases are impossible by using the fact that the restriction of ρ to N is trivial.

Case 1: $b\rho 0$ and $a\rho a$ imply that $b + a\rho 0 + a$ hence $1\rho a$ which contradicts the fact that the restriction of ρ to N is trivial.

Case 2: $b\rho 1$ implies that $a \times b\rho a \times 1$ hence $0\rho a$ which contradicts the fact that the restriction of ρ to N is trivial.

Case 3: $b\rho a$ and $a\rho a$ imply that $b + a\rho a + a$ hence $1\rho a$ which contradicts the fact that the restriction of ρ to N is trivial.

This proves that the restriction of ρ to N is nontrivial and so N is essential in M.

• Let us prove that $N = \{0; 1; a\}$ is not semiessential. If $0 \neq b \in M$, for all $r \in R = \{0; 1; a\}$, we have $r \times b = 0$ or $r \times b = b$. This proves that N is not semiessential in M.

Proposition 2.18. *Let M be a left R-semimodule. Suppose that* $K_1 \leq M_1 \leq M$; $K_2 \leq M_2 \leq M$ *with* M_i *substractive* $\forall i \in \{1; 2\}$ *and* $M = M_1 \overline{\oplus} M_2$. *Then* $(K_1 \overline{\oplus} K_2) \trianglelefteq_s (M_1 \overline{\oplus} M_2) \Leftrightarrow (K_1 \trianglelefteq_s M_1$ *and* $K_2 \trianglelefteq_s M_2)$.

Proof.

• $(\Rightarrow)$ Suppose, for example $K_1 \ntrianglelefteq M_1$. Then there exists a subsemimodule $L_1 \neq 0$ of M_1 such that $L_1 \cap K_1 = 0$. So prove that $L_1 \cap (K_1 + K_2) = 0$.

Let $l_1 \in L_1 \cap (K_1 + K_2)$. Then there exist $(k_1; k_2) \in K_1 \times K_2$ such that $l_1 = k_1 + k_2$. We have $l_1 \in L_1 \leq M_1$; $k_1 \in K_1 \leq M_1$ and as M_1 is substractive, therefore $k_2 \in M_1$, we know also $k_2 \in M_2$ so $k_2 = 0$, $k_2 = 0 \Rightarrow l_1 = k_1 = 0$. Consequently, $L_1 \cap (K_1 \overline{\oplus} K_2) = 0$. Thus $(K_1 \overline{\oplus} K_2) \ntrianglelefteq_s (M_1 \overline{\oplus} M_2)$.

• $(\Leftarrow)$ Suppose that $K_i \trianglelefteq_s M_1$ for all $i \in \{1, 2\}$. Let $0 \neq x \in M_1 \overline{\oplus} M_2$. Then there exists $(0, 0) \neq (x_1, x_2) \in M_1 \times M_2$ such that $0 \neq x = x_1 + x_2$. Without loss of generality, we suppose that $0 \neq x_1 \in M_1$. Since $K_1 \trianglelefteq_s M_1$, from Lemma 2.2, there exists an $r_1 \in R$ such that $r_1x_1 \in K_1$ and $r_1x_1 \neq 0$.

- If $r_1x_2 \in K_2$, then $r_1x_1 + r_1x_2 \in K_1 + K_2$, therefore $r_1(x_1 + x_2) \in K_1 \overline{\oplus} K_2$ with $r_1(x_1 + x_2) \neq 0$, because if $r_1x_1 + r_1x_2 = 0$ with M_2 substractive and $M_2 \cap M_1 = 0$, then $r_1x_1 = 0$, this is absurd.

Consequently, $(K_1 \overline{\oplus} K_2) \trianglelefteq_s (M_1 \overline{\oplus} M_2)$.

- If r_1x_2 is not in K_2, then there exists $r_2 \in R$ such that $0 \neq r_2r_1x_2 \in K_2$. We know $r_2r_1x_1 \in K_1$, then $r_2r_1(x_1 + x_2) \in K_1 \overline{\oplus} K_2$. If we put $r = r_2r_1$, then there exists $r \in R$ such that $r(x_1 + x_2) \in K_1 \overline{\oplus} K_2$ with $r(x_1 + x_2) \neq 0$ because if

$rx_1 + rx_2 = 0$ with M_1 substractive and $M_2 \bigcap M_1 = 0$, then $rx_2 = 0$ which implies that $rx_1 = 0$, this is absurd. Therefore $(K_1 \overline{\oplus} K_2) \trianglelefteq_s (M_1 \overline{\oplus} M_2)$.

Thus $K_i \trianglelefteq_s M_i$ for all $i \in \{1; 2\} \Rightarrow (K_1 \overline{\oplus} K_2) \trianglelefteq_s (M_1 \overline{\oplus} M_2)$. □

Definition 2.19. Let N be a subsemimodule of a left R-semimodule M. A subsemimodule N' of M is called *M-complement* of N if N' is maximal and $N \bigcap N' = 0$.

In the following, we give a nontrivial example of a strongly substractive subsemimodule.

Example 2.20. Let R be the semiring $R([0, 1], \max, \min, 0, 1)$ in which $[0, 1]$ is the unit interval on the real line with addition (respectively, multiplication) of x and y and is defined by $x \oplus y = \max(x, y)$ (respectively, $x \oplus y = \min(x, y)$). R is a left R-semimodule. Consider the ideal $N = R \backslash \{1\}$, N is a subsemimodule of R. If $x \oplus y = \max(x, y) \in N$, then $x < 1$ and $y < 1$ hence $x \in N$ and $y \in N$. Therefore, N is strongly substractive.

Proposition 2.21. (1) *Every subsemimodule N of a left R-semimodule M has an M-complement.*

(2) *If N' is an M-complement of a strongly substractive subsemimodule N of M, then* $N \overline{\oplus} N' \trianglelefteq_s M$.

Proof. (1) Let $£ = \{A \leq M; A \bigcap N = 0\}$, $0 \in £$ therefore $£ \neq \varnothing$. $(£; \subseteq)$ is an ordered poset. Let $(A_n)_{n \in \mathbb{N}}$ be an ordered subchain of $£$ such that $A_i \subset A_{i+1}$; $\forall i \in \mathbb{N}$.

Let $A = \bigcup_{n \in \mathbb{N}} A_n$, $A \leq M$ and $\forall n \in \mathbb{N}$, $A_n \bigcap N = 0 \Rightarrow \bigcup_{n \in \mathbb{N}} (A_n \bigcap N) = 0$. Therefore $\left(\bigcup_{n \in \mathbb{N}} A_n\right) \bigcap N = 0$. Consequently, $A \bigcap N = 0$ and thus $A \in £$. For $n \in \mathbb{N}$, $A_n \subset A$ therefore A is the upper bound of $\{A_n; n \in \mathbb{N}\}$. Let $K \in £$ be such that $A_n \subset K$, $\forall n \in \mathbb{N}$. Then we have $A = \bigcup_{n \in \mathbb{N}} A_n \subset K$. So $£$ is a nonempty inductive poset. Therefore $£$ has at least one maximal element N'. Also, N' is an M-complement of N.

(2) Let $L \leq M$ be such that $(N \overline{\oplus} N') \cap L = 0$,

$$(N \overline{\oplus} N') \cap L = 0 \Rightarrow N' \cap L = 0.$$

We have also $(N \overline{\oplus} N') \cap L = 0 \Rightarrow N \cap L = 0.$

Let $n \in N \cap (N' + L)$. Then there exist $n' \in N'$, $l \in L$ such that $n = n' + l$.

$n' + l \in N$ with N strongly substractive, therefore $n' \in N$ and $l \in N$. We have $N \cap N' = 0$ and $N \cap L = 0$ therefore $n' = l = 0$ and so $n = 0$.

Thus $N \cap (N' + L) = 0$. On the other hand, we have $N' \subset (N' + L)$ and N' is an *M*-complement of *N*, therefore $N' + L = N' \Rightarrow L \subset N'$, but $N' \cap L = 0$, then $L = 0$. Thus $(N \overline{\oplus} N') \trianglelefteq_s M$. □

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