Far East Journal of Mathematical Sciences (FJMS)
Volume 53, Number 2, 2011, Pages 113-169
Published Online: July 9, 2011
This paper is available online at http://pphmj.com/journals/fjms.htm

CONSTRUCTING (ω_1, β)-MORASSES FOR $\omega_1 \leq \beta$

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Abstract

Let $\kappa \in Card$ and $L_\kappa[X]$ be such that the fine structure theory, condensation and $Card^{L_\kappa[X]} = Card \cap \kappa$ hold. Then it is possible to prove the existence of morasses. In particular, I will prove that there is a κ-standard morass, a notion that I introduced in a previous paper. This shows the consistency of (ω_1, β)-morasses for all $\beta \geq \omega_1$.

1. Introduction

R. Jensen formulated in the 1970's the concept of an (ω_α, β)-morass whereby objects of size $\omega_{\alpha+\beta}$ could be constructed by a directed system of objects of size less than ω_α. He defined the notion of an (ω_α, β)-morass only for the case that $\beta < \omega_\alpha$. I introduced in a previous paper [6] a definition of an (ω_α, β)-morass for the case that $\omega_1 \leq \beta$.

This definition of an (ω_1, β)-morass for the case that $\omega_1 \leq \beta$ seems to be an

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2010 Mathematics Subject Classification: 03E05.

Keywords and phrases: higher gap morasses, inner model, condensation, fine structure, coherence

Submitted by Qi Feng

Received August 24, 2009; Revised December 17, 2010

axiomatic description of the condensation property of Gödel's constructible universe L and the whole fine structure theory of it. I was, however, not able to formulate and prove this fact in form of a mathematical statement. Therefore, I defined a seemingly innocent strengthening of the notion of an (ω_1, β)-morass, which I actually expect to be equivalent to the notion of (ω_1, β)-morass. I call this strengthening an $\omega_{1+\beta}$-standard morass. As will be seen, if we construct a morass in the usual way in L, the properties of a standard morass hold automatically.

Using the notion of a standard morass, I was able to prove a theorem which can be interpreted as saying that standard morasses fully cover the condensation property and fine structure of L. More precisely, I was able to show the following [6].

Theorem. *Let $\kappa \geq \omega_1$ be a cardinal and assume that a κ-standard morass exists. Then there exists a predicate X such that $Card \cap \kappa = Card^{L_\kappa[X]}$ and $L_\kappa[X]$ satisfies amenability, coherence and condensation.*

Let me explain this. The predicate X is a sequence $X = \langle X_\nu \mid \nu \in S^X \rangle$, where $S^X \subseteq Lim \cap \kappa$, and $L_\kappa[X]$ is endowed with the following hierarchy: Let $I_\nu = \langle J^X_\nu, X \restriction \nu \rangle$ for $\nu \in Lim - S^X$ and $I_\nu = \langle J^X_\nu, X \restriction \nu, X_\nu \rangle$ for $\nu \in S^X$, where $X_\nu \subseteq J^X_\nu$ and

$$J^X_0 = \emptyset,$$

$$J^X_{\nu+\omega} = rud(I^X_\nu),$$

$$J^X_\lambda = \bigcup\{J^X_\nu \mid \nu \in \lambda\} \text{ for } \lambda \in Lim^2 := Lim(Lim),$$

where $rud(I^X_\nu)$ is the rudimentary closure of $J^X_\nu = \bigcup\{J^X_\nu\}$ relative to $X \restriction \nu$ if $\nu \in Lim - S^X$ and relative to $X \restriction \nu$ and X_ν if $\nu \in S^X$. Now, the properties of $L_\kappa[X]$ are defined as follows:

(**Amenability**) The structures I_ν are amenable.

(**Coherence**) If $\nu \in S^X$, $H \prec_1 I_\nu$ and $\lambda = \sup(H \cap On)$, then $\lambda \in S^X$ and $X_\lambda = X_\nu \cap J^X_\lambda$.

(**Condensation**) If $\lambda \in S^X$ and $H \prec_1 I_\nu$, then there is some $\mu \in S^X$ such that $H \cong I_\mu$.

Moreover, if we let $\beta(\nu)$ be the least β such that $J^X_{\beta+\omega} \models \nu$ singular, then $S^X = \{\beta(\nu) \mid \nu$ singular in $I_\kappa\}$.

As will be seen, these properties suffice to develop the fine structure theory. In this sense, the theorem shows indeed what I claimed. In the present paper, I shall show the converse:

Theorem. *If $L_\kappa[X]$, $\kappa \in Card$, satisfies condensation, coherence, amenability, $S^X = \{\beta(\nu) \mid \nu$ singular in $I_\kappa\}$ and $Card^{L_\kappa[X]} = Card \cap \kappa$, then there is a κ-standard morass.*

Since L itself satisfies the properties of $L_\kappa[X]$, this also shows that the existence of κ-standard morasses and (ω_1, β)-morasses is consistent for all $\kappa \geq \omega_2$ and all $\lambda \geq \omega_1$.

Most results that can be proved in L from condensation and the fine structure theory also hold in the structures $L_\kappa[X]$ of the above form. As examples, I proved in my dissertation the following two theorems whose proofs can also be seen as applications of morasses:

Theorem. *Let $\lambda \geq \omega_1$ be a cardinal, $S^X \subseteq Lim \cap \lambda$, $Card \cap \lambda = Card^{L_\lambda[X]}$ and $X = \langle X_\nu \mid \nu \in S^X \rangle$ be a sequence such that amenability, coherence, condensation and $S^X = \{\beta(\nu) \mid \nu$ singular in $I_\kappa\}$ hold. Then $\square_k$ holds for all infinite cardinals $\kappa < \lambda$.*

Theorem. *Let $S^X \subseteq Lim$ and $X = \langle X_\nu \mid \nu \in S^X \rangle$ be a sequence such that amenability, coherence, condensation and $S^X = \{\beta(\nu) \mid \nu$ singular in $L[X]\}$ hold. Then the weak covering lemma holds for $L[X]$. That is, if there is no non-trival, elementary embedding $\pi : L[X] \to L[X]$, $\kappa \in Card^{L[X]} - \omega_2$ and $\tau = (\kappa^+)^{L[X]}$, then*

$$\tau < \kappa^+ \Rightarrow cf(\tau) = card(\kappa).$$

2. The Inner Model $L[X]$

We say a function $f : V^n \to V$ is *rudimentary* for some structure $\mathfrak{W} = \langle W, X_i \rangle$ if it is generated by the following schemata:

$$f(x_1, ..., x_n) = x_i \text{ for } 1 \le i \le n,$$

$$f(x_1, ..., x_n) = \{x_i, x_j\} \text{ for } 1 \le i, j \le n,$$

$$f(x_1, ..., x_n) = x_i - x_j \text{ for } 1 \le i, j \le n,$$

$$f(x_1, ..., x_n) = h(g_1(x_1, ..., x_n), ..., g_n(x_1, ..., x_n)),$$

where $h, g_1, ..., g_n$ are rudimentary

$$f(y, x_2, ..., x_n) = \bigcup\{g(z, x_2, ..., x_n) \mid z \in y\},$$

where g is rudimentary

$$f(x_1, ..., x_n) = X_i \cap x_j, \text{ where } 1 \le j \le n.$$

Lemma 1. *A function is rudimentary iff it is a composition of the following functions*:

$$F_0(x, y) = \{x, y\},$$

$$F_1(x, y) = x - y,$$

$$F_2(x, y) = x \times y,$$

$$F_3(x, y) = \{\langle u, z, v\rangle \mid z \in x \text{ and } \langle u, v\rangle \in y\},$$

$$F_4(x, y) = \{\langle z, u, v\rangle \mid z \in x \text{ and } \langle u, v\rangle \in y\},$$

$$F_5(x, y) = \bigcup x,$$

$$F_6(x, y) = dom(x),$$

$$F_7(x, y) = \in \cap (x \times x),$$

$$F_8(x, y) = \{x[\{z\}] \mid z \in y\},$$

$$F_{9+i}(x, y) = x \cap X_i$$

for the predicates X_i *of the structure under consideration*.

Proof. See, for example, in [3]. □

A relation $R \subseteq V^n$ is called *rudimentary* if there is a rudimentary function $f : V^n \to V$ such that $R(x_i) \Leftrightarrow f(x_i) \neq \varnothing$.

Lemma 2. *Every relation that is* Σ_0 *over the considered structure is rudimentary*.

Proof. Let χ_R be the characteristic function of R. The claim follows from the facts (i)-(vi):

(i) R rudimentary $\Leftrightarrow \chi_R$ rudimentary.

$\Leftarrow$ is clear. Conversely, $\chi_R = \bigcup\{g(y) \mid y \in f(x_i)\}$, where $g(y) = 1$ is constant and $R(x_i) \Leftrightarrow f(x_i) \neq \varnothing$.

(ii) If R is rudimentary, then $\neg R$ is also rudimentary.

Since $\chi_{\neg R} = 1 - \chi_R$.

(iii) $x \in y$ and $x = y$ are rudimentary.

By $x \notin y \Leftrightarrow \{x\} - y \neq \varnothing$, $x \neq y \Leftrightarrow (x - y) \bigcup (y - x) \neq \varnothing$ and (ii).

(iv) If $R(y, x_i)$ is rudimentary, then $(\exists z \in y) R(z, x_i)$ and $(\forall z \in y) R(z, x_i)$ are rudimentary.

If $R(y, x_i) \Leftrightarrow f(y, x_i) \neq \varnothing$, then

$$(\exists z \in y) R(z, x_i) \Leftrightarrow \bigcup\{f(z, x_i) \mid z \in y\} \neq \varnothing.$$

The second claim follows from this by (ii).

(v) If $R_1, R_2 \subseteq V^n$ are rudimentary, then so are $R_1 \vee R_2$ and $R_1 \wedge R_2$.

Because $f(x, y) = x \bigcup y$ is rudimentary, $(R_1 \vee R_2)(x_i) \Leftrightarrow \chi_{R_1}(x_i) \bigcup \chi_{R_2}(x_i) \neq \varnothing$ is rudimentary. The second claim follows from that by (ii).

(vi) $x \in X_i$ is rudimentary.

Since $\{x\} \bigcap X_i \neq \varnothing \Leftrightarrow x \in X_i$. □

For a converse of this lemma, we define:

A function f is called *simple* if $R(f(x_i), y_k)$ is Σ_0 for every Σ_0-relation $R(z, y_k)$.

Lemma 3. *A function f is simple iff*

(i) $z \in f(x_i)$ *is* Σ_0,

(ii) $A(z)$ *is* $\Sigma_0 \Rightarrow (\exists z \in f(x_i))A(z)$ *is* Σ_0.

Proof. If f is simple, then (i) and (ii) hold, because these are instances of the definition. The converse is proved by induction on Σ_0-formulas, e.g., if $R(z, y_k) :\Leftrightarrow z = y_k$, then $R(f(x_i), y_k) \Leftrightarrow f(x_i) = y_k \Leftrightarrow (\forall z \in f(x_i))(z \in y_k)$ and $(\forall z \in y_k)(z \in f(x_i))$. Thus we need (i) and (ii). The other cases are similar. $\square$

Lemma 4. *Every rudimentary function is* Σ_0 *in the parameters* X_i.

Proof. By induction, one proves that the rudimentary functions that are generated without the schema $f(x_1, ..., x_n) = X_i \cap x_j$ are simple. For this, one uses Lemma 3. But since the function $f(x, y) = x \cap y$ is one of those, the claim holds. $\square$

Thus every rudimentary relation is Σ_0 in the parameters X_i, but not necessarily Σ_0 with the X_i as predicates. An example is the relation $\{x, y\} \in X_0$.

A structure is said to be *rudimentary closed* if its underlying set is closed under all rudimentary functions.

Lemma 5. *If W is rudimentary closed and* $H \prec_1 \mathfrak{W}$, *then H and the collapse of H are also rudimentary closed*.

Proof. That is clear, since the functions $F_0, ..., F_{9+i}$ are Σ_0 with the predicates X_i. $\square$

Let T_N be the set of Σ_0 formulae of our language $\{\in, X_1, ..., X_N\}$ having exactly one free variable. By Lemma 2, there is a rudimentary function f for every Σ_0 formula ψ such that $\psi(x_*) \Leftrightarrow f(x_*) \neq \emptyset$. By Lemma 1, we have

$$x_0 = f(x_*) = F_{k_1}(x_1, x_2),$$

$$\begin{aligned} \text{where } x_1 &= F_{k_2}(x_3, x_4) \\ x_2 &= F_{k_3}(x_5, x_6) \\ \text{and } x_3 &= \cdots . \end{aligned}$$

Of course, x_* appears at some point.

Therefore, we may define an effective Gödel coding

$$T_N \to G, \qquad \psi_u \mapsto u$$

as follows $(m, n \text{ possibly} = *)$:

$$\langle k, l, m, n\rangle \in u :\Leftrightarrow x_k = F_l(x_m, x_n).$$

Let $\models^{\Sigma_0}_{\mathfrak{W}}(u, x_*) :\Leftrightarrow$

ψ_u is Σ_0 formula with exactly one free variable and $\mathfrak{W} \models \psi_u(x_*)$.

Lemma 6. *If* $\mathfrak{W}$ *is transitive and rudimentary closed, then* $\models^{\Sigma_0}_{\mathfrak{W}}(x, y)$ *is* Σ_1*-definable over* $\mathfrak{W}$. *The definition of* $\models^{\Sigma_0}_{\mathfrak{W}}(u, x_*)$ *depends only on the number of predicates of* $\mathfrak{W}$. *That is, it is uniform for all structures of the same type.*

Proof. Whether $\models^{\Sigma_0}_{\mathfrak{W}}(u, x_*)$ holds, may be computed directly. First, one computes the x_k which only depend on x_*. For those $\langle k, l, *, *\rangle \in u$. Then one computes the x_i which only depend on x_m and x_n such that $m, n \in \{k \mid \langle k, l, *, *\rangle \in u\}$ – etc. Since $\mathfrak{W}$ is rudimentary closed, this process only breaks off, when one has computed $x_0 = f(x_*)$. And $\models^{\Sigma_0}_{\mathfrak{W}}(u, x_*)$ holds iff $x_0 = f(x_0) \neq \emptyset$.

More formally speaking: $\models^{\Sigma_0}_{\mathfrak{W}}(x, x_*)$ holds iff there is some sequence $\langle x_i \mid i \in d\rangle$, $d = \{k \mid \langle k, l, m, n\rangle \in u\}$ such that

$$\langle k, l, m, n\rangle \in u \Rightarrow x_k = F_l(x_m, x_n) \text{ and } x_0 \neq \emptyset.$$

Hence $\models^{\Sigma_0}_{\mathfrak{W}}$ is Σ_1. □

If $\mathfrak{W}$ is a structure, then let $rud(\mathfrak{W})$ be the closure of $W \cup \{W\}$ under the functions which are rudimentary for $\mathfrak{W}$.

Lemma 7. *If* $\mathfrak{W}$ *is transitive, then so is* $rud(\mathfrak{W})$.

Proof. By induction on the definition of the rudimentary functions. □

Lemma 8. *Let* $\mathfrak{W}$ *be a transitive structure with underlying set W. Then*

$$rud(\mathfrak{W}) \cap \mathfrak{P}(W) = Def(\mathfrak{W}).$$

Proof. First, let $A \in Def(\mathfrak{W})$. Then A is Σ_0 over $\langle W \cup \{W\}, X_i \rangle$, i.e. there are parameters $p_i \in W \cup \{W\}$ and some Σ_0 formula φ such that $x \in A \Leftrightarrow \varphi(x, p_i)$. But by Lemma 2, every Σ_0 relation is rudimentary. Thus there is a rudimentary function f such that $x \in A \Leftrightarrow f(x, p_i) \neq \emptyset$. Let $g(z, x) = \{x\}$ and define $h(y, x) = \bigcup\{g(z, x) | z \in y\}$. Then $h(f(x, p_i), x) = \bigcup\{g(z, x) | z \in f(x, p_i)\}$ is rudimentary, $h(f(x, p_i), x) = \emptyset$ if $x \notin A$ and $h(f(x, p_0), x) = \{x\}$ if $x \in A$. Finally, let $H(y, p_i) = \bigcup\{h(f(x, p_i), x) | x \in y\}$. Then H is rudimentary and $A = H(W, p_i)$. So we are done.

Conversely, let $A \in rud(\mathfrak{W}) \cap \mathfrak{P}(W)$. Then there is a rudimentary function f and some $a \in W$ such that $A = f(a, W)$. By Lemma 4 and Lemma 3, there exists Σ_0 formula such that $x \in f(a, W) \Leftrightarrow \psi(x, a, W, X_i)$. By Σ_0 absoluteness, $A = \{x \in W | W \cup \{W, X_i\} \models \psi(x, a, W, X_i)\}$, since $X_i \subseteq W$. Therefore, there is a formula φ such that $A = \{x \in W | \mathfrak{W} \models \varphi(x, a)\}$. □

Let $\kappa \in Card - \omega_1$, $S^X \subseteq Lim \cap \kappa$ and $\langle X_\nu \restriction \nu \in S^X \rangle$ be a sequence.

For $\nu \in Lim - S^X$, let $I_\nu = \langle J_\nu^X, X \restriction \nu, X_\nu \rangle$ and let $I_\nu = \langle J_\nu^X, X \restriction \nu, X_\nu \rangle$ for $\nu \in S^X$ such that $X_\nu \subseteq J_\nu^X$, where

$$J_0^X = \emptyset,$$

$$J_{\nu+\omega}^X = rud(I_\nu),$$

$$J_\lambda^X = \bigcup\{J_\nu^X \mid \nu \in \lambda\} \text{ if } \lambda \in Lim^2 := Lim(Lim).$$

Obviously, $L_\kappa[X] = \bigcup\{J_\nu^X \mid \nu \in \kappa\}$.

We say that $L_\kappa[X]$ is *amenable* if I_ν is rudimentary closed for all $\nu \in S^X$.

Lemma 9. (i) *Every* J_ν^X *is transitive*,

(ii) $\mu < \nu \Rightarrow J_\mu^X \in J_\nu^X$,

(iii) $rank(J_\nu^X) = J_\nu^X \cap On = \nu$.

Proof. That are three easy proofs by induction. □

Sometimes we need levels between J_ν^X and $J_{\nu+\omega}^X$. To make those transitive, we define

$$G_i(x, y, z) = F_i(x, y) \text{ for } i \le 8,$$

$$G_9(x, y, z) = x \cap X,$$

$$G_{10}(x, y, z) = \langle x, y\rangle,$$

$$G_{11}(x, y, z) = x[y],$$

$$G_{12}(x, y, z) = \{\langle x. y\rangle\},$$

$$G_{13}(x, y, z) = \langle x, y, z\rangle,$$

$$G_{14}(x, , z) = \{\langle x. y\rangle, z\}.$$

Let

$$S_0 = \emptyset,$$

$$S_{\mu+1} = S_\mu \cup \{S_\mu\} \cup \bigcup\{G_i[(S_\mu \cup \{S_\mu\})^3] \mid i \in 15\},$$

$$S_\lambda = \bigcup\{S_\mu \mid \mu \in \lambda\} \text{ if } \lambda \in Lim.$$

Lemma 10. *The sequence* $\langle I_\mu \mid \mu \in Lim \cap \nu\rangle$ *is* (*uniformly*) Σ_1*-definable over* I_ν.

Proof. By definition $J_\mu^X = S_\mu$ for $\mu \in Lim$, that is, the sequence $\langle J_\mu^X \mid \mu \in Lim \cap \nu\rangle$ is the solution of the recursion defining Σ_0 restricted to *Lim*.

Since the recursion condition is Σ_0 over I_ν, the solution is Σ_1. It is Σ_1 over I_ν if the existential quantifier can be restricted to J_ν^X. Hence we must prove $\langle S_\mu \mid \mu \in \tau \rangle \in J_\nu^X$ for $\tau \in \nu$. This is done by induction on ν. The base case $\nu = 0$ and the limit step are clear. For the successor step, note that $S_{\mu+1}$ is a rudimentary function of S_μ and μ, and use the rudimentary closedness of J_ν^X. □

Lemma 11. *There are well-orderings* $<_\nu$ *of the sets* J_ν^X *such that*

(i) $\mu < \nu \Rightarrow <_\mu \subseteq <_\nu$,

(ii) $<_{\nu+1}$ *is an end-extension of* $<_\nu$,

(iii) *the sequence* $\langle <_\mu \mid \mu \in Lim \cap \nu \rangle$ *is (uniformly)* Σ_1*-definable over* I_ν,

(iv) $<_\nu$ *is (uniformly)* Σ_1*-definable over* I_ν,

(v) *the function* $pr_\nu(x) = \{z \mid z <_\nu x\}$ *is (uniformly)* Σ_1*-definable over* I_ν.

Proof. Define well-orderings $<_\mu$ of S_μ by recursion:

(I) $<_0 = \varnothing$.

(II) (1) For $x, y \in S_\mu$, let $x <_{\mu+1} y \Leftrightarrow x <_\mu y$.

(2) $x \in S_\mu$ and $y \notin S_\mu \Rightarrow y <_{\mu+1} y$,

$y \in S_\mu$ and $x \notin S_\mu \Rightarrow y <_{\mu+1} x$.

(3) If $x, y \notin S_\mu$, then there is an $i \in 15$ and $x_1, x_2, x_3 \in S_\mu$ such that $x = G_i(x_1, x_2, x_3)$. And there is a $j \in 15$ and $y_1, y_2, y_3 \in S_\mu$ such that $y = G_j(y_1, y_2, y_3)$. First, choose *i* and *j* minimal, then x_1 and y_1, then x_2 and y_2, and finally x_3 and y_3.

Set:

(a) $x <_{\mu+1} y$ if $i < j$,

$y <_{\mu+1} x$ if $i = j$.

(b) $x_1 <_\mu x_1$ if $i = j$ and $x_1 <_\mu x_1$,

$y <_{\mu+1} y$ if $i = j$ and $y_1 <_\mu x_1$.

(c) $x_1 <_{\mu+1} y$ if $i = j$ and $x_1 = y_1$ and $x_2 <_\mu y_2$,

$y <_{\mu+1} x$ if $i = j$ and $x_1 = y_1$ and $y_2 <_\mu x_2$.

(d) $x <_{\mu+1} y$ if $i = j$ and $x_1 = y_1$ and $x_2 = y_2$ and $x_3 <_\mu y_3$,

$y <_{\mu+1} x$ if $i = j$ and $x_1 = y_1$ and $y_2 = x_2$ and $y_3 <_\mu x_3$.

(III) $<_\lambda = \bigcup\{<_\mu \mid \mu \in \lambda\}$.

The properties (i) to (v) are obvious. For the Σ_1-definability, one needs the argument from Lemma 10. □

Lemma 12. *The rudimentary closed* $\langle J_\nu^X, X \restriction \nu, A\rangle$ *have a canonical* Σ_1*-Skolem function* h.

Proof. Let $\langle \psi_i \mid i \in \omega\rangle$ be an effective enumeration of the Σ_0 formulae with three free variables. Intuitively, we would define:

$$h(i, x) \simeq (z)_0$$

for

$$\text{the } <_\nu\text{-least } z \in J_\nu^X \text{ such that } \langle J_\nu^X, X \restriction \nu, A\rangle \models \psi_i((z)_0, x, (z)_1).$$

Formally, we define:

By Lemma 11(v), let θ be a Σ_0 formula such that

$$w = \{v \mid v <_\nu z\} \Leftrightarrow \langle J_\nu^X, X \restriction \nu, A\rangle \models (\exists t)\theta(w, z, t).$$

Let u_i be the Gödel coding of

$$\theta((s)_1, (s)_0, (s)_2) \wedge \psi(((s)_0)_0, s_3, ((s)_0)_1) \wedge (\forall v \in (s)_1) \neg \psi_i((v)_0, (s)_3, (v)_1)$$

and

$$y = h(i, x) \Leftrightarrow (\exists s)(((s_0)_0 = y \wedge (s)_3 = x \models^{\Sigma_0}_{\langle J_\nu^X, X \restriction \nu, A\rangle} (u_i, s)).$$

This has the desired properties. Note Lemma 6! □

I will denote this Σ_1-Skolem function by $h_{\nu, A}$. Let $h_\nu := h_{\nu, \varnothing}$.

Let us say that $L_\kappa[X]$ has *condensation* if the following holds:

If $\nu \in S^X$ and $H \prec_1 I_\nu$, then there is some $\mu \in S^X$ such that $H \cong I_\mu$.

From now on, suppose that $L_\kappa[X]$ is amenable and has condensation.

Set $I_\nu^0 = \langle J_\nu^X, X \restriction \nu \rangle$ for all $\nu \in Lim \cap \kappa$.

Lemma 13 (Gödel's pairing function)**.** *There is a bijection* $\Phi : On^2 \to On$ *such that* $\Phi(\alpha, \beta) \geq \alpha, \beta$ *for all* α, β *and* $\Phi^{-1} \restriction \alpha$ *is uniformly* Σ_1*-definable over* I_α^0 *for all* $\alpha \in Lim$*.*

Proof. Define a well-ordering $<^*$ on On^2 by

$$\langle \alpha, \beta \rangle <^* \langle \gamma, \delta \rangle$$

iff

$$\max(\alpha, \beta) < \max(\gamma, \delta) \text{ or}$$

$$\max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha < \gamma \text{ or}$$

$$\max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta.$$

Let $\Phi : \langle On^2, <^* \rangle \cong \langle On, < \rangle$. Then Φ may be defined by the recursion

$$\Phi(0, \beta) = \sup\{\Phi(\nu, \nu) | \nu < \beta\},$$

$$\Phi(\alpha, \beta) = \Phi(0, \beta) + \alpha \text{ if } \alpha < \beta,$$

$$\Phi(\alpha, \beta) = \Phi(0, \alpha) + \alpha + \beta \text{ if } \alpha \geq \beta.$$ □

So there is a uniform map from α onto $\alpha \times \alpha$ for all α that are closed under Gödel's pairing function. Such a map exists for all $\alpha \in Lim.$ But then we have to give up uniformity.

Lemma 14. *For all* $\alpha \in Lim$, *there exists a function from* α *onto* $\alpha \times \alpha$ *that is* Σ_1*-definable over* I_α^0.

Proof. by induction on $\alpha \in Lim.$ If α is closed under Gödel's pairing function, then Lemma 13 does the job. Therefore, if $\alpha = \beta + \omega$ for some $\beta \in Lim,$ we may assume $\beta \neq 0.$ But then there is some over I^0_α Σ_1-definable bijection $j : \alpha \to \beta.$ And by the induction hypothesis, there is an over I^0_α Σ_1-definable function from β onto $\beta \times \beta.$ Thus there exists a Σ_1 formula $\varphi(x, y, p)$ and *a* parameter $p \in J^X_\beta$ such that there is some $x \in \beta$ satisfying $\varphi(x, y, p)$ for all $y \in \beta \times \beta.$ So we get an over I^0_β Σ_1-definable injective function $g : \beta \times \beta \to \beta$ from the Σ_1-Skolem function. Hence $f(\langle \nu, \tau \rangle) = g(\langle j(\nu), j(\tau) \rangle)$ defines an injective function $f : \alpha^2 \to \beta$ which is Σ_1-definable over $I^0_\alpha.$ An *h* which is as needed may be defined by

$$h(\nu) = f^{-1}(\nu) \text{ if } \nu \in rng(f),$$

$$h(\nu) = \langle 0, 0 \rangle \text{ else.}$$

For $rng(f) = rng(g) \in J^X_\alpha$.

Now, assume $\alpha \in Lim^2$ is not closed under Gödel's pairing function. Then $\nu, \tau \in \alpha$ for $\langle \nu, \tau \rangle = \Phi^{-1}(\alpha),$ and $c := \{z | z <^* \langle \nu, \tau \rangle\}$ lies in J^X_α. Thus $\Phi^{-1} \upharpoonright c : c \to \alpha$ is an over I^0_α Σ_1-definable bijection. Pick a $\gamma \in Lim$ such that $\nu, \tau < \gamma.$ Then $\Phi^{-1} \upharpoonright \alpha : \alpha \to \gamma^2$ is an over I^0_α Σ_1-definable injective function. Like in the first case, there exists an injective function $g : \gamma \times \gamma \to \gamma$ in J^X_α by the induction hypothesis. So $f(\langle \xi, \zeta \rangle) = g(\langle g\Phi^{-1}(\xi), g\Phi^{-1}(\zeta) \rangle)$ defines an over I^0_α Σ_1-definable bijection $f : \alpha^2 \to d$ such that $d := g[g[c] \times g[c]].$ Again, we define *h* by

$$h(\xi) = f^{-1}(\xi) \text{ if } \xi \in d,$$

$$h(\xi) = \langle 0, 0 \rangle \text{ else.} \qquad \square$$

Lemma 15. *Let* $\alpha \in Lim - \omega + 1.$ *Then there is some over* I^0_α Σ_1*-definable function from* α *onto* J^X_α. *This function is uniformly definable for all* α *closed under Gödel's pairing function*.

Proof. Let $f : \alpha \to \alpha \times \alpha$ be a surjective function which is Σ_1-definable over I^0_α with parameter p. Let p be minimal with respect to the canonical well-ordering such that such an f exists. Define f^0, f^1 by $f\langle\nu\rangle = \langle f^0(\nu), f^1(\nu)\rangle$ and, by induction, define $f_1 = id \restriction \alpha$ and $f_{n+1}(\nu) = \langle f^0(\nu), f_n \circ f^1(\nu)\rangle$. Let $h := h_\alpha$ be the canonical Σ_1-Skolem function and $H = h[\omega \times (\alpha \times \{p\})]$. Then H is closed under ordered pairs. For, if $y_1 = h(j_1, \langle\nu_1, p\rangle)$, $y_2 = h(j_2, \langle\nu_2, p\rangle)$ and $\langle\nu_1, \nu_2\rangle = f(\tau)$, then $\langle y_1, y_2\rangle$ is Σ_1-definable over I^0_α with the parameters τ, p. Hence it is in H. Since H is closed under ordered pairs, we have $H \prec_1 I^0_\alpha$. Let $\sigma : H \to I^0_\beta$ be the collapse of H. Then $\alpha = \beta$, because $\alpha \subseteq H$ and $\sigma \restriction \alpha = id \restriction \alpha$. Thus $\sigma[f] = f$, and $\sigma[f]$ is Σ_1-definable over I^0_α with the parameter $\sigma(p)$. Since σ is a collapse, $\sigma(p) \leq p$. So $\sigma(p) = p$ by the minimality of p. In general, $\pi(h(i, x)) \simeq h(i, \pi(x))$ for Σ_1-elementary π. Therefore, $\sigma(h(i, \langle\nu, p\rangle)) \simeq h(i, \langle\nu, p\rangle)$ holds in our case for all $i \in \omega$ and $\nu \in \alpha$. But then $\sigma \restriction H = id \restriction H$ and $H = J^X_\alpha$. Thus we may define the needed surjective map by $g \circ f_3$, where

$$g(i, \nu, \tau) = y \text{ if } (\exists z \in S_\tau)\varphi(z, y, i, \langle\nu, p\rangle),$$

$$g(i, \nu, \tau) = \emptyset \text{ else.}$$

Here, S_τ shall be defined as in Lemma 10 and

$$y = h(i, x) \Leftrightarrow (\exists t \in J^X_\alpha)\varphi(t, i, x, y). \qquad \square$$

Let $\langle I^0_\nu, A\rangle := \langle J^X_\nu, X \restriction \nu, A\rangle$.

The idea of the fine structure theory is to code Σ_n predicates over large structures in Σ_1 predicates over smaller structures. In the simplest case, one codes the Σ_1 information of the given structure I^0_β in a rudimentary closed structure $\langle I^0_\rho, A\rangle$, i.e., we want to have something like:

Over I^0_β, there exists a Σ_1 function f such that

$$f[J^X_\rho] = J^X_\beta.$$

For the Σ_1 formulae φ_1,

$$\langle i,\ x\rangle \in A \Leftrightarrow I^0_\beta \models \varphi_i(f(x))$$

holds. And

$$\langle I^0_\rho,\ A\rangle \text{ is rudimentary closed.}$$

Now, suppose we have such an $\langle I^0_\rho,\ A\rangle$. Then every $B \subseteq J^X_\rho$ that is Σ_1-definable over I^0_β is of the form

$$B = \{x \mid A(i, \langle x,\ p\rangle)\} \text{ for some } i \in \omega,\ p \in J^X_\rho.$$

So $\langle I^0_\rho,\ B\rangle$ is rudimentary closed for all $B \in \Sigma_1(I^0_\beta) \cap \mathfrak{P}(J^X_\rho)$.

The ρ is uniquely determined.

Lemma 16. *Let* $\beta > \omega$ *and* $\langle I^0_\rho, C\rangle$ *be rudimentary closed*. *Then there is at most one* $\rho \in Lim$ *such that*

$$\langle I^0_\rho, C\rangle \textit{ is rudimentary closed for all } C \in \Sigma_1(\langle I^0_\beta,\ B\rangle) \cap \mathfrak{P}(J^X_\rho)$$

and

there is an over $\langle I^0_\rho,\ B\rangle \Sigma_1$*-definable function f such that* $f[J^X_\rho] = J^X_\beta$.

Proof. Assume $\rho < \overline{\rho}$ both had these properties. Let f be an over $\langle I^0_\beta,\ B\rangle$ Σ_1-definable function such that $f[J^X_\rho] = J^X_\beta$ and $C = \{x \in J^X_\rho \mid x \notin f(x)\}$. Then $C \subseteq J^X_\rho$ is Σ_1-definable over $\langle I^0_\beta,\ B\rangle$ So $\langle I^0_{\overline{\rho}},\ C\rangle$ is rudimentary closed. But then $C = C \cap J^X_\rho \in J^X_{\overline{\rho}}$. Hence there is an $x \in J^X_\rho$ such that $C = f(x)$. From this, the contradiction $x \in f(x) \Leftrightarrow x \in C \Leftrightarrow x \notin f(x)$ follows. □

The uniquely determined ρ from Lemma 16 is called the *projectum* of $\langle I^0_\beta,\ B\rangle$. If there is some over $\langle I^0_\beta,\ B\rangle$ Σ_1-definable function f such that $f[J^X_\rho] = J^X_\beta$, then $h_{\beta,B}[\omega \times (J^X_\rho \times \{p\})] = J^X_\beta$ for a $p \in J^X_\beta$. Using the canonical function $h_{\beta,B}$, we can define a canonical A:

Let p be minimal with respect to the canonical well-ordering such that the above property holds. Define

$$A = \{\langle i, x\rangle \mid i \in \omega \text{ and } x \in J^X_\rho \text{ and } \langle I^0_\beta, B\rangle \vDash \varphi_i(x, p)\}.$$

We say p is the *standard parameter* of $\langle I^0_\beta, B\rangle$ and A the standard code of it.

Lemma 17. *Let* $\beta > 0$ *and* $\langle I^0_\beta, B\rangle$ *be rudimentary closed. Let* ρ *be the projectum and A the standard code of it. Then for all* $m \geq 1$, *the following holds*:

$$\Sigma_{1+m}(\langle I^0_\beta, B\rangle) \cap \mathfrak{P}(J^X_\rho) = \Sigma_m(\langle I^0_\rho, A\rangle).$$

Proof. First, let $R \in \Sigma_{1+m}(\langle I^0_\beta, B\rangle) \cap \mathfrak{P}(J^X_\rho)$ and let m be even. Let P be a relation being Σ_1-definable over $\langle I^0_\beta, B\rangle$ with parameter q_1 such that, for $x \in J^X_\rho$, $R(x)$ holds $\exists y_0 \forall y_1 \exists y_3 \cdots \forall y_{m-1} P(y_i, x)$. Let f be some over $\langle I^0_\beta, B\rangle$ with parameter $q_2 \Sigma_1$-definable function such that $f[J^X_\rho] = J^X_\beta$. Define $Q(z_i, x)$ by $z_i, x \in J^X_\rho$ and $(\exists y_i)(y_i = f(z_i) \text{and } P(y_i, x))$. Let p be the standard parameter of $\langle I^0_\beta, B\rangle$. Then, by definition, there is some $u \in J^X_\rho$ such that $\langle q_1, q_2\rangle$ is Σ_1-definable in $\langle I^0_\beta, B\rangle$ with the parameters u, p, i.e., there is some $i \in \omega$ such that $Q(z_i, x)$ holds $z_i, x \in J^X_\rho$ and $\langle I^0_\beta, B\rangle \vDash \varphi_i(\langle z_i, x, u\rangle, p)$, i.e., iff $z_i, x \in J^X_\rho$ and $A(i, \langle z_i, x, u\rangle)$. Analogously there is a $j \in \omega$ and a $v \in J^X_\rho$ such that $z \in dom(f) \cap J^X_\rho$ iff $z \in J^X_\rho$ and $A(j, \langle z, v\rangle)$. Abbreviate this by $D(z)$. But then, for $x \in J^X_\rho$, $R(x)$ holds iff

$$\exists y_0 \forall y_1 \exists y_3 \cdots \forall y_{m-1}(D(z_0) \wedge \cdots \wedge D(z_{m-2}) \, and \, (D(z_1) \wedge D(z_3) \wedge \cdots \wedge) \Rightarrow Q(z_i, x)).$$

So the claim holds. If m is odd, then we proceed correspondingly. Thus $\Sigma_{1+m}(\langle I^0_\beta, B\rangle) \cap \mathfrak{P}(J^X_\rho) \subseteq \Sigma_m(\langle I^0_\rho, A\rangle)$ is proved.

Conversely, let φ be a Σ_0 formula and $q \in J^X_\rho$ such that, for all $x \in J^X_\rho$, $R(x)$ holds iff $\langle I^0_\rho, A\rangle \vDash \varphi(x, q)$. Since $\langle I^0_\rho, A\rangle$ is rudimentary closed, $R(x)$ holds iff $(\exists u \in J_\rho^X)(\exists a \in J_\rho^X)$ $(u$ transitive *and* $x \in u$ *and* $q \in u$ *and* $a = A \cap u$ *and* $\langle u, a\rangle \models \varphi(x, q))$. Write $a = A \cap u$ as formula: $(\forall v \in a)(v \in u$ *and* $v \in A)$ *and* $(\forall v \in u)(v \in A \Rightarrow v \in a)$. If $m = 1$, we are done provided we can show that this is Σ_2 over $\langle I_\beta^0, B\rangle$. If $m > 1$, then the claim follows immediately by induction. The second part is Π_1. So we only have to prove that the first part is Σ_2 over $\langle I_\beta^0, B\rangle$. By the definition of A, $v \in A$ is Σ_1-definable over $\langle I_\beta^0, B\rangle$, i.e., there is some Σ_0 formula and some parameter p such that $v \in A \Leftrightarrow \langle I_\beta^0, B\rangle \models (\exists y)\psi(v, y, p)$. Now, we have two cases.

In the first case, there is no over $\langle I_\beta^0, B\rangle$ Σ_1-definable function from some $\gamma < \rho$ cofinal in β. Then $(\forall v \in a)(v \in A)$ is Σ_2 over $\langle I_\beta^0, B\rangle$ because some kind of replacement axiom holds, and $(\forall v \in a)(\exists y)\psi(v, y, p)$ is over $\langle I_\beta^0, B\rangle$ equivalent to $(\exists z)(\forall v \in a)(\exists y \in z)\psi(v, y, p)$. For $\rho = \omega$, this is obvious. If $\rho \neq \omega$, then $\rho \in Lim^2$ and we can pick $\gamma < \rho$ such that $a \in J_\gamma^X$. Let $j : \gamma \to J_\gamma^X$ an over I_γ Σ_1-definable surjection, and g an over $\langle I_\beta^0, B\rangle$-definable function that maps $v \in J_\beta^X$ to $g(v) \in J_\beta^X$ such that $\psi(v, g(v), p)$ if such an element exists. We can find such a function with the help of the Σ_1-Skolem function. Now, define a function $f : \gamma \to \beta$ by

$f(\nu) =$ the least $\tau < \beta$ such that $g \circ j(\nu) \in S_\tau$ if $j(\nu) \in a$

$g(\nu) = 0$ else.

Since f is Σ_1, there is, in the given case, a $\delta < \beta$ such that $f[\gamma] \subseteq \delta$. So we have as collecting set $z = S_\delta$, and the equivalence is clear.

Now, let us come to the second case. Let $\gamma < \rho$ be minimal such that there is some over $\langle I_\beta^0, B\rangle$ Σ_1-definable function g from cofinal in β. Then $(\forall v \in a)(\exists y)\psi(v, y, p)$ is equivalent to $(\forall v \in a)(\exists \nu \in \gamma)(\exists y \in S_{g(\nu)})\psi(v, y, p)$. If we define a predicate $C \subseteq J_\rho^X$ by $\langle v, \nu\rangle \in C \Leftrightarrow y \in S_{g(\nu)}$ *and* $\psi(v, y, p)$, then

$\langle I^0_\beta, B\rangle \models (\forall v \in a)(\exists y)\psi(v, y, p)$ is equivalent to $\langle I^0_\beta, C\rangle \models (\forall v \in a)(\exists \nu \in \gamma)(\exists y)$ $\cdot(\langle v, \nu\rangle \in C)$. But this holds iff $\langle I^0_\beta, B\rangle \models (\exists w)$ (w transitive *and* $a, \gamma \in w$ *and* $\langle w, C \cap w\rangle \models (\forall v \in a)(\exists \nu \in y)(\exists y)(\langle v, \nu\rangle \in C \cap w)$). Since C is Σ_1 over $\langle I^0_\beta, B\rangle$, $\langle I^0_\rho, C\rangle$ is rudimentary closed by the definition of the projectum, i.e., the statement is equivalent to $\langle I^0_\rho, C\rangle \models (\exists w)(\exists c)$ (w transitive *and* $a, \gamma \in w$ *and* $c = C \cap w$ *and* $\langle w, c\rangle \models (\forall v \in a)(\exists \nu \in y)(\exists y)(\langle v, \nu\rangle \in c)$). So, to prove that this is Σ_2, it suffices to show that $c = C \cap w$ is Σ_2. In its full form, this is $(\forall z)(z \in a \Leftrightarrow z \in w$ *and* $z \in C)$. But $z \in C$ is even Δ_1 over $\langle I^0_\beta, B\rangle$ by the definition. So we are finished.

$\square$

Lemma 18. (a) *Let* $\pi : \langle J^X_{\bar\beta}, X \upharpoonright \bar\beta, \bar B\rangle \to \langle J^X_\beta, X \upharpoonright \beta, B\rangle$ *be* Σ_0*-elementary and* $\pi[\bar\beta]$ *be cofinal in* β. *Then* π *is even* Σ_1*-elementary*.

(b) *Let* $\langle J^X_{\bar\nu}, X \upharpoonright \bar\nu, \bar A\rangle$ *be rudimentary closed and* $\pi : \langle J^X_{\bar\nu}, X \upharpoonright \bar\nu\rangle \to \langle J^X_\nu, Y \upharpoonright \nu\rangle$ *be* Σ_0*-elementary and cofinal*. *Then there is a uniquely determined* $A \subseteq J^Y_\nu$ *such that* $\pi : \langle J^X_{\bar\nu}, X \upharpoonright \bar\nu, \bar A\rangle \to \langle J^X_\nu, X \upharpoonright \nu, A\rangle$ *is* Σ_0*-elementary and* $\langle J^X_\nu, X \upharpoonright \nu, A\rangle$ *is rudimentary closed*.

Proof. (a) Let φ be a Σ_0 formula such that $\langle J^X_\beta, X \upharpoonright \beta, B\rangle \models (\exists z)\varphi(z, \pi(x_i))$. Since $\pi[\bar\beta]$ is cofinal in β, there is *a* $\nu \in \bar\beta$ such that $\langle J^X_\beta, X \upharpoonright \beta, B\rangle \models (\exists z \in S_{\pi(\nu)})\varphi(z, \pi(x_i))$. Here, the S_ν is defined as in Lemma 10. If $\pi(S_\nu) = S_{\pi(\nu)}$, then $\langle J^X_\beta, X \upharpoonright \beta, B\rangle \models (\exists z \in \pi(S_\nu))\varphi(z, \pi(x_i))$. So, by the Σ_0-elementarity of π, $\langle J^X_{\bar\beta}, X \upharpoonright \bar\beta, \bar B\rangle \models (\exists z \in S_\nu)\varphi(z, x_i)$, i.e., $\langle J^X_{\bar\beta}, X \upharpoonright \bar\beta, \bar B\rangle \models (\exists z)\varphi(z, x_i)$. The converse is trivial.

It remains to prove $\pi(S_\nu) = S_{\pi(\nu)}$. This is done by induction on ν. If $\nu = 0$ or $\nu \notin Lim$, then the claim is obvious by the definition of S_ν and the induction hypothesis. So let $\lambda \in Lim$ and $M := \pi(S_\lambda)$. Then M is transitive by the Σ_0-elementarity of π. And since $\lambda \in Lim$ (i.e. $S_\lambda = J^X_\lambda$), $\langle S_\nu \mid \nu < \lambda\rangle$ is definable over

$\langle J_\lambda^X, X \upharpoonright \lambda \rangle$ by (the proof of) Lemma 10. Let φ be the formula $(\forall x)(\exists \nu)(x \in S_\nu)$. Since π is Σ_0-elementary, $\pi \upharpoonright S_\lambda : \langle J_\lambda^X, X \upharpoonright \lambda \rangle \to \langle M, (X \upharpoonright \lambda) \cap M \rangle$ is elementary. Thus, if $\langle J_\lambda^X, X \upharpoonright \lambda \rangle \models \varphi$, then also $\langle M, (X \upharpoonright \lambda) \cap M \rangle \models \varphi$. Since M is transitive, we get $M = S_\tau$ for a $\tau \in Lim$. And, by $\pi(\lambda) = \pi(S_\lambda \cap On) = S_\tau \cap On = \tau$, it follows that $\pi(S_\lambda) = S_{\pi(\lambda)}$.

(b) Since $\langle J_{\bar{\nu}}^X, X \upharpoonright \bar{\nu}, \bar{A} \rangle$ is rudimentary closed, $\bar{A} \cap S_\mu \in J_{\bar{\nu}}^X$ for all $\mu < \bar{\nu}$, where S_μ is defined as in Lemma 10. As in the proof of (a), $\pi(S_\mu) = S_{\pi(\mu)}$. So we need $\pi(\bar{A} \cap S_\mu) = A \cap S_{\pi(\mu)}$ to get that $\pi : \langle J_{\bar{\nu}}^X, X \upharpoonright \bar{\nu}, \bar{A} \rangle \to \langle J_\nu^X, X \upharpoonright \nu, A \rangle$ is Σ_0-elementary. Since π is cofinal, we necessarily obtain $A = \bigcup \{ \pi(\bar{A} \cap S_\mu) \mid \mu < \bar{\nu} \}$. But then $\langle J_\nu^Y, X \upharpoonright \nu, A \rangle$ is rudimentary closed. For, if $x \in J_\nu^X$, we can choose some $\mu < \bar{\nu}$ such that $x \in S_{\pi(\mu)}$. And $x \cap A = x \cap (A \cap S_{\pi(\mu)}) = x \cap \pi(\bar{A} \cap S_\mu)$ $\in J_\nu^X$. Now, let $\langle J_{\bar{\nu}}^X, X \upharpoonright \bar{\nu}, \bar{A} \rangle \models \varphi(x_i)$, where φ is a Σ_0 formula and $u \in J_{\bar{\nu}}^X$ is transitive such that $x_i \in u$. Then $\langle u, X \upharpoonright \bar{\nu} \cap u, A \cap u \rangle \models \varphi(x_i)$ holds. Since $\pi : \langle J_{\bar{\nu}}^Y, X \upharpoonright \bar{\nu} \rangle \to \langle J_\nu^Y, X \upharpoonright \nu \rangle$ is Σ_0-elementary, $\langle \pi(u), Y \upharpoonright \nu \cap \pi(u), A \cap \pi(u) \rangle$ $\models \varphi(\pi(x_i))$. Because $\pi(u)$ is transitive, we get $\langle J_\nu^Y, X \upharpoonright \nu \rangle \models \varphi(\pi(x_i))$. This argument works as well for the converse. □

Write $Cond_B(I_\beta^0)$ if there exists for all $H \prec_1 \langle I_\beta^0, B \rangle$ some $\bar{\beta}$ and some $\bar{B}$ such that $H \cong \langle I_{\bar{\beta}}^0, \bar{B} \rangle$.

Lemma 19 (Extension of embeddings)**.** *Let* $\beta > \omega$, $m \geq 0$ *and* $\langle I_\beta^0, B \rangle$ *be a rudimentary closed structure. Let* $Cond_B(I_\beta^0)$ *hold. Let* ρ *be the projectum of* $\langle I_\beta^0, B \rangle$, A *the standard code and* p *the standard parameter of* $\langle I_\beta^0, B \rangle$. *Then* $Cond_A(I_\rho^0)$ *holds. And if* $\langle I_{\bar{\rho}}^0, \bar{A} \rangle$ *is rudimentary closed and* $\pi : \langle I_{\bar{\rho}}^0, \bar{A} \rangle \to \langle I_\rho^0, A \rangle$ *is* Σ_m*-elementary, then there is an uniquely determined* Σ_{m+1}*-elementary extension* $\tilde{\pi} : \langle I_{\bar{\beta}}^0, \bar{B} \rangle \to \langle I_\beta^0, B \rangle$ *of* π *where* $\bar{\rho}$ *is the projectum of* $\langle I_{\bar{\beta}}^0, \bar{B} \rangle$, $\bar{A}$ *is the standard code and* $\tilde{\pi}^{-1}(p)$ *is the standard parameter of* $\langle I_{\bar{\beta}}^0, \bar{B} \rangle$.

Proof. Let $H = h_{\beta,B}[\omega\times(rng(\pi)\times\{p\})] \prec_1 \langle I^0_\beta, B\rangle$ and $\tilde{\pi} : \langle I^0_{\bar{\beta}}, \bar{B}\rangle \to \langle I^0_\beta, B\rangle$ be the uncollapse of *H*.

(1) $\tilde{\pi}$ is an extension of π

Let $\tilde{\rho} = \sup(\pi[\bar{\rho}])$ and $\tilde{A} = A \cap J^X_{\tilde{\rho}}$. Then $\pi : \langle J^Y_{\bar{\rho}}, X \upharpoonright \bar{\rho}\bar{A}\rangle \to \langle J^X_{\tilde{\rho}}, X \upharpoonright \tilde{\rho}, \tilde{A}\rangle$ is Σ_0-elementary, and by Lemma 18, it is even Σ_1-elementary. We have $rng(\pi) = H \cap J^X_{\tilde{\rho}}$. Obviously, $rng(\pi) \subseteq H \cap J^X_{\tilde{\rho}}$. So let $y \in H \cap J^X_{\tilde{\rho}}$. Then there is an $i \in \omega$ and an $x \in rng(\pi)$ such that *y* is the unique $y \in J^X_\beta$ that satisfies $\langle I^0_\beta, B\rangle \vDash \varphi_i(\langle y, x\rangle, p)$. So by definition of *A*, *y* is the unique $y \in J^X_\beta$ such that $\tilde{A}(i, \langle y, x\rangle)$. But $x \in rng(\pi)$ and $\pi : \langle J^Y_{\bar{\rho}}, X \upharpoonright \bar{\rho}, \bar{A}\rangle \to \langle J^X_{\tilde{\rho}}, X \upharpoonright \tilde{\rho}, \tilde{A}\rangle$ is Σ_1-elementary. Therefore $y \in rng(\pi)$. So we have proved that *H* is an $\in$-end-extension of $rng(\pi)$. Since π is the collapse of $rng(\pi)$ and $\tilde{\pi}$ the collapse of *H*, we obtain $\pi \subseteq \tilde{\pi}$.

(2) $\tilde{\pi} : \langle I^0_{\bar{\beta}}, \bar{B}\rangle \to \langle I^0_\beta, B\rangle$ is Σ_{m+1}-elementary

We must prove $H \prec_{m+1} \langle I^0_\beta, B\rangle$. If $m = 0$, this is clear. So let $m > 0$ and let *y* be Σ_{m+1}-definable in $\langle I^0_\beta, B\rangle$ with parameters from $rng(\pi) \cup \{p\}$. Then we have to show $y \in H$. Let φ be a Σ_{m+1} formula and $x_i \in rng(\pi)$ such that *y* is uniquely determined by $\langle I^0_\beta, B\rangle \vDash \varphi(y, x_i, p)$. Let $\tilde{h}(\langle i, x\rangle) \simeq h(i, \langle x, p\rangle)$. Then $\tilde{h}[J^X_\rho] = J^X_\beta$ by the definition of *p*. So there is a $z \in J^X_\rho$ such that $y = \tilde{h}(z)$. If such a *z* lies in $J^X_\rho \cap H$, then also $y \in H$, since $z, p \in H \prec_1 \langle I^0_\beta, B\rangle$. Let $D = dom(\tilde{h}) \cap J^X_\rho$. Then it suffices to show

$$(*) \quad (\exists z_0 \in D)(\forall z_1 \in D)\cdots\langle I^0_\beta, B\rangle \vDash \psi(\tilde{h}(z_i), \tilde{h}(z), x_i p)$$

for some $z \in H \cap J^X_\rho$, where ψ is Σ_1 for even *m* and Π_1 for odd *m* such that $\varphi(y, x_i, p) \Leftrightarrow \langle I^0_\beta, B\rangle \vDash (\exists z_0)(\forall z_1)\cdots\psi(z_i, y, x_i, p)$. First, let *m* be even. Since *A* is the standard code, there is an $i_0 \in \omega$ such that $z \in D \Leftrightarrow A(i_0, x)$ holds for all

$z \in J_\rho^X$ – and a $j_0 \in \omega$ such that, for all z_i, $z \in D\langle I_\beta^0, B\rangle \vDash \psi(\tilde{h}(z_i), \tilde{h}(z), x_i p)$ Thus $(*)$ is, for $z \in J_\rho^X$, equivalent with an obvious Σ_m formula. If m is odd, then write in $(*)\cdots\neg\langle I_\beta^0, B\rangle \vDash \neg\psi(\cdots)$. Then $\neg\psi$ is Σ_1 and we can proceed as above. Eventually $\pi : \langle I_{\bar\rho}^0, \bar{A}\rangle \to \langle I_\rho^0, A\rangle$ is Σ_m-elementary by the hypothesis and $\pi \subseteq \tilde{\pi}$ by (1) – i.e., $H \cap J_\rho^X \prec_m \langle I_\rho^0, A\rangle$. Since there is a $z \in J_\rho^X$ which satisfies $(*)$ and x_i, $p \in H \cap J_\rho^X$, there exists such a $z \in H \cap J_\rho^X$. Let $H \prec_1 \langle I_\rho^0, A\rangle$. Let π be the uncollapse of H. Then π has a Σ_1-elementary extension $\tilde{\pi} = \langle I_{\bar\beta}^0, \bar{B}\rangle \to \langle I_\beta^0, B\rangle$. So $H \cong \langle I_{\bar\rho}^0, \bar{A}\rangle$ for some $\bar\rho$ and $\bar{A}$, i.e., $Cond_A(I_\rho^0)$.

(3) $\tilde{A} = \{\langle i, x\rangle \mid i \in \omega \text{ and } x \in J_{\bar\rho}^X \text{ and} \langle I_{\bar\beta}^0, \bar{B}\rangle \vDash \varphi_i(x, \tilde{\pi}^{-1}(p))\}$

Since $\pi : \langle I_{\bar\rho}^0, \bar{A}\rangle \to \langle I_\rho^0, A\rangle$ is Σ_0-elementary, $\bar{A}(i, x) \Leftrightarrow A(i, \pi(x))$ for $x \in J_{\bar\rho}^X$. Since A is the standard code of $\langle I_\beta^0, \beta\rangle$, $A(i, \pi(x)) \Leftrightarrow \langle I_\beta^0, B\rangle \vDash \varphi_i(\pi(x), p)$. Finally, $\langle I_\beta^0, B\rangle \vDash \varphi_i(\pi(x), p) \Leftrightarrow \langle I_{\bar\beta}^0, \bar{B}\rangle \vDash \varphi_j(x, \tilde{\pi}^{-1}(p))$, because $\tilde{\pi} : \langle I_{\bar\beta}^0, \bar{B}\rangle \to \langle I_\beta^0, B\rangle$ is Σ_1-elementary.

(4) $\bar\rho$ is the projectum of $\langle I_{\bar\beta}^0, \bar{B}\rangle$

By the definition of H, $J_{\bar\beta}^X = h_{\bar\beta, \bar{B}}[\omega \times (J_{\bar\rho}^X \times \{\tilde{\pi}^{-1}(p)\})]$. So $f(\langle i, x\rangle) \simeq h_{\bar\beta, \bar{B}}(i, \langle x, \tilde{\pi}^{-1}(p)\rangle)$ is a over $\langle I_{\bar\beta}^0, \bar{B}\rangle$ Σ_1-definable function such that $f[J_{\bar\rho}^X] = J_{\bar\beta}^X$. It remains to prove that $\langle I_{\bar\rho}^0, C\rangle$ is rudimentary closed for all $C \in \Sigma_1(\langle I_{\bar\beta}^0, \bar{B}\rangle) \cap \mathfrak{P}(J_{\bar\rho}^X)$. By the definition of H, there exists an $i \in \omega$ and a $y \in J_{\bar\rho}^X$ such that $x \in C \Leftrightarrow \langle I_{\bar\beta}^0, \bar{B}\rangle \vDash \varphi_i(\langle x, y\rangle, \tilde{\pi}^{-1}(p))$ for all $x \in J_{\bar\rho}^X$. Thus, by (3), $x \in C \Leftrightarrow \bar{A}(i, \langle x, y\rangle)$. For $u \in J_{\bar\rho}^X$, let $v = \{\langle i, \langle x, y\rangle\rangle \mid x \in u\}$. Then $v \in J_{\bar\rho}^X$ and $\bar{A} \cap v \in J_{\bar\rho}^X$, because $\langle I_{\bar\rho}^0, \bar{A}\rangle$ is rudimentary closed by the hypothesis. But $x \in C \cap u$ holds iff $\langle i, \langle x, y\rangle\rangle \in \bar{A} \cap v$. Finally, $J_{\bar\rho}^X$ is rudimentary closed and therefore $C \cap u \in J_{\bar\rho}^X$.

(5) $\tilde{\pi}^{-1}(p)$ is the standard parameter of $\langle I^0_{\bar{\beta}}, \bar{B}\rangle$

By the definition of H, $J^X_{\bar{\beta}} = h_{\bar{\beta}, \bar{B}}[\omega \times (J^X_{\bar{\rho}} \times \{\tilde{\pi}^{-1}(p)\})]$ and, by (4), $\bar{\rho}$ is the projectum of $\langle I^0_{\bar{\beta}}, \bar{B}\rangle$. So we just have to prove that $\tilde{\pi}^{-1}(p)$ is the least with this property. Suppose that $\bar{p}' < \tilde{\pi}^{-1}(p)$ had this property as well. Then there were an $i \in \omega$ and an $x \in J^X_{\bar{\rho}}$ such that $\pi^{-1}(p) = h_{\bar{\beta}, \bar{B}}(i, \langle x, \bar{p}'\rangle)$. Since $\tilde{\pi} : \langle I^0_{\bar{\beta}}, \bar{B}\rangle \to \langle I^0_{\beta}, B\rangle$ is Σ_1-elementary, we had $p = h_{\beta, B}(i, \langle x, p'\rangle)$ for $p' = \pi(\bar{p}') < p$. And so also $h_{\beta, B}[\omega \times (J^X_{\rho} \times \{p'\})] = J^X_{\beta}$. That contradicts the definition of p.

(6) Uniqueness

Assume $\langle I^0_{\bar{\beta}_0}, \bar{B}_0\rangle$ and $\langle I^0_{\bar{\beta}_1}, \bar{B}_1\rangle$ both have $\bar{\rho}$ as projectum and $\bar{A}$ as standard code. Let $\bar{p}_i$ be the standard parameter of $\langle I^0_{\bar{\beta}_i}, \bar{B}_i\rangle$. Then, for all $j \in \omega$ and $x \in J^X_{\bar{\rho}}$, $\langle I^0_{\bar{\beta}_0}, \bar{B}_0\rangle \models \varphi_j(x, \bar{p}_0)$ iff $\bar{A}(j, x)$ iff $\langle I^0_{\bar{\beta}_1}, \bar{B}_1\rangle \models \varphi_j(x, \bar{p}_1)$. So $\sigma(h_{\bar{\beta}_0, \bar{B}_0}(j, (x, \bar{p}_0))) \simeq h_{\tilde{\beta}_1, \bar{B}_1}(j, \langle x, \bar{p}_1\rangle)$ defines an isomorphism $\sigma : \langle I^0_{\bar{\beta}_0}, \bar{B}_0\rangle \cong \langle I^0_{\bar{\beta}_0}, \bar{B}_0\rangle$, because, for both $h_{\bar{\beta}_i, \bar{B}_i}[\omega \times (J^X_{\bar{\rho}} \times \{\bar{p}_i\})] = J^X_{\bar{\beta}_i}$ holds. But since both structures are transitive, σ must be the identity. Finally, let $\tilde{\pi}_0 : \langle I^0_{\bar{\beta}}, \bar{B}\rangle \to \langle I^0_{\beta}, B\rangle$ and $\tilde{\pi}_1 : \langle I^0_{\bar{\beta}}, \bar{B}\rangle \to \langle I^0_{\beta}, B\rangle$ be Σ_1-elementary extensions of π. Let $\bar{p}$ be the standard parameter of $\langle I^0_{\bar{\beta}}, \bar{B}\rangle$. Then, for every $y \in J^X_{\bar{\beta}}$, there is an $x \in J^X_{\bar{\rho}}$ and a $j \in \omega$ such that $y = h_{\bar{\beta}, \bar{B}}(j, \langle x, \bar{p}\rangle)$ and $\tilde{\pi}_0(y) = h_{\beta, B}(j, \pi(x), \pi(p)) = \tilde{\pi}_1(y)$. Thus $\tilde{\pi}_0 = \tilde{\pi}_1$. □

To code the Σ_n information of I_β, where $\beta \in S^X$ in a structure $\langle I^0_\rho, A\rangle$, one iterates this process.

For $n \geq 0$, $\beta \in S^X$, let

$$\rho^0 = \beta,\ p^0 = \emptyset,\ A^0 = X_\beta,$$

$$\rho^{n+1} = \text{the projectum of } \langle I^0_{\rho^n}, A^n\rangle,$$

p^{n+1} = the standard parameter of $\langle I^0_{\rho^n}, A^n \rangle$,

A^{n+1} = the standard code of $\langle I^0_{\rho^n}, A^n \rangle$.

Call

ρ^n the nth projectum of β,

p^n the nth (standard) parameter of β,

A^n the nth (standard) code of β.

By Lemma 17, $A^n \subseteq J^X_{\rho^n}$ is Σ_n-definable over I_β and, for all $m \geq 1$,

$$\Sigma_{n+m}(I_\beta) \cap \mathfrak{P}(J^X_{\rho^n}) = \Sigma_m(\langle I^0_{\rho^n}, A^n \rangle).$$

From Lemma 19, we get by induction:

For $\beta > \omega$, $n \geq 1$, $m \geq 0$, let ρ^n be the nth projectum and A^n be the nth code of β. Let $\langle I^0_{\overline{\rho}}, \overline{A} \rangle$ be a rudimentary closed structure and $\pi := \langle I^0_{\overline{\rho}}, \overline{A} \rangle \to \langle I^0_{\rho^n}, A^n \rangle$ be Σ_m-elementary. Then:

(1) There is a unique $\overline{\beta} \geq \overline{\rho}$ such that $\overline{\rho}$ is the nth projectum and $\overline{A}$ is the nth code of $\overline{\beta}$.

For $k \leq n$, let

ρ^k be the kth projectum of β,

p^k be the kth parameter of β,

A^k be the kth code of β

and

$\overline{\rho}^k$ be the kth projectum of $\overline{\beta}$,

$\overline{p}^k$ be the kth parameter of $\overline{\beta}$,

$\overline{A}^k$ be the kth code of $\overline{\beta}$.

(2) There exists a unique extension $\tilde{\pi}$ of π such that, for all $0 \le k \le n$,

$$\tilde{\pi} \upharpoonright \langle I^0_{\bar{\rho}^k}, A^k \rangle \to \langle I^0_{\rho^k}, A^k \rangle \text{ is } \Sigma_{m+n-k}\text{-elementary}$$

$$\text{and } \tilde{\pi}(\bar{p}^k) = p^k.$$

Lemma 20. *Let* $\omega < \beta \in S^X$. *Then all projecta of* β *exist.*

Proof. By induction on n. That ρ^0 exists is clear. So suppose that the first projecta $\rho^0, \ldots, \rho^{n-1}, \rho := \rho^n$, the parameters $p^0, \ldots, p^n$ and the codes $A^0, \ldots, A^{n-1}$, $A := A^n$ of β exist. Let $\gamma \in Lim$ be minimal such that there is some over $\langle I^0_\rho, A \rangle$ Σ_1-definable function f such that $f[J^X_\gamma] = J^X_\rho$. Let $C \in \Sigma_1(\langle I^0_\rho, A \rangle) \cap \mathfrak{P}(J^X_\gamma)$. We have to prove that $\langle I^0_\gamma, C \rangle$ is rudimentary closed. If $\gamma = \omega$, then $J^X_\gamma = H_\omega$, and this is obvious. If $\gamma > \omega$, then $\gamma \in Lim^2$ by the definition of γ. Then it suffices to show $C \cap J^X_\delta \in J^X_\gamma$ for $\delta \in Lim \cap \gamma$. Let $B := C \cap J^X_\delta$ be definable over $\langle I^0_\rho, A \rangle$ with parameter q. Since obviously $\gamma \le \rho$, $C \cap J^X_\delta$ is Σ_n-definable over I_β with parameters $p_1, \ldots, p^n, q$ by Lemma 17. So let φ be a Σ_n formula such that $x \in C \Leftrightarrow I_\beta \models \varphi(x, p^1, \ldots, p^n, q)$. Let

$$H_{n+1} := h_{\rho^n, A^n}[\omega \times (J^X_\delta \times \{q\})],$$

$$H_n := h_{\rho^{n-1}, A^{n-1}}[\omega \times (H_n \times \{p^n\})],$$

$$H_{n-1} := h_{\rho^{n-2}, A^{n-2}}[\omega \times (H_{n-1} \times \{p^{n-1}\})],$$

etc.

Since $L[X]$ has condensation, there is an I_μ such that $H_1 \cong I_\mu$. Let π be the uncollapse of H_1. Then π is the extension of the collapse of H_{n+1} defined in the proof of Lemma 19. Therefore, it is Σ_{n+1}-elementary. Since $B \subseteq J^X_\delta$ and $\pi \upharpoonright J^X_\delta = id \upharpoonright J^X_\delta$, we get $x \in B \Leftrightarrow I_\mu \models \varphi(x, \pi^{-1}(p^1), \ldots, \pi^{-1}(p^n), \pi^{-1}(q))$. So B is indeed already Σ_n-definable over I_μ. Thus $B \in J^X_{\mu+1}$ by Lemma 8. But now we are done because $\mu < \rho$. For, if

$$h_{n+1}(\langle i, x\rangle) = h_{\rho^n, A^n}(i, \langle x, p\rangle),$$

$$h_n(\langle i, x\rangle) = h_{\rho^{n-1}, A^{n-1}}(i, \langle x, p^n\rangle),$$

etc.,

then the function $h = h_1 \circ \cdots \circ h_{n+1}$ is Σ_{n+1}-definable over I_β. Thus the function $\bar{h} = \pi[h \cap (H_1 \times H_1)]$ is Σ_{n+1}-definable over I_μ and $\bar{h}[J^X_\delta] = J^X_\mu$. So $\bar{h} \cap (J^X_\rho)^2$ is Σ_1-definable over $\langle I^0_\rho, A\rangle$ by Lemma 17 and Lemma 19. And by the definition of γ, there is an over $\langle I^0_\rho, A\rangle$ Σ_1-definable function f such that $f[J^X_\gamma] = J^X_\rho$. So if we had $\mu \geq \rho$, then $f \circ \bar{h}$ was an over $\langle I^0_\rho, A\rangle$ Σ_1-definable function such that $(f \circ \bar{h})[J^X_\delta] = J^X_\rho$. That contradicts the minimality of γ. □

Let $\omega < \nu \in S^X$, ρ^n be the nth projectum of ν, p^n be the nth parameter and A^n be the nth Code. Let

$$h_{n+1}(\langle i, x\rangle) = h_{\rho^n, A^n}(i, x),$$

$$h_n(\langle i, x\rangle) = h_{\rho^{n-1}, A^{n-1}}(i, \langle x, p^n\rangle),$$

etc.

Then define

$$h^{n+1}_\nu = h_1 \circ \cdots \circ h_{n+1}.$$

We have:

(1) h^n_ν is Σ_n-definable over I_ν

(2) $h^n_\nu[\omega \times Q] \prec_n I_\nu$, if $Q \subseteq J^X_{\rho^{n-1}}$ is closed under ordered pairs.

Lemma 21. *Let* $\omega < \beta \in S^X$ *and* $n \geq 1$. *Then*

(1) *the least ordinal* $\gamma \in Lim$ *such that there is a over* $I_\beta \Sigma_n$ *-definable function f such that* $f[J^X_\gamma] = J^X_\beta$,

(2) *the last ordinal* $\gamma \in Lim$ *such that* $\langle I^0_\gamma, C\rangle$ *is rudimentary closed for all* $C \in \Sigma_n(I_\beta) \cap \mathfrak{P}(J^X_\gamma)$,

(3) *the least ordinal* $\gamma \in Lim$ *such that* $\mathfrak{P}(\gamma) \cap \Sigma_n(I_\beta) \not\subseteq J^X_\beta$, *is the nth projectum of* β.

Proof. (1) By the definition of the *n*th projectum, there is an over $\langle I^0_{\rho^{n-1}}, A^{n-1}\rangle$ Σ_1-definable f^n such that $f^n[J^X_{\rho^n}] = J^X_{\rho^{n-1}}$, an over $\langle I^0_{\rho^{n-2}}, A^{n-2}\rangle$ Σ_1-definable f^{n-1} such that $f^{n-1}[J^X_{\rho^{n-1}}] = J^X_{\rho^{n-2}}$, etc. But then f^k is Σ_k-definable over I_β by Lemma 17. So $f = f^1 \circ f^2 \circ \cdots \circ f^n$ is Σ_n-definable over I_β and $f[J^X_{\rho^n}] = J^X_\beta$.

On the other hand, the projectum $\overline{\rho}$ of a rudimentary closed structure $\langle I^0_\beta, B\rangle$ is the least $\overline{\rho}$ such that there is an over $\langle I^0_\beta, B\rangle$ Σ_1-definable function *f* such that $f[J^X_{\overline{\rho}}] = J^X_\beta$. For, suppose there is no such $\rho < \overline{\rho}$ such that such an *f*, $f[J^X_\rho] = J^X_\beta$, exists. Then the proof of Lemma 16 provides a contradiction. So if there was a $\gamma < \rho^n$ such that there is an over $I_\beta\, \Sigma_n$-definable function *f* such that $f[J^X_\rho] = J^X_\beta$, then $g := f \cap (J^X_{\rho^{n-1}})^2$ would be an over $\langle I^0_{\rho^{n-1}}, A^{n-1}\rangle$ Σ_1-definable function such that $g[J^X_\gamma] = J^X_{\rho^{n-1}}$. But this is impossible.

(2) By the definition of the *n*th projectum, $\langle I^0_{\rho^n}, C\rangle$ is rudimentary closed for all $C \in \Sigma_1(\langle I^0_{\rho^{n-1}}, A^{n-1}\rangle) \cap \mathfrak{P}(J^X_{\rho^n})$. But by Lemma 17, $\Sigma_1(\langle I^0_{\rho^{n-1}}, A^{n-1}\rangle) = \Sigma_n(I_\beta) \cap \mathfrak{P}(J^X_{\rho^{n-1}})$. So, since $\rho^n \leq \rho^{n-1}$, $\langle I^0_{\rho^n}, C\rangle$ is rudimentary closed for all $C \in \Sigma_n(I_\beta) \cap \mathfrak{P}(J^X_{\rho^n})$.

Assume γ was a larger ordinal $\in Lim$ having this property. Let f be, by (1), an over I_β Σ_n-definable function such that $f[J^X_{\rho^n}] = J^X_\beta$. Set $C = \{u \in J^X_{\rho^n} \mid u \notin f(u)\}$. Then C is Σ_n-definable over I_β and $C \subseteq J^X_{\rho^n}$. So $\langle J^X_\gamma, C\rangle$ was rudimentary closed. And therefore $C = C \cap J^X_{\rho^n} \in J^X_\gamma \subseteq J^X_\beta$ and $C = f(u)$ for some $u \in J^X_{\rho^n}$. But this implies the contradiction that $u \in f(u) \Leftrightarrow u \in C \Leftrightarrow u \notin f(u)$.

(3) Let $\rho := \rho^n$ and f by (1) an over I_β Σ_n-definable function such that $f[J^X_{\rho^n}] = J^X_\beta$. Let j be an over I^0_ρ Σ_1-definable function from ρ onto J^X_ρ. Let $C = \{\nu \in \rho \mid \nu \notin f \circ j(n)\}$. Then C is an over I_β Σ_n-definable subset of ρ. If $C \in J^X_\beta$, then there would be a $\nu \in \rho$ such that $C = f \circ j(\nu)$, and we had the contradiction $\nu \in C \Leftrightarrow \nu \notin f \circ j(\nu) \Leftrightarrow \nu \notin C$. Thus $\mathfrak{P}(\rho) \cap \Sigma_n(I_\beta) \not\subseteq J^X_\beta$. But if $\gamma \in Lim \cap \rho$ and $D \in \mathfrak{P}(\gamma) \cap \Sigma_n(I_\beta)$, then $D = D \cap J^X_\gamma \in J^X_\rho \subseteq J^X_\beta$. So $\mathfrak{P}(\gamma) \cap \Sigma_n(I_\beta) \subseteq J^X_\beta$. □

3. Morasses

Let $\omega_1 \leq \beta$, $S = Lim \cap \omega_{1+\beta}$ and $\kappa := \omega_{1+\beta}$.

We write *Card* for the class of cardinals and *RCard* for the class of regular cardinals.

Let $\triangleleft$ be a binary relation on S such that:

(a) If $\nu \triangleleft \tau$, then $\nu < \tau$.

For all $\nu \in S - RCard$, $\{\tau \mid \nu \triangleleft \tau\}$ is closed.

For $\nu \in S - RCard$, there is a largest μ such that $\nu \trianglelefteq \mu$.

Let μ_ν be this largest μ with $\nu \trianglelefteq \mu$.

Let

$$\nu \sqsubseteq \tau :\Leftrightarrow \nu \in Lim(\{\delta \mid \delta \triangleleft \tau\}) \cup \{\delta \mid \delta \trianglelefteq \tau\}.$$

(b) $\sqsubseteq$ is a (many-rooted) tree.

Hence, if $\nu \notin RCard$ is a successor in $\sqsubset$, then μ_ν is the largest μ such that $\nu \sqsubseteq \mu$. To see this, let μ^*_ν be the largest μ such that $\nu \sqsubseteq \mu$. It is clear that $\mu_\nu \leq \mu^*_\nu$, since $\nu \trianglelefteq \mu$ implies $\nu \sqsubseteq \mu$. So assume that $\mu_\nu < \mu^*_\nu$. Then $\nu \ntriangleleft \mu^*_\nu$ by the definition of μ_ν. Hence $\nu \in Lim(\{\delta \mid \delta \triangleleft \mu^*_\nu\})$ and $\nu \in Lim(\{\delta \mid \delta \sqsubseteq \mu^*_\nu\})$. Therefore, $\nu \in Lim(\sqsubseteq)$ since ν is a tree. That contradicts our assumption that $\sqsubseteq$ is a successor in $\sqsubset$.

For $\alpha \in S$, let $|\alpha|$ be the rank of α in this tree. Let

$$S^+ := \{\nu \in S \mid \nu \text{ is a successor in } \sqsubset\},$$

$$S^0 := \{\alpha \in S \mid |\alpha| = 0\},$$

$$\hat{S}^+ := \{\mu_\tau \mid \tau \in S^+ - RCard\},$$

$$\hat{S} := \{\mu_\tau \mid \tau \in S - RCard\}.$$

Let $S_\alpha := \{\nu \in S \mid \nu$ is a direct successor of α in $\sqsubset\}$. For $\nu \in S^+$, let α_ν be the direct predecessor of ν in $\sqsubset$. For $\nu \in S^0$, let $\alpha_\nu := 0$. For $\nu \notin S^+ \cup S^0$, let $\alpha_\nu := \nu$.

(c) For $\nu, \tau \in (S^+ \cup S^0)$-$RCard$ such that $\alpha_\nu = \alpha_\tau$, suppose:

$$\nu < \tau \Rightarrow \mu_\nu < \tau.$$

For all $\alpha \in S$, suppose:

(d) S_α is closed.

(e) $card(S_\alpha) \leq \alpha^+$,

$card(S_\alpha) \leq card(\alpha)$ if $card(\alpha) < \alpha$.

(f) $\omega_1 = \max(S^0) = \sup(S^0 \cap \omega_1)$,

$\omega_{1+i+1} = \max(S_{\omega_{1+i}}) = \sup(S_{\omega_{1+i}} \cap \omega_{1+i+1})$ for all $i < \beta$.

Let $D = \langle D_\nu \mid \nu \in \hat{S} \rangle$ be a sequence such that $D_\nu \subseteq J_\nu^D$.

Let an $\langle S, \lhd, D \rangle$-maplet f be a triple $\langle \bar{\nu}, |f|, \nu \rangle$ such that $\bar{\nu}, \nu \in S - RCard$ and $|f| : J^D_{\mu_{\bar{\nu}}} \to J^D_{\mu_\nu}$.

Let $f\langle \bar{\nu}, |f|, \nu \rangle$ be an $\langle S, \lhd, D \rangle$-maplet. Then we define $d(f)$ and $r(f)$ by $d(f) = \bar{\nu}$ and $r(f) = \nu$. Set $f(x) := |f|(x)$ for $x \in J^D_{\mu_{\bar{\nu}}}$ and $f(\mu_{\bar{\nu}}) := \mu_\nu$. But $dom(f)$, $rng(f)$, $f \upharpoonright X$, etc. keep their usual set-theoretical meaning, i.e., $dom(f) = dom(|f|)$, $rng(f) = rng(|f|)$, $f \upharpoonright X$, etc.

For $\bar{\tau} \leq \mu_{\bar{\nu}}$, let $f^{(\bar{\tau})} = \langle \bar{\tau}, |f| \upharpoonright J^D_{\mu_{\bar{\tau}}} \rangle$, where $\tau = f(\bar{\tau})$. Of course, $f^{(\bar{\tau})}$ needs not to be a maplet. The same is true for the following definitions. Let $f^{-1} = \langle \nu, |f|^{-1} \bar{\nu} \rangle$. For $g = \langle \nu, |g|, \nu' \rangle$ and $f = \langle \bar{\nu}, |f|, \nu \rangle$, let $g \circ f = \langle \bar{\nu}, |g| \circ |f|, \nu' \rangle$. If $g = \langle \nu', |g|, \nu \rangle$ and $f = \langle \bar{\nu}, |f|, \nu \rangle$ such that $rng(f) \subseteq rng(g)$, then set $g^{-1} f = \langle \nu, |f|^{-1} |f|, \nu' \rangle$. Finally, set $id_\nu = \langle \nu, id \upharpoonright J^D_{\mu_\nu} \nu \rangle$.

Let $\mathfrak{F}$ be a set of $(S, \lhd, D)$-maplets $f = \langle \bar{\nu}, |f|, \nu \rangle$ such that the following holds:

(0) $f(\bar{\nu}) = \nu$, $f(\alpha_{\bar{\nu}}) = \alpha_\nu$ and $|f|$ is order-preserving.

(1) For $f \neq id_{\bar{\nu}}$, there is some $\beta \sqsubseteq \alpha_{\bar{\nu}}$ such that $f \upharpoonright \beta = id \upharpoonright \beta$ and $f(\beta) > \beta$.

(2) If $\bar{\tau} \in S^+$ and $\bar{\nu} \sqsubset \bar{\tau} \sqsubseteq \mu_{\bar{\nu}}$, then $f^{(\bar{\tau})} \in \mathfrak{F}$.

(3) If $f, g \in \mathfrak{F}$ and $d(g) = r(f)$, then $g \circ f \in \mathfrak{F}$.

(4) If f, $g \in \mathfrak{F}$, $r(g) = r(f)$ and $rng(f) \subseteq rng(g)$, then $g^{-1} \circ f \in \mathfrak{F}$.

We write $f : \bar{\nu} \Rightarrow \nu$ if $f = \langle \bar{\nu}, |f|, \nu \rangle \in \mathfrak{F}$. If $f \in \mathfrak{F}$ and $r(f) = \nu$, then we write $f \Rightarrow \nu$. The uniquely determined β in (1) shall be denoted by $\beta(f)$. Say $f \in \mathfrak{F}$ is minimal for a property $P(f)$ if $P(g)$ holds and $P(g)$ implies $g^{-1} f \in \mathfrak{F}$.

Let

$f_{(u, x, \nu)} =$ the unique minimal $f \in \mathfrak{F}$ for $f \Rightarrow \nu$ and $u \cup \{x\} \subseteq rng(f)$, if such an f exists. The axioms of the morass will guarantee that $f_{(u,x,\nu)}$ always exists if $\nu \in S - RCard^{L_\kappa[D]}$ Therefore, we will always assume and explicitly mention that $\nu \in S - RCard^{L_\kappa[D]}$ when $f_{(u,x,\nu)}$ is mentioned.

Say $\nu \in S - RCard^{L_\kappa[D]}$ is independent if $d(f_{(\beta,0,\nu)}) < \alpha_\nu$ holds for all $\beta < \alpha_\nu$.

For $\tau \sqsubseteq \nu \in S - RCard^{L_\kappa[D]}$, say ν is ξ-dependent on τ if $f_{(\alpha_\tau,\xi,\nu)} = id_\nu$.

For $f \in \mathfrak{F}$, let $\lambda(f) := \sup(f[d(f)])$.

For $\nu \in S - RCard^{L_\kappa[D]}$ let

$$C_\nu = \{\lambda(f) < \nu \mid f \Rightarrow \nu\},$$

$$\Lambda(x,\nu) = \{\lambda(f_{(\beta,x,\nu)}) < \nu \mid \beta < \nu\}.$$

It will be shown that C_ν and $\Lambda(x,\nu)$ are closed in ν.

Recursively define a function $q_\nu : k_\nu + 1 \to On$, where $k_\nu \in \omega$:

$$q_\nu(0) = 0,$$

$$q_\nu(k+1) = \max(\Lambda(q_\nu \restriction (k+1), \nu))$$

if $\max(\Lambda(q_\nu \restriction (k+1), \nu))$ exists. The axioms will guarantee that this recursion breaks off (see Lemma 4 of [6]), i.e. there is some k_ν such that either

$$\Lambda(q_\nu \restriction (k_\nu + 1), \nu) = \varnothing$$

or

$$\Lambda(q_\nu \restriction (k_\nu + 1), \nu) \text{ is unbounded in } \nu.$$

Define by recursion on $1 \le n \in \omega$, simultaneously for all $\nu \in S - RCard^{L_\kappa[D]}$, $\beta \in \nu$ and $x \in J^D_{\mu_\nu}$ the following notions:

$$f^1_{(\beta,x,\nu)} = f_{(\beta,x,\nu)},$$

$\tau(n, \nu) =$ the least $\tau \in S^0 \cup S^+ \cup \hat{S}$ such that for some $x \in J^D_{\mu_\nu}$,

$$f^1_{(\alpha_\tau, x, \nu)} = id_\nu,$$

$x(n, \nu) =$ the least $x \in J^D_{\mu_\nu}$ such that $f^n_{(\alpha_{\tau(n,\nu)}, x, \nu)} = id_\nu$,

$K^n_\nu = \{d(f^n_{(\beta, x(n,\nu), \nu)}) < \alpha_{\tau(n,\nu)} \mid \beta < \nu\}$,

$f \Rightarrow_n \nu$ iff $f \Rightarrow \nu$ and for all $1 \leq m < n$,

$$rng(f) \cap J^D_{\alpha_{\tau(m,\nu)}} \prec_1 \langle J^D_{\alpha_{\tau(m,\nu)}}, D \restriction \alpha_{\tau(m,\nu)}, K^m_\nu \rangle,$$

$$x(m, \nu) \in rng(f),$$

$f^n_{(u,\nu)} =$ the minimal $f \Rightarrow_n \nu$ such that $u \subseteq rng(f)$,

$f^n_{(\beta, x, \nu)} = f^n_{(\beta \cup \{x\}, \nu)}$,

$f : \bar{\nu} \Rightarrow_n \nu :\Leftrightarrow f \Rightarrow_n \nu$ and $f : \bar{\nu} \Rightarrow \nu$.

Here definitions are to be understood in Kleene's sense, i.e., that the left side is defined iff the right side is, and in that case, both are equal.

Let

$n_\nu =$ the least n such that $f^n_{(\gamma, x, \mu_\nu)}$ is confinal in ν for some $x \in J^D_{\mu_\nu}$, $\gamma \sqsubset \nu$,

$x_\nu =$ the least x such that $f^{n_\nu}_{(\alpha_\nu, x, \mu_\nu)} = id_{\mu_\nu}$.

Let

$\alpha^*_\nu = \alpha_\nu$ if $\nu \in S^+$,

$\alpha^*_\nu = \sup\{\alpha < \nu \mid \beta(f^{n_\nu}_{(\alpha, x_\nu, \mu_\nu)}) = \alpha\}$ if $\nu \notin S^+$.

Let $P_\nu := \{x_\tau \mid \nu \sqsubset \tau \sqsubseteq \mu_\nu, \tau \in S^+\} \cup \{x_\nu\}$.

We say that $\mathfrak{M} = \langle S, \triangleleft, \mathfrak{F}, D\rangle$ is an (ω_1, β)-*morass* if the following axioms hold:

(MP - minimum principle)

If $\nu \in S - RCard^{L_\kappa[D]}$ and $x \in J^D_{\mu_\nu}$, then $f_{(0, x, \nu)}$ exists.

(LP1 - first logical preservation axiom)

If $f : \bar{\nu} \Rightarrow \nu$, then $|f| : \langle J^D_{\mu_{\bar{\nu}}}, D\restriction \mu_{\bar{\nu}}\rangle \to \langle J^D_{\mu_\nu}, D\restriction \mu_\nu\rangle$ is Σ_1-elementary.

(LP2 - second logical preservation axiom)

Let $f : \bar{\nu} \Rightarrow \nu$ and $f(\bar{x}) = x$. Then

$$(f\restriction J^D_{\bar{\nu}}) : \langle J^D_{\bar{\nu}}, D\restriction \bar{\nu}, \Lambda(\bar{x}, \bar{\nu})\rangle \to \langle J^D_\nu, D\restriction \nu, \Lambda(x, \nu)\rangle$$

is Σ_0-elementary.

(CP1 - first continuity principle)

For $i \leq j < \lambda$, let $f_i : \nu_i \Rightarrow \nu$ and $g_{ij} : \nu_i \Rightarrow \nu_j$ such that $g_{ij} = f_j^{-1} f_i$. Let $\langle g_i \mid i < \lambda\rangle$ be the transitive, direct limit of the directed system $\langle g_{ij} \mid i \leq j < \lambda\rangle$ and $h_{g_i} = f_i$ for all $i < \lambda$. Then $g_i, h \in \mathfrak{F}$.

(CP2 - second continuity principle)

Let $f : \bar{\nu} \Rightarrow \nu$ and $\lambda = \sup(f[\bar{\nu}])$. If, for some $\bar{\lambda}$, $h : \langle J^{\bar{D}}_{\bar{\lambda}}, \bar{D}\rangle \to \langle J^D_\lambda, \bar{D}\restriction \lambda\rangle$ is Σ_1-elementary and $rng(f\restriction J^D_{\bar{\nu}}) \subseteq rng(h)$, then there is some $g : \bar{\lambda} \Rightarrow \lambda$ such that $g\restriction J^{\bar{D}}_{\bar{\lambda}} = h$.

(CP3 - third continuity principle)

If $C_\nu = \{\lambda(f) < \nu \mid f \Rightarrow \nu\}$ is unbounded in $\nu \in S - RCard^{L_\kappa[D]}$, then the following holds for all $x \in J^D_{\mu_\nu}$:

$rng(f_{(0, x, \nu)}) = \bigcup\{rng(f_{(0, x, \lambda)}) \mid \lambda \in C_\nu\}$.

(DP1 - first dependency axiom)

If $\mu_\nu < \mu_{\alpha_\nu}$, then $\nu \in S - RCard^{L_\kappa[D]}$ is independent.

(DP2 - second dependency axiom)

If $\nu \in S - RCard^{L_\kappa[D]}$ is η-dependent on $\tau \sqsubseteq \nu$, $\tau \in S^+$, $f : \overline{\nu} \Rightarrow \nu$, $f(\overline{\tau}) = \tau$ and $\eta \in rng(f)$, then $f^{(\overline{\tau})} : \overline{\tau} \Rightarrow \tau$.

(DP3 - third dependency axiom)

For $\nu \in \hat{S} - RCard^{L_\kappa[D]}$ and $1 \leq n \in \omega$, the following holds:

(a) If $f^n_{(\alpha_\tau, x, \nu)} = id_\nu$, $\tau \in S^+ \cup S^0$ and $\tau \sqsubseteq \nu$, then $\mu_\nu = \mu_\tau$.

(b) If $\beta < \alpha_{\tau(n, \nu)}$, then also $d(f^n_{(\beta, x(n, \nu), \nu)}) < \alpha_{\tau(n, \nu)}$.

(DF - definability axiom)

(a) If $f_{(0, z_0, \nu)} = id_\nu$ for some $\nu \in \hat{S} - RCard^{L_\kappa[D]}$ and $z_0 \in J^D_{\mu_\nu}$, then

$$\{\langle z, x, f_{(0, z, \nu)}(x)\rangle \mid z \in J^D_{\mu_\nu}, x \in dom(f_{(0, z, \nu)})\}$$

is uniformly definable over $\langle J^D_{\mu_\nu}, D \upharpoonright \mu_\nu, D_{\mu_\nu}\rangle$.

(b) For all $\nu \in S - RCard^{L_\kappa[D]}$, $x \in J^D_{\mu_\nu}$, the following holds:

$$f_{(0, x, \nu)} = f^{n_\nu}_{(0, \langle x, \nu, \alpha^*_\nu, P_\nu\rangle, \mu_\nu)}.$$

This finishes the definition of an (ω_1, β)-morass.

A consequence of the axioms is $(\times)$ by [6]:

Theorem.

$$\{\langle z, \tau, x, f_{(0, z, \tau)}(x)\rangle \mid \tau < \nu, \mu_\tau = \nu, z \in J^D_{\mu_\tau}, x \in dom(f_{(0, z, \tau)})\}$$

$$\cup \{\langle z, x, f_{(0, z, \nu)}(x)\rangle \mid \mu_\nu = \nu, z \in J^D_{\mu_\nu}, x \in dom(f_{(0, z, \nu)})\}$$

$$\cup (\sqsubset \cap \nu^2)$$

is for all $\nu \in S$ *uniformly definable over* $\langle J^D_\nu, D \upharpoonright \nu, D_\nu\rangle$.

A structure $\mathfrak{M} = \langle S, \triangleleft, \mathfrak{F}, D\rangle$ is called an $\omega_{1+\beta}$-*standard morass* if it satisfies all axioms of an (ω_1, β)-morass except (DF) which is replaced by:

$$\nu \triangleleft \tau \Rightarrow \nu \text{ is regular in } J^D_\tau$$

and there are functions $\sigma_{(x,\nu)}$ for $\nu \in \hat{S}$ and $x \in J^D_\nu$ such that:

(MP)$^+$

$$\sigma_{(x,\nu)}[\omega] = rng(f_{(0,x,\nu)})$$

(CP1)$^+$

If $f : \bar{\nu} \Rightarrow \nu$ and $f(\bar{x}) = x$, then $\sigma_{(x,\nu)} = f \circ \sigma_{(\bar{x},\bar{\nu})}$.

(CP3)$^+$

If C_ν is unbounded in ν, then $\sigma_{(x,\nu)} = \bigcup\{\sigma_{(x,\lambda)} \mid \lambda \in C_\nu,\ x \in J^D_\lambda\}$.

(DF)$^+$

(a) If $f_{(0,x,\nu)} = id_\nu$ for some $x \in J^D_\nu$, then

$$\{\langle i, z, \sigma_{(z,\nu)}(i)\rangle \mid z \in J^D_\nu,\ i \in dom(\sigma_{(z,\nu)})\}$$

is uniformly definable over $\langle J^D_{\mu_\nu}, D \upharpoonright \mu_\nu, D_{\mu_\nu}\rangle$.

(b) If C_ν is unbounded in ν, then $D_\nu = C_\nu$. If it is bounded, then $D_\nu = \{\langle i, \sigma_{(q_\nu,\nu)}(i)\rangle \mid i \in dom(\sigma_{(q_\nu,\nu)})\}$.

Now, I am going to construct a κ-standard morass.

Let $\beta(\nu)$ be the least β such that $J^X_{\beta+1} \models \nu$ singular.

Let $L_\kappa[X]$ satisfy amenability, condensation and coherence such that $S^X = \{\beta(\nu) \mid \nu \text{ singular in } L_\kappa[X]\}$ and $Card^{L_\kappa[X]} = Card \cap \kappa$.

Let

$$\nu \triangleleft \tau :\Leftrightarrow \nu \text{ regular in } I_\tau.$$

Let

$$E = Lim - RCard^{L_\kappa[X]}.$$

For $\nu \in E$, let

$\beta(\nu) =$ the least β such that there is a cofinal $f : a \to \nu \in Def(I_\beta)$ and $a \subseteq \nu' < \nu$,

$n(\nu) =$ the least $n \geq 1$ such that such an f is Σ_n-definable over $I_{\beta(\nu)}$,

$\rho(\nu) =$ the $(n(\nu) - 1)$ th projectum of $I_{\beta(\nu)}$,

$A_\nu =$ the $(n(\nu) - 1)$ th standard code of $I_{\beta(\nu)}$,

$\gamma(\nu) =$ the $n(\nu)$ th projectum of $I_{\beta(\nu)}$.

If $\nu \in S^+ - Card$, then the $n(\nu)$ th projectum of $\beta(\nu)$ is less or equal $\alpha_\nu :=$ the largest cardinal in I_ν: Since α_ν is the largest cardinal in I_ν, there is, by definition of $\beta(\nu)$ and $n(\nu)$, some over $I_{\beta(\nu)}$ $\Sigma_{n(\nu)}$-definable function f such that $f[\alpha_\nu]$ is cofinal in ν. But, since ν is regular in $\beta(\nu)$, f cannot be an element of $J^X_{\beta(\nu)}$. So $\mathfrak{P}(\nu \times \nu) \cap \Sigma_{n(\nu)}(I_{\beta(\nu)}) \not\subseteq J^X_{\beta(\nu)}$. By Lemma 14, also $\mathfrak{P}(\nu) \cap \Sigma_{n(\nu)}(I_{\beta(\nu)}) \not\subseteq J^X_{\beta(\nu)}$. Using Lemma 21(3), we get $\gamma \leq \nu$, i.e., there is an over $I_{\beta(\nu)}$ $\Sigma_{n(\nu)}$-definable function g such that $g[\nu] = J^X_{\beta(\nu)}$. On the other hand, there is, for every $\tau < \nu$ in J^X_ν, a surjection from α_ν onto τ, because α_ν is the largest cardinal in I_ν. Let f_τ be the $<_\nu$-least such. Define $j_1(\sigma, \tau) = f_{f(\tau)}(\sigma)$ for $\sigma, \tau < \nu$. Then j_1 is $\Sigma_{n(\nu)}$-definable over $I_{\beta(\nu)}$ and $j_1[\alpha_\nu \times \alpha_\nu] = \nu$. By Lemma 15, we obtain an over $I_{\beta(\nu)}$ $\Sigma_{n(\nu)}$-definable function j_2 from a subset of α_ν onto ν. Thus $g \circ j_2$ is an over $I_{\beta(\nu)}$ $\Sigma_{n(\nu)}$-definable map such that $g \circ j_2[\alpha_\nu] = J^X_{\beta(\nu)}$.

Moreover, $\alpha_\nu < \nu \leq \rho(\nu)$: By definition of $\rho(\nu)$, there is an over $I_{\beta(\nu)}$ $\Sigma_{n(\nu)-1^-}$-definable function f such that $f[\rho(\nu)] = \beta(\nu)$ if $n(\nu) > 1$. But ν is $\Sigma_{n(\nu)-1^-}$-regular over $I_{\beta(\nu)}$. Thus $\nu \leq \rho(\nu)$. If $n(\nu) = 1$, then $\rho(\nu) = \beta(\nu) \geq \nu$.

By the first inequality, there is a q such that every $x \in J^X_{\rho(\nu)}$ is Σ_1-definable in $\langle I^0_{\rho(\nu)}, A_\nu \rangle$ with parameters from $\alpha_\nu \cup \{q\}$. Let p_ν be the $<_{\rho(\nu)}$-least such.

Obviously, $p_\tau \leq p_\nu$ if $\nu \sqsubseteq \tau \sqsubseteq \mu_\nu$.

Thus $P_\nu := \{p_\tau \mid \nu \sqsubseteq \tau \sqsubseteq \mu_\nu,\ \tau \in S^+\}$ is finite.

Now, let $\nu \in E - S^+$. By definition of $\beta(\nu)$, there exists no cofinal $f : a \to \nu$ in J^X_β such that a $a \subseteq \nu' < \nu$. So $\mathfrak{P}(\nu \times \nu) \cap \Sigma_{n(\nu)}(I_{\beta(\nu)}) \not\subseteq J^X_{\beta(\nu)}$. Then, by Lemma 14, $\mathfrak{P}(\nu) \cap \Sigma_{n(\nu)}(I_{\beta(\nu)}) \not\subseteq J^X_{\beta(\nu)}$. Hence, by Lemma 21(3),

$$\gamma(\nu) \leq \nu.$$

Assume $\rho(\nu) < \nu$. Then there was an over $I_{\beta(\nu)}$ $\Sigma_{n(\nu)-1}$-definable f such that $f[\rho(\nu)] = \nu$. But this contradicts the definition of $n(\nu)$. So

$$\nu \leq \rho(\nu).$$

Using Lemma 21(1), it follows from the first inequality that there is some over $I_{\beta(\nu)}$ $\Sigma_{n(\nu)}$-definable function f such that $f[J^X_\nu] = J^X_{\beta(\nu)}$. So there is a $p \in J^X_{\rho(\nu)}$ such that every $x \in J^X_{\rho(\nu)}$ is Σ_1-definable in $\langle I^0_{\rho(\nu)}, A_\nu \rangle$ with parameters from $\nu \cup \{p\}$. Let p_ν be the least such.

Let

$$\alpha^*_\nu = \sup\{\alpha < \nu \mid h_{\rho(\nu), A_\nu}[\omega \times (J^X_\alpha \times \{p_\nu\})] \cap \nu = \alpha\}.$$

Then $\alpha^*_\nu < \nu$ because, by definition of $\beta(\nu)$, there exists a $\nu' < \nu$ and a $p \in J^X_{\rho(\nu)}$ such that $h_{\rho(\nu), A_\nu}[\omega \times (J^X_{\nu'} \times \{p_\nu\})]$. is cofinal in ν. But p is in $h_{\rho(\nu), A_\nu}[\omega \times (J^X_\nu \times \{p_\nu\})]$. So there is an $\alpha < \nu$ such that $h_{\rho(\nu), A_\nu}[\omega \times (J^X_\alpha \times \{p_\nu\})]$ $\cap\, \nu$ is cofinal in ν. Thus $\alpha^*_\nu < \alpha < \nu$.

If $\nu \in S^+$, then we set $\alpha^*_\nu := \alpha_\nu$.

For $\nu \in E$, let $f : \overline{\nu} \Rightarrow \nu$ iff, for some f^*,

(1) $f = \langle \overline{\nu}, f^* \upharpoonright J^D_{\mu_{\overline{\nu}}}, \nu \rangle$,

(2) $f^* : I_{\mu_{\overline{\nu}}} \to I_{\mu_\nu}$ is $\Sigma_{n(\nu)}$-elementary,

(3) $\alpha^*_\nu, p_\nu, \alpha^*_{\mu_\nu}, P_\nu \in rng(f^*)$,

(4) $\nu \in rng(f^*)$ if $\nu < \mu_\nu$,

(5) $f(\overline{\nu}) = \nu$ and $\overline{\nu} \in S^+ \Leftrightarrow \nu \in S^+$.

By this, $\mathfrak{F}$ is defined.

Set $D = X$.

Let P^*_ν be minimal such that $h^{n(\nu)-1}_{\mu_\nu}(i, P^*_\nu) = P_\nu$ for an $i \in \omega$.

Let $\alpha^{**}_{\mu_\nu}$ be minimal such that $h^{n(\nu)-1}_{\mu_\nu}(i, \alpha^{**}_{\mu_\nu}) = \alpha^*_{\mu_\nu}$ for some $i \in \omega$.

Set

$$\nu^* = \varnothing \text{ if } \nu = \rho(\nu),$$

$$\nu^* = \nu \text{ if } \nu < \rho(\nu).$$

For $\tau \in On$, let S_τ be defined as in Lemma 10. For $\tau \in On$, $E_i \subseteq S_\tau$ and a Σ_0 formula φ, let

$h^\varphi_{\tau, E_i}(x_1, ..., x_m)$ the least $x_0 \in S_\tau$ w.r.t. the canonical well-ordering such that $\langle S_\tau, E_i \rangle \models \varphi(x_i)$ if such an element exists,

and

$$h^\varphi_{\tau, E_i}(x_1, ..., x_m) = \varnothing \text{ else.}$$

For $\tau \in On$ such that $\nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_{\mu_\nu}, P^*_\nu \in S_\tau$, let $H_\nu(\alpha, \tau)$ be the closure of

$S_\alpha \cup \{\nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_{\mu_\nu}, P^*_\nu\}$ under all $h^\varphi_{\tau, X \cap S_\tau, A_\nu \cap S_\tau}$. Then

$$H_\nu(\alpha, \tau) \prec_1 \langle S_\tau, X \cap S_\tau, A_\nu \cap S_\tau, \{\nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_{\mu_\nu}, P^*_\nu\}\rangle$$

by the definition of $h^\varphi_{\tau, X \cap S_\tau, A_\nu \cap S_\tau}$. Let $M_\nu(\alpha, \tau)$ be the collapse of $H_\nu(\alpha, \tau)$. Let τ_0 be the minimal τ such that $\nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_{\mu_\nu}, P^*_\nu \in S_\tau$. Define by induction for $\tau_0 \leq \tau < \rho(\nu)$:

$$\alpha(\tau_0) = \alpha_\nu,$$

$$\alpha(\tau + 1) = \sup(M_\nu(\alpha(\tau), \tau + 1) \cap \nu),$$

$$\alpha(\lambda) = \sup\{\alpha(\tau) \mid \tau < \lambda\} \text{ if } \lambda \in Lim.$$

Set

$$B_\nu = \{\langle \alpha(\tau), M_\nu(\alpha(\tau), \tau)\rangle \mid \tau_0 < \tau \in \rho(\nu)\} \text{ if } \nu < \rho(\nu),$$

$$B_\nu = \{0\} \times A_\nu \cup \{\langle 1, \nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_{\mu_\nu}, P^*_\nu\rangle\} \text{ else.}$$

Lemma 22. *$B_\nu \subseteq J^X_\nu$ and $\langle I^0_\nu, B_\nu\rangle$ is rudimentary closed.*

Proof. If $\nu = \rho(\nu)$, then both claims are clear. Otherwise, we first prove $M^\nu(\alpha, \tau) \in J^X_\nu$ for all $\alpha < \nu$ and all $\tau \in \rho(\nu)$ such that $\tau_0 \leq \tau < \rho(\nu)$. Let such a τ be given and $\tau' \in \rho(\nu) - Lim$ be such that $X \cap S_\tau, A_\nu \cap S_\tau \in S_{\tau'}$ (rudimentary closedness of $\langle I^0_{\rho(\nu)}, A_\nu\rangle$). Let $\eta := \sup(\tau' \cap Lim)$. Let H be the closure of

$$\alpha \cup \{\nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_{\mu_\nu}, P^*_\nu, X \cap S_\tau, S_\tau, A_\nu \cap S_\tau, \eta\}$$

under all $h^\varphi_{\tau'}$. Let $\sigma : H \cong S$ be the collapse of H and $\sigma(\eta) = \overline{\eta}$. If $\eta \in S^X$, then $S = S_{\overline{\tau}'}$ for some $\overline{\tau}'$ by the condensation property of $L[X]$. If $\eta \notin S^X$, then $S = S^{X \upharpoonright \overline{\eta}}_{\overline{\tau}'}$ for some $\overline{\tau}'$, where $S^{X \upharpoonright \overline{\eta}}_{\overline{\tau}'}$ is defined like $S_{\overline{\tau}'}$ with $X \upharpoonright \overline{\eta}$ instead of X. The reason is that, even if $\eta \notin S^X$, it is the supremum of points in S^X, because $S^X = \{\beta(\nu) \mid \nu \text{ singular in } L_\kappa[X]\}$. In both cases, $S \in J^X_{\rho_\nu}$ and there is a function in $I_{\overline{\eta}+\omega}$ that maps

$$\alpha \cup \{\sigma(\nu^*), \sigma(\alpha_\nu^*), \sigma(p_\nu), \sigma(\alpha_{\mu_\nu}^{**}), \sigma(P_\nu^*), \sigma(X \cap S_\tau), \sigma(S_\tau), \sigma(A_\nu \cap S_\tau), \sigma(\eta)\}$$

onto S. So ν would be singular in $J_{\rho_\nu}^X$ if $\nu \leq \overline{\tau}'$. But this contradicts the definition of $\beta(\nu)$. Therefore,

$$\sigma(\nu^*), \sigma(\alpha_\nu^*), \sigma(p_\nu), \sigma(\alpha_{\mu_\nu}^{**}), \sigma(P_\nu^*),$$

$$\sigma(X \cap S_\tau), \sigma(S_\tau), \sigma(A_\nu \cap S_\tau), \sigma(\eta) \in J_\nu^X.$$

Let $\overline{H}_\nu(\alpha, \tau)$ be the closure of

$$S_\alpha \cup \{\sigma(\nu^*), \sigma(\alpha_\nu^*), \sigma(p_\nu), \sigma(\alpha_{\mu_\nu}^{**}), \sigma(P_\nu^*),$$

$$\sigma(X \cap S_\tau), \sigma(S_\tau), \sigma(A_\nu \cap S_\tau), \sigma(\eta)\}$$

under all $h^\varphi_{\sigma(S_\tau), \sigma(X \cap S_\tau), \sigma(A_\nu \cap S_\tau)}$, where these are defined like h^φ_{τ, E_i} but with $\sigma(S_\tau)$ instead of S_τ. Then

$$\overline{H}_\nu(\alpha, \tau) \prec_1 \langle \sigma(S_\tau), \sigma(X \cap S_\tau), \sigma(A_\nu \cap S_\tau), \{\sigma(\nu^*),$$

$$\sigma(\alpha_\nu^*), \sigma(p_\nu), \sigma(\alpha_{\mu_\nu}^{**}), \sigma(P_\nu^*), \sigma(X \cap S_\tau), \sigma(S_\tau), \sigma(A_\nu \cap S_\tau), \sigma(\eta)\}$$

and $M_\nu(\alpha, \tau)$ is the collapse of $\overline{H}_\nu(\alpha, \tau)$. Since $\nu < \rho(\nu)$ and ν is a cardinal in $I_{\beta(\nu)}$, $J_\nu^X \models ZF^-$. So we can form the collapse inside J_ν^X. Thus $M_\nu(\alpha, \tau) \in J_\nu^X$.

Now, we turn to rudimentary closedness. Since B_ν is unbounded in ν, it suffices to prove that the initial segments of B_ν are elements of J_ν^X. Such an initial segment is of the form $\langle M_\nu(\alpha(\tau), \tau) | \tau < \gamma \rangle$, where $\gamma < \rho(\nu)$, and we have $H_\nu(\alpha(\tau), \delta_\tau) = H_\nu(\alpha(\tau), \tau)$, where δ_τ is for $\tau < \gamma$ the least $\eta \geq \tau$ such that $\eta \in H_\nu(\alpha(\tau), \gamma) \cup \{\gamma\}$. Since $\delta_\tau \in H_\nu(\alpha(\tau), \gamma) \prec_1 \langle S_\gamma, X \cap S_\gamma, A_\nu \cap S_\gamma, \{\cdots\} \rangle$, $(H_\nu(\alpha(\tau), \delta_\tau))^{H_\nu(\alpha(\gamma), \gamma)} = H_\nu(\alpha(\tau), \tau)$. Let $\pi : M_\nu(\alpha(\gamma), \gamma) \to S_\gamma$ be the uncollapse of $H_\nu(\alpha(\gamma), \gamma)$. Then, by the Σ_1-elementarity of π, $M_\nu(\alpha(\tau), \tau) = M_\nu(\alpha(\tau), \delta_\tau)$

is the collapse of $(H(\alpha(\tau), \pi^{-1}(\delta_\tau)))^{M_\nu(\alpha(\gamma),\gamma)}$. So $\langle M_\nu(\alpha(\tau), \tau) | \tau < \gamma\rangle$ is definable from $M_\nu(\alpha(\gamma), \gamma) \in J_\nu^X$. □

Lemma 23. *For* $x, y_i \in J_\nu^X$, *the following are equivalent*:

(i) x *is* Σ_1*-definable in* $\langle I^0_{\rho(\nu)}, A_\nu\rangle$ *with the parameters* y_i, ν^*, α^*_ν, p_ν, $\alpha^{**}_{\mu_\nu}$, P^*_ν.

(ii) x *is* Σ_1*-definable in* $\langle I^0_\nu, B_\nu\rangle$ *with the parameters* y_i.

Proof. For $\nu = \rho(\nu)$, this is clear. Otherwise, let first x be uniquely determined in $\langle I^0_{\rho(\nu)}, A_\nu\rangle$ by $(\exists z)\psi(z, x, \langle y_i, \nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_{\mu_\nu}, P^*_\nu\rangle)$, where is a Σ_0 formula. That is equivalent to $(\exists\tau)(\exists z \in S_\tau)\psi(z, x, \langle y_i, \nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_{\mu_\nu}, P^*_\nu\rangle)$ and that again to $(\exists\tau)H_\nu(\alpha(\tau), \tau) \models (\exists z)\psi(z, x, \langle y_i, \nu^*, \alpha^*_\nu, p_\nu, \alpha^{**}_{\mu_\nu}, P^*_\nu\rangle)$. If τ is large enough, the y_i are not moved by the collapsing map, since then $y_i \in J^X_{\alpha(\tau)} \subseteq H_\nu(\alpha(\tau), \tau)$. Let $\bar{\nu}$, α, p, α', P be the images of ν^*, α^*_ν, p_ν, $\alpha^{**}_{\mu_\nu}$, P^*_ν under the collapse. Then

$$(\exists\tau)(y_i \in J^X_{\alpha(\tau)} \textit{ and } M_\nu(\alpha(\tau), \tau) \models (\exists z)\psi(z, x, \langle y_i, \bar{\nu}, \alpha, p, \alpha', P\rangle))$$

defines x. So it is definable in $\langle I^0_\nu, B_\nu\rangle$.

Since B_ν and the satisfaction relation of $\langle I^0_\gamma, B\rangle$ are Σ_1-definable over $\langle I^0_{\rho(\nu)}, A_\nu\rangle$, the converse is clear. □

Lemma 24. *Let* $H \prec_1 \langle I^0_\nu, B_\nu\rangle$ *for a* $\nu \in E$ *and* $\pi : \langle I^0_\mu, B\rangle \to \langle I^0_\nu, B_\nu\rangle$ *be the uncollapse of H. Then* $\mu \in E$ *and* $B = B_\mu$.

Proof. First, we extend π like in Lemma 19. Let

$$M = \{x \in J^X_{\rho(\nu)} \,|\, x \text{ is } \Sigma_1\text{-definable in } \langle I^0_{\rho(\nu)}, A_\nu\rangle \text{ with parameters from}$$
$$rng(\pi) \cup \{p_\nu, \nu^*, \alpha^*_\nu, \alpha^{**}_{\mu_\nu}, P^*_\nu\}\}.$$

Then $rng(\pi) = M \cap J^X_\nu$. For, if $x \in M \cap J^X_\nu$, then there are by definition of M $y_i \in rgn(\pi)$ such that x is Σ_1-definable in $\langle I^0_{\rho(\nu)}, A_\nu \rangle$ with the parameters y_i and p_ν, ν^*, $\alpha^{**}_{\mu_\nu}$, P^*_ν. Thus it is Σ_1-definable in $\langle I^0_\nu, B_\nu \rangle$ with the y_i by Lemma 23. Therefore, $x \in rng(\pi)$ because $y_i \in rng(\pi) \prec_1 \langle I^0_\nu, B_\nu \rangle$. Let $\hat{\pi} : \langle I^0_\rho, A \rangle \to \langle I^0_{\rho(\nu)}, A_\nu \rangle$ be the uncollapse of M. Then $\hat{\pi}$ is an extension of π, since $M \cap J^X_\nu$ is an $\in$-initial segment of M and $rng(\pi) = M \cap J^X_\nu$. In addition, there is by Lemma 19, a $\Sigma_{n(\nu)}$-elementary extension $\tilde{\pi} : I_\beta \to I_{\beta(\nu)}$ such that ρ is the $(n(\nu) - 1)$ th projectum of I_β and A is the $(n(\nu) - 1)$ th standard code of it. Let $\tilde{\pi}(p) = p_\nu$ and $\tilde{\pi}(\alpha) = \alpha^*_\nu$. And we have $\tilde{\pi}(\mu) = \nu$ if $\nu < \beta(\nu)$. In this case, $\nu \in rng(\pi)$ by the definition of ν^*. Since $\tilde{\pi}$ is Σ_1-elementary, cardinals of J^X_μ are mapped on cardinals of J^X_ν.

Assume $\nu \in S^+$. Suppose there was a cardinal $\tau > \alpha$ of J^X_μ. Then $\pi(\tau) > \alpha_\tau$ was a cardinal in J^X_ν. But this is a contradiction.

Next, we note that μ is $\Sigma_{n(\nu)}$-singular over I_β. If $\nu \in S^+$, then, by the definition of p_ν, $J^X_\rho = h_{\rho, A}[\omega \times (\alpha \times \{p\})]$ is clear. So there is an over $\langle I^0_\nu, A \rangle$ Σ_1-definable function from α cofinal into μ. But since ρ is the $(n(\nu) - 1)$ th projectum and A is the $(n(\nu) - 1)$ th code of it, this function is Σ_n-definable over I_β. Now, suppose $\nu \notin S^+$. Let $\lambda := \sup(\pi[\mu])$. Since $\lambda > \alpha^*_\nu$, there is a $\gamma < \lambda$ such that

$$\sup(h_{\rho(\nu), A_\nu}[\omega \times (J^X_\gamma \times \{q_\nu\})] \cap \nu) \geq \lambda.$$

And since $rng(\pi)$ is cofinal in λ, there is such a $\gamma \in rng(\pi)$. Let $\gamma = \pi(\bar{\gamma})$. By the Σ_1-elementarity of $\tilde{\pi}$, $\bar{\gamma} < \mu$ and setting $\tilde{\pi}(q) = q_\nu$ we have for every $\eta < \mu$,

$$\langle I_\rho, A \rangle \models (\exists x \in J^X_{\bar{\gamma}})(\exists i) h_{\rho, A}(i, \langle x, p \rangle) > \eta.$$

Hence $h_{\rho, A}[\omega \times (J^X_{\bar{\gamma}} \times \{q\})]$ is cofinal in μ. This shows $\mu \in E$.

On the other hand, μ is $\Sigma_{n(\nu)-1}$-regular over I_β if $n(\nu) > 1$. Assume there was an over I_β $\Sigma_{n(\nu)-1}$-definable function f and some $x \in \mu$ such that $f[x]$ was cofinal in μ, i.e., $(\forall y \in \mu)(\exists z \in x)(f(x) > y)$ would hold in I_β. Over I_β, $(\exists z \in x)(f(z) > y)$ is $\Sigma_{n(\nu)-1}$. So it is Σ_0 over $\langle I^0_\rho, A\rangle$. But then also $(\forall y \in \mu)(\exists z \in x)(f(z) > y)$ is Σ_0 over $\langle I^0_\rho, A\rangle$ if $\mu < \rho$. Hence it is $\Sigma_{n(\nu)}$ over I_β. But then the same would hold for $\tilde{\pi}(x)$ in $I_{\beta(\nu)}$. This contradicts the definition of $n(\nu)$! Now, let $\mu = \rho$. Since λ is the largest cardinal in I_μ, we had in f also an over I_β $\Sigma_{n(\nu)-1}$-definable function from α onto ρ and therefore one from α onto β. But this contradicts Lemma 21 and the fact that ρ is the $(n(\nu)-1)$ th projectum of β. If $n(\nu) = 1$, then we get with the same argument that μ is regular in I_β.

The previous two paragraphs show $\beta = \beta(\mu)$ and $n(\mu) = n(\nu)$. We are done if we can also show that $\alpha = \alpha^*_\mu$, $\pi(\alpha^{**}_{\mu_\mu}) = \alpha^{**}_{\mu_\nu}$, $p = p_\mu$, $\pi(P^*_\mu) = P^*_\nu$, because $\tilde{\pi}$ is Σ_1-elementary, $\tilde{\pi}(h^\varphi_{\tau, X \cap S_\tau, A_\mu, A_\mu \cap S_\tau}(x_i)) = h^\varphi_{\tilde{\pi}(\tau), X \cap S_{\tilde{\pi}(\tau)}, A_\nu \cap S_{\tilde{\pi}(\tau)}}(x_i)$ for all Σ_1 formulas φ and $x_i \in S_\tau$.

For $\nu \in S^+$, $\alpha = \alpha_\mu$ was shown above. So let $\nu \notin S^+$. By the Σ_1-elementarity of $\tilde{\pi}$, we have for all $\alpha \in \mu$,

$$h_{\rho, A}[\omega \times (J^X_\alpha \times \{p\})] \cap \mu = \alpha \Leftrightarrow h_{\rho(\nu), A_\nu}[\omega \times (J^X_{\pi(\alpha)} \times \{p_\nu\})] \cap \nu = \pi(\alpha).$$

The same argument proves $\pi(\alpha^{**}_{\mu_\mu}) = \alpha^{**}_{\mu_\nu}$. Finally, $p = p_\mu$ and $\pi(P^*_\mu) = P^*_\nu$ can be shown as in (5) in the proof of Lemma 19. □

Lemma 25. *Let* $H \prec_1 \langle I^0_\nu, B_\nu\rangle$ *and* $\lambda = \sup(H \cap \nu)$ *for a* $\nu \in E$. *Then* $\lambda \in E$ *and* $B_\nu \cap J^X_\lambda = B_\lambda$.

Proof. Let $\pi_0 : \langle I^0_\mu, B_\mu\rangle \to \langle I^0_\lambda, B_\nu \cap J^X_\lambda\rangle$ be the uncollapse of H and let $\pi_1 : \langle I^0_\lambda, B_\nu \cap J^X_\lambda\rangle \to \langle I^0_\nu, B_\nu\rangle$ be the identity. Since $L[X]$ has coherence, π_0 and π_1 are Σ_0-elementary. By Lemma 18, π_0 is even Σ_1-elementary, because it is cofinal. To show $B_\lambda = B_\nu \cap J_\lambda^X$, we extend π_0 and π_1 to $\hat{\pi}_0 : \langle I^0_{\rho(\mu)}, A_\mu\rangle \to \langle I^0_\rho, A\rangle$ and $\hat{\pi}_1 : \langle I^0_\rho, A\rangle \to \langle I^0_{\rho(\mu)}, A_\nu\rangle$ in such a way that $\hat{\pi}_0$ is Σ_1-elementary and $\hat{\pi}_1$ is Σ_0-elementary. Then we know from Lemma 19 that ρ is the $(n(\nu)-1)$ th projectum of some β and A is the $(n(\nu)-1)$ th code of it. So there is a $\Sigma_{n(\nu)}$-elementary extension of $\tilde{\pi}_0 : I_{\overline{\beta}} \to I_\beta$. We can again use the argument from Lemma 24 to show that λ is $\Sigma_{n(\nu)-1}$-regular over I_β. But on the other hand, λ is as supremum of $H \cap On\, \Sigma_{n(\nu)}$-singular over I_β. From this, we conclude as in the proof of Lemma 24 that $B_\lambda = B_\nu \cap J_\lambda^X$.

First, suppose $\nu \in S^+$. Since $\alpha_\nu \in H \prec_1 \langle I^0_\nu, B_\nu\rangle$, $\alpha_\nu < \lambda \leq \nu$. Since $I_\nu \vDash (\alpha_\nu$ is the largest cardinal), we therefore have $\lambda \notin Card$. In addition, α_ν is the largest cardinal in I_λ. Assume τ was the next larger cardinal. Then τ was Σ_1-definable in I_λ with parameter α_ν and some $\tau' \in H$ and hence it was in H. By the Σ_1-elementarity of π_0, $\pi_0^{-1}(\tau) > \pi_0^{-1}(\alpha_\nu) = \alpha_\mu$ was also a cardinal in I_μ. But this contradicts the definition of α_μ.

But now to $B_\lambda = B_\nu \cap J_\lambda^X$. First, assume $\nu \notin S^+$. Let $\pi = \pi_1 \circ \pi_0 : \langle I^0_\mu, B_\mu\rangle \to \langle I^0_\nu, B_\nu\rangle$ and $\hat{\pi} : \langle I^0_{\rho(\mu)}, A_\mu\rangle \to \langle I^0_{\rho(\nu)}, A_\nu\rangle$ be the extension constructed in the proof of Lemma 24. Let $\gamma = \sup(rng(\hat{\pi}))$. Then $\hat{\pi}' = \hat{\pi} \cap (J^X_{\rho(\mu)} \times J^X_\gamma) : \langle I^0_{\rho(\mu)}, A_\mu\rangle \to \langle I^0_\gamma, A_\nu \cap J^X_\gamma\rangle$ is Σ_0-elementary, by coherence of $L_\kappa[X]$, and cofinal. Thus $\hat{\pi}'$ is Σ_1-elementary. Let $H' = h_{\gamma, A_\nu \cap J^X_\gamma}[\omega \times (J^X_\lambda \times \{p_\nu\})]$ and $\hat{\pi}_1 : \langle I^0_\rho, A\rangle \to \langle I^0_{\rho(\nu)}, A_\nu\rangle$ be the uncollapse of H'. Then $H = rng(\hat{\pi}') \subseteq H'$. To see this, let $z \in rng(\hat{\pi}')$ and $z = \hat{\pi}'(y)$. Then, by definition of p_μ, there is an $x \in J^X_\mu$ and an $i \in \omega$ such that $y = h_{\rho(\mu), A_\mu}(i, \langle x, p_\mu\rangle)$. By the Σ_1-elementarity of $\hat{\pi}'$, we therefore have $z = h_{\gamma, A_\nu \cap J^X_\gamma}(i, \langle \hat{\pi}'(x), \hat{\pi}'(p_\mu)\rangle)$. But $\hat{\pi}'(p_\mu) = \hat{\pi}(p_\mu) = p_\nu$ and $\hat{\pi}'(x) \in J^X_\lambda$.

In addition, $\sup(H' \cap \nu) = \lambda$. That $\sup(H' \cap \nu) \geq \lambda$ is clear. Conversely, let

$x \in H' \cap \nu$, i.e., $x = h_{\gamma, A_\nu \cap J_\gamma^X}(i, \langle y, p_\nu \rangle)$ for some $i \in \omega$ and a $y \in J_\lambda^X$. Then x is uniquely determined by $\langle I_\gamma^0, A_\nu \cap J_\gamma^X \rangle \models (\exists z)\psi_i(z, x, \langle y, p_\nu \rangle)$. But such a z exists already in a $H_\nu^0(\alpha, \tau)$, where $H_\nu^0(\alpha, \tau)$ is the closure of S_α under all $h^\varphi_{\tau, X \cap S_\tau, A_\nu \cap S_\tau}$. Since $\gamma = \sup(rng(\hat{\pi}))$ and $\lambda = \sup(rng(\pi))$ we can pick such $\tau \in rng(\hat{\pi})$ and $\alpha \in rng(\pi)$. Let $\overline{\tau} = \hat{\pi}^{-1}(\tau)$ and $\overline{\alpha} = \hat{\pi}^{-1}(\alpha)$. Let $\vartheta = \sup(\nu \cap H_\nu^0(\alpha, \tau))$ and $\overline{\vartheta} = \sup(\mu \cap H_\mu^0(\overline{\alpha}, \overline{\tau}))$. Since ν is regular in $I_{\rho(\nu)}$, $\vartheta < \nu$. Analogously, $\overline{\vartheta} < \mu$. But of course $\hat{\pi}(\overline{\vartheta}) = \vartheta$. So $x < \vartheta = \hat{\pi}(\overline{\vartheta}) < \sup(\hat{\pi}[\mu]) = \lambda$.

If $\nu \in S^+$, then we may define H' as $h_{\gamma, A_\nu \cap J_\gamma^X}[\omega \times (J^X_{\alpha_\nu} \times \{p_\nu\})]$ and still conclude that $H = rng(\hat{\pi}') \subseteq H'$ and $\sup(H' \cap \nu) = \lambda$ by the definition of p_ν.

By Lemma 19, $\hat{\pi} : \langle I_\rho^0, A \rangle \to \langle I^0_{\rho(\nu)}, A_\nu \rangle$ may be extended to a $\Sigma_{n(\nu)-1}$-elementary embedding $\tilde{\pi}_1 : I_\beta \to I_{\beta(\nu)}$ such that ρ is the $(n(\nu) - 1)$th projectum of I_β and A is the $(n(\nu) - 1)$th standard code of it. Let $\hat{\pi}_0 = \hat{\pi}_1^{-1} \circ \hat{\pi}$. Then $\hat{\pi}_0 : \langle I^0_{\rho(\mu)}, A_\mu \rangle \to \langle I_\rho^0, A \rangle$ is Σ_0-elementary, by the coherence of $L_\kappa[X]$, and cofinal. Thus it is Σ_1-elementary by Lemma 18. Applying again Lemma 19, we get a $\Sigma_{n(\nu)}$-elementary $\tilde{\pi}_0 : I_{\beta(\mu)} \to I_\beta$.

As in Lemma 24, it suffices to prove $\beta = \beta(\lambda)$, $n(\nu) = n(\lambda)$, $\rho = \rho(\lambda)$, $A = A_\lambda$, $\hat{\pi}_1^{-1}(p_\nu) = p_\lambda$, $\hat{\pi}_1^{-1}(P_\nu^*) = P_\lambda^*$, $\alpha_\nu^* = \alpha_\lambda^*$ and $\hat{\pi}_1^{-1}(\alpha^{**}_{\mu_\nu}) = \alpha^{**}_{\mu_\lambda}$. So, if $n(\nu) > 1$, we have to show that λ is $\Sigma_{n(\nu)-1}$-regular over I_β. If $n(\nu) = 1$, then $I_\beta \models (\lambda$ regular) suffices. In addition, λ must be $\Sigma_{n(\nu)}$-singular over I_β. For regularity, consider $\tilde{\pi}_0$ and, as in Lemma 24, the least $x \in \lambda$ proving the opposite if such an x exists. This is again Σ_n-definable and therefore in $rng(\tilde{\pi}_0)$. But then $\tilde{\pi}_0^{-1}(x)$ had the same property in $I_{\beta(\mu)}$. Contradiction!

Now, assume $\nu \in S^+$. Since $I_\nu \models (\alpha_\nu$ is the largest cardinal), $H' \cap \nu$ is transitive. Thus $H' \cap \nu = \lambda$. Since $\hat{\pi}_1 : \langle I_\rho^0, A \rangle \to \langle I_\gamma^0, A \cap J_\gamma^X \rangle$ is Σ_1-elementary and $\lambda \subseteq H' = rng(\hat{\pi}_1)$, we have $\lambda = \lambda \cap h_{\rho, A}[\omega \times (J^X_{\alpha_\nu} \times \{\hat{\pi}_1^{-1}(p_\nu)\})]$, i.e., there is a Σ_1-map over $\langle I_\rho, A\rangle$ from α_ν onto λ. But this is then $\Sigma_{n(\nu)}$-definable over I_β and λ is $\Sigma_{n(\nu)}$-singular over I_β.

If $\nu \notin S^+$, then the fact that λ is $\Sigma_{n(\nu)}$-singular over I_β, $\alpha^*_\nu = \alpha^*_\lambda$ and $\hat{\pi}_1^{-1}(\alpha^{**}_{\mu_\nu}) = \alpha^{**}_{\mu_\lambda}$ may be seen as in Lemma 24 because $\pi_0(\alpha^*_\mu) = \alpha^*_\nu \in rng(\pi_0)$.

That $\hat{\pi}_1^{-1}(p_\nu) = p_\lambda$ and $\hat{\pi}_1^{-1}(P^*_\nu) = P^*_\lambda$ can again be proved as in (5) in the proof of Lemma 19. □

Lemma 26. *Let* $\nu \in E$ *and*

$$\Lambda(\xi, \nu) = \{\sup(h_{\nu, B_\nu}[\omega \times (J^X_\beta \times \{\xi\})] \cap \nu) < \nu \,|\, \beta \in Lim \cap \nu\}.$$

Let $\overline{\eta} < \overline{\nu}$ *and* $\pi : \langle I^0_{\overline{\nu}}, B\rangle \to \langle I^0_\nu, B_\nu\rangle$ *be* Σ_1-*elementary. Then* $\Lambda(\overline{\xi}, \overline{\nu}) \cap \overline{\eta} \in J^X_{\overline{\nu}}$ *and* $\pi(\Lambda(\overline{\xi}, \overline{\nu}) \cap \overline{\eta}) = \Lambda(\xi, \nu) \cap \pi(\overline{\eta})$ *where* $\pi(\overline{\xi}) = \xi$ *and* $\pi(\overline{\eta}) = \eta$.

Proof. (1) Let $\lambda \in \Lambda(\xi, \nu)$. Then $\Lambda(\xi, \lambda) = \Lambda(\xi, \nu) \cap \lambda$.

Let β_0 be minimal such that

$$\sup(h_{\nu, B_\nu}[\omega \times (J^X_{\beta_0} \times \{\xi\})] \cap \nu) = \lambda.$$

Then, by Lemma 25, for all $\beta \leq \beta_0$,

$$h_{\lambda, B_\lambda}[\omega \times (J^X_\beta \times \{\xi\})] = h_{\nu, B_\nu}[\omega \times (J^X_\beta \times \{\xi\})]$$

and for all $\beta_0 \leq \beta$,

$$\begin{aligned} h_{\lambda, B_\lambda}[\omega \times (J^X_{\beta_0} \times \{\xi\})] &\subseteq h_{\lambda, B_\lambda}[\omega \times (J^X_\beta \times \{\xi\})] \\ &\subseteq h_{\nu, B_\nu}[\omega \times (J^X_\beta \times \{\xi\})]. \end{aligned}$$

So $\Lambda(\xi, \lambda) = \Lambda(\xi, \nu) \cap \lambda$.

(2) $\Lambda(\overline{\xi}, \overline{\nu}) \cap \overline{\eta} \in J^X_{\overline{\nu}}$

Let $\overline{\lambda} := \sup((\overline{\xi}, \overline{\nu}) \cap \overline{\eta} + 1)$. Then, by (1), $\Lambda(\overline{\xi}, \overline{\nu}) \cap \overline{\eta} + 1 = \Lambda(\overline{\xi}, \overline{\nu}) \cup \{\overline{\lambda}\}$. But $\Lambda(\overline{\xi}, \overline{\nu})$ is definable over $I_{\beta(\overline{\lambda})}$. Since $\beta(\overline{\lambda}) < \overline{\nu}$, we get $\Lambda(\overline{\xi}, \overline{\nu}) \cap \overline{\eta} + 1 \in J^X_{\overline{\nu}}$.

(3) Let $\sup(h_{\overline{\nu}, B_{\overline{\nu}}}[\omega \times (J^X_{\overline{\beta}} \times \{\overline{\xi}\})]) < \overline{\nu}$ and $\pi(\overline{\beta}) = \beta$. Then

$$\pi(\sup(h_{\overline{\nu}, B_{\overline{\nu}}}[\omega \times (J^X_{\overline{\beta}} \times \{\overline{\beta}\})] \cap \overline{\nu})) = \sup(h_{\nu, B_\nu}[\omega \times (J^X_{\beta} \times \{\xi\})] \cap \nu).$$

Let $\overline{\lambda} := \sup(h_{\overline{\nu}, B_{\overline{\nu}}}[\omega \times (J^X_{\overline{\beta}} \times \{\overline{\xi}\})] \cap \overline{\nu})$. Then

$$\langle I^0_{\overline{\nu}}, B_{\overline{\nu}} \rangle \models \neg(\exists \overline{\lambda} < \theta)(\exists i \in \omega)(\exists \xi_i < \overline{\beta})(\theta = h_{\overline{\nu}, B_{\overline{\nu}}}(i, \langle \xi_i, \overline{\xi} \rangle)).$$

So

$$\langle I^0_{\nu}, B_{\nu} \rangle \models \neg(\exists \lambda < \theta)(\exists i \in \omega)(\exists \xi_i < \beta)(\theta = h_{\nu, B_\nu}(i, \langle \xi_i, \xi \rangle)),$$

where $\pi(\overline{\lambda}) = \lambda$, i.e., $\sup(h_{\nu, B_\nu}[\omega \times (J^X_{\beta} \times \{\xi\})] \cap \nu) \leq \lambda$. But $(\pi \restriction J^X_{\overline{\lambda}}) : \langle I^0_{\overline{\lambda}}, B_{\overline{\lambda}} \rangle \to \langle I^0_{\lambda}, B_{\lambda} \rangle$ is elementary. So, if

$$\langle I^0_{\overline{\lambda}}, B_{\overline{\lambda}} \rangle \models (\forall \eta)(\exists \xi_i \in \overline{\beta})(\exists n \in \omega)(n \leq h_{\overline{\lambda}, B_{\overline{\lambda}}}(n, \langle \xi_i, \overline{\xi} \rangle)),$$

then

$$\langle I^0_{\lambda}, B_{\lambda} \rangle \models (\forall \eta)(\exists \xi_i \in \beta)(\exists n \in \omega)(n \leq h_{\lambda, B_{\lambda}}(n, \langle \xi_i, \xi \rangle)).$$

But by Lemma 25, $h_{\lambda, B_\lambda}[\omega \times (J^X_{\beta} \times \{\xi\})] \subseteq h_{\nu, B_\nu}[\omega \times (J^X_{\beta} \times \{\xi\})]$, i.e., it is indeed $\lambda = \sup(h_{\nu, B_\nu}[\omega \times (J^X_{\beta} \times \{\xi\})] \cap \nu)$.

(4) $\pi(\Lambda(\overline{\xi}, \overline{\nu}) \cap \overline{\eta}) = \Lambda(\xi, \nu) \cap \pi(\overline{\eta})$

For $\overline{\lambda} \in \Lambda(\overline{\xi}, \overline{\nu})$,

$$\pi(\Lambda(\overline{\xi}, \overline{\nu}) \cap \overline{\lambda})$$

by (1)

$$= \pi(\Lambda(\overline{\xi}, \overline{\lambda}))$$

by Σ_1-elementarity of π

$$= \Lambda(\xi, \pi(\overline{\lambda})).$$

by (1) and (3),

$$= \Lambda(\xi, \nu) \cap \pi(\overline{\lambda}).$$

So, if $\Lambda(\overline{\xi}, \overline{\lambda})$ is cofinal in $\overline{\nu}$, then we are finished. But if there exists $\overline{\lambda} := \max(\Lambda(\overline{\xi}, \overline{\nu}))$, then, by (1) and (2), $\Lambda(\overline{\xi}, \overline{\lambda}) \in J^X_{\overline{\nu}}$, and it suffices to show $\pi(\Lambda(\overline{\xi}, \overline{\nu})) = \Lambda(\xi, \nu)$. To this end, let $\overline{\beta}$ be maximal such that $\overline{\lambda} = \sup(h_{\overline{\nu}, B_{\overline{\nu}}}[\omega \times (J^X_{\overline{\beta}} \times \{\overline{\xi}\})] \cap \overline{\nu})$, i.e., $h_{\overline{\nu}, B_{\overline{\nu}}}[\omega \times (J^X_{\overline{\beta}+1} \times \{\overline{\xi}\})]$ is cofinal in $\overline{\nu}$. So, since $\pi[h_{\overline{\nu}, B_{\overline{\nu}}}[\omega \times (J^X_{\overline{\beta}+1} \times \{\overline{\xi}\})]] \subseteq h_{\nu, B_\nu}[\omega \times (J^X_{\beta+1} \times \{\xi\})]$, where

$$\pi(\overline{\xi}) = \beta, \ \sup(rng(\pi) \cap \nu) \leq \sup(h_{\nu, B_\nu}[\omega \times (J^X_{\beta+1} \times \{\xi\})] \cap \nu).$$

Hence indeed $\pi(\Lambda(\overline{\xi}, \overline{\nu})) = \Lambda(\xi, \nu)$. □

Lemma 27. *Let* $\nu \in E$, $H \prec_1 \langle I^0_\lambda, B_\lambda \rangle$ *and* $\lambda = \sup(H \cap \nu)$. *Let* $h : I^0_{\overline{\lambda}} \to I^0_\lambda$ *be* Σ_1*-elementary and* $H \subseteq rng(h)$. *Then* $\lambda \in E$ *and* $h : \langle I^0_{\overline{\lambda}}, B_{\overline{\lambda}} \rangle \to \langle I^0_\lambda, B_\lambda \rangle$ *is* Σ_1*-elementary*.

Proof. By Lemma 25, $B_\lambda = B_\nu \cap J^X_\lambda$. So it suffices, by Lemma 24, to show $rng(h) \prec_1 \langle I^0_\lambda, B_\lambda \rangle$. Let $x_i \in rng(h)$ and $\langle I^0_\lambda, B_\lambda \rangle \models (\exists z)\psi(z, x_i)$ for a Σ_0 formula ψ. Then we have to prove that there exists a $z \in rng(h)$ such that $\langle I^0_\lambda, B_\lambda \rangle \models \psi(z, x_i)$. Since $\lambda = \sup(H \cap \nu)$, there is a $\eta \in H \cap Lim$ such that $\langle I^0_\eta, B_\lambda \cap J^X_\eta \rangle \models (\exists z)\psi(z, x_i)$. And since $H \prec_1 \langle I^0_\nu, B_\nu \rangle$, we have $\langle I^0_\lambda, B_\lambda \cap J^X_\eta \rangle \in H \subseteq rng(h)$. So also

$$rng(h) \models (\langle I^0_\eta, B_\lambda \cap J^X_\eta \rangle \models (\exists z)\psi(z, x_i))$$

because $rng(h) \prec_1 I^0_\lambda$. Hence there is a $z \in rng(h)$ such that $\langle I^0_\eta, B_\lambda \cap J^X_\eta \rangle \models \psi(z, x_i)$, i.e., $\langle I^0_\eta, B_\lambda \rangle \models \psi(z, x_i)$. □

Lemma 28. *Let* $f : \overline{\nu} \Rightarrow \nu$, $\overline{\nu} \sqsubset \overline{\tau} \sqsubseteq \mu_{\overline{\nu}}$ *and* $f(\overline{\tau}) = \tau$. *If* $\overline{\tau} \in S^+ \cup \hat{S}$ *is independent, then* $(f \restriction J^D_{\alpha\overline{\tau}}) : \langle J^D_{\alpha_{\overline{\tau}}}, D_{\alpha_{\overline{\tau}}}, K_{\overline{\tau}} \rangle \to \langle J^D_{\alpha_\tau}, D_{\alpha_\tau}, K_\tau \rangle$ *is* Σ_1*-elementary.*

Proof. If $\overline{\tau} = \mu_{\overline{\tau}} < \mu_{\overline{\nu}}$, then the claim holds since $|f| : I_{\mu_{\overline{\nu}}} \to I_{\mu_\nu}$ is Σ_1-elementary. If $\mu_\tau = \mu_\nu$ and $n(\tau) = n(\nu)$, then $P_\tau \subseteq P_\nu$. I.e. τ is dependent on ν. Thus $\overline{\tau}$ is not independent. So let $\mu := \mu_\tau = \mu_\nu$, $n := n(\tau) < n(\nu)$ and $\tau \in S^+ \cup \hat{S}$ be independent. Then, by the definition of the parameters, α_τ is the nth projectum of μ.

Let

$$\gamma_\beta := crit(f_{(\beta, 0, \tau)}) < \alpha_\tau$$

for a β and

$$H_\beta := \text{the } \Sigma_n\text{-hull of } \beta \cup P_\tau \cup \{\alpha^*_\mu, \tau\} \text{ in } I_\mu,$$

i.e., $H_\beta = h^n_\mu[\omega \times (J^X_\beta \times \{\alpha'_\mu, \tau', P'_\tau\})]$, where

$\alpha'_\mu :=$ minimal such that $h^n_\mu(i, \alpha'_\mu) = \alpha^*_\mu$ for an $i \in \omega$,

$P'_\tau :=$ minimal such that $h^n_\mu(i, P'_\tau) = P_\tau$ for an $i \in \omega$,

$\tau' :=$ minimal such that $h^n_\mu(i, \tau') = \tau$ for an $i \in \omega$ (resp. $\tau' := 0$ for $\tau = \mu$).

For the standard parameters are in P_τ.

So H_β is Σ_n-definable over I_μ with the parameters $\{\beta, \tau, \alpha^*_\mu\} \cup P_\tau$. Let

$\rho := \alpha_\tau =$ the nth projectum of μ,

$A :=$ the nth standard code of μ,

$p := \langle \alpha'_\mu, \tau', P'_\tau \rangle$.

So $H_\beta \cap J^X_\rho$ is Σ_0-definable over $\langle I^0_\rho, A \rangle$ with parameters β and p (fine structure theory!).

And γ_β is defined by

$$\gamma_\beta \notin H_\beta \quad \text{and} \quad (\forall \delta \in \gamma_\beta)(\delta \in H_\beta),$$

i.e., γ_β is also Σ_0-definable over $\langle I^0_\rho, A\rangle$ with parameters β and p.

Let $f_0 := f_{(\beta, 0, \tau)}$ for a β, $\overline{\tau}_0 := d(f_0) < \alpha_\tau$ and $\gamma := crit(f_0) < \alpha_\tau$. Let $f_1 := f_{(\beta, \gamma, \tau)}$, $\overline{\tau}_1 := d(f_1) < \alpha_\tau$ and $\delta := crit(f_1) < \alpha_\tau$. Then $\mu_{\overline{\tau}_1}$ is the direct successor of $\mu_{\overline{\tau}_0}$ in K_τ. So $f_{(\beta, \gamma, \overline{\tau}_1)} = id_{\overline{\tau}_1}$. Hence $\mu_\eta = \mu_{\overline{\tau}_1}$ holds for the minimal $\eta \in S^+ \cup S^0$ such that $\gamma < \eta \sqsubseteq \delta$. Thus

$$\mu' \in K^+_\tau := K_\tau - (Lim(K_\tau) \cup \{\min(K_\tau)\})$$

$$\Leftrightarrow$$

$$(\exists \beta, \gamma, \delta, \eta)(\gamma = \gamma_\beta \textit{ and } \delta = \gamma_{(\gamma_\beta + 1)}$$

$$\textit{and } \eta \in S^+ \cup S^0 \text{ minimal such that } \gamma < \eta \sqsubseteq \delta \textit{ and } \mu' = \mu_\eta).$$

Therefore, K^+_τ is Σ_1-definable over $\langle I^0_\rho, A\rangle$ with parameter p.

Now, consider $\langle I^0_{\alpha_\tau}, K_\tau\rangle \models \varphi(x)$, where φ is a Σ_1 formula. Then, since K_τ is unbounded in α_τ,

$$\langle I^0_{\alpha_\tau}, K_\tau\rangle \models \varphi(x)$$

$$\Leftrightarrow$$

$$(\exists \gamma)(\gamma \in K^+_\tau \textit{ and } \langle I^0_\alpha, K_\gamma\rangle \models \varphi(x)).$$

So $\langle I^0_{\alpha_\tau}, K_\tau\rangle \models \varphi(x)$ is Σ_1 over $\langle I^0_\rho, A\rangle$ with parameter p, resp. Σ_{n+1} over I_μ with parameters α^*_μ, τ, P_τ. But since $n = n(\tau) < n(\nu)$, f is at least Σ_{n+1}-elementary. In addition, $f(\alpha^*_{\overline{\tau}}) = \alpha^*_\tau$, $f(\overline{\tau}) = \tau$, $f(P_{\overline{\tau}}) = P_\tau$. So, for $x \in rng(f)$, $\langle I^0_{\alpha_{\overline{\tau}}}, K_{\overline{\tau}}\rangle \models \varphi(f^{-1}(x))$ holds $\langle I^0_{\alpha_\tau}, K_\tau\rangle \models \varphi(x)$. □

Theorem 29. $\mathfrak{M} := \langle S, \lhd, \mathfrak{F}, D\rangle$ *is a* κ*-standard morass*.

Proof. Set

$$\sigma_{(\xi,\nu)}(i) = h_\nu^{n(\nu)}(i, \langle \xi, \alpha_\nu^*, p_\nu \rangle).$$

Then D is uniquely determined by the axioms of standard morasses and

(1) D^ν is uniformly definable over $\langle J_\nu^X, X \restriction \nu, X_\nu \rangle$,

(2) X_ν is uniformly definable over $\langle J_\nu^D, D_\nu, D^\nu \rangle$.

(1) is clear. For (2), assume first that $v \in \hat{S}$ and $f_{(0, q_\nu, \nu)} = id_\nu$. Since the set $\{i \mid \sigma_{(q_\nu, \nu)}(i) \in X_\nu\}$ is $\Sigma_{n(\nu)}$-definable over $\langle J_\nu^X, X \restriction \nu, X_\nu \rangle$ with the parameters $p_\nu, \alpha_\nu^*, q_\nu$, there is a $j \in \omega$ such that

$$\sigma_{(q_\nu, \nu)}(\langle i, j \rangle) \text{ exists } \Leftrightarrow \sigma_{(q_\nu, \nu)}(i) \in X_\nu.$$

Using this j, we have

$$X_\nu = \{\sigma_{(q_\nu, \nu)}(i) \mid \langle i, j \rangle \in dom(\sigma_{(q_\nu, \nu)})\}.$$

So, in case that $f_{(0, q_\nu, \nu)} = id_\nu$, there is the desired definition of X_ν.

Let $\nu \in \hat{S}$, $f_{(0, q_\nu, \nu)} : \bar{\nu} \Rightarrow \nu$ cofinal and $f(\bar{q}) = q_\nu$. Then $f_{(0, \bar{q}, \bar{\nu})} = id_{\bar{\nu}}$. And by Lemma 6(b) of [6], $\bar{q} = q_{\bar{\nu}}$. So, if $\bar{\nu} = \nu$, then $f_{(0, q_\nu, \nu)} = id_\nu$. Thus let $\bar{\nu} < \nu$. Then $f_{(0, q_\nu, \nu)}(x) = y$ is defined by: There is a $\bar{\nu} \le \nu$ such that, for all $r, s \in \omega$,

$$\sigma_{(q_{\bar{\nu}}, \bar{\nu})}(r) \le \sigma_{(q_{\bar{\nu}}, \bar{\nu})}(s) \Leftrightarrow \sigma_{(q_\nu, \nu)}(r) \le \sigma_{(q_\nu, \nu)}(s)$$

holds *and* for all $z \in J_{\bar{\nu}}^X$ there is an $s \in \omega$ such that

$$z = \sigma_{(q_{\bar{\nu}}, \bar{\nu})}(s)$$

and there is an $s \in \omega$ such that

$$\sigma_{(q_{\bar{\nu}}, \bar{\nu})}(s) = x \Leftrightarrow \sigma_{(q_\nu, \nu)}(s) = y.$$

And since $\langle J_\nu^X, X_\nu \rangle$ is rudimentary closed,

$$X_\nu = \bigcup \{ f(X_{\bar{\nu}} \cap \eta) \mid \eta < \bar{\nu} \}.$$

Finally, if $\nu \in \hat{S}$ and $f_{(0, q_\nu, \nu)}$ is not cofinal in ν, then C_ν is unbounded in ν and

$$X_\nu = \bigcup\{X_\lambda \mid \lambda \in C_\nu\}$$

by the coherence of $L_\kappa[X]$.

So (2) holds. From this, $(\mathrm{DF})^+$ follows.

By (1) and (2), $J_\nu^X = J_\nu^D$ for all $\nu \in Lim$, and for all $H \subseteq J_\nu^X = J_\nu^D$,

$$H \prec_1 \langle J_\nu^X, X \restriction \nu \rangle \Leftrightarrow H \prec_1 \langle J_\nu^D, D_\nu \rangle.$$

Now, we check the axioms.

(MP) and $(\mathrm{MP})^+$

$\mid f_{(0, \xi, \nu)} \mid$ is the uncollapse of $h_{\mu_\nu}^{n(\nu)}[\omega \times \{\xi^*, \nu^*, \alpha_\nu^*, \alpha_{\mu_\nu}^{**}, P_\nu^*\}^{<\omega}]$, where ξ^* is minimal such that $h_{\mu_\nu}^{n(\nu)-1}(i, \xi^*) = \xi$. Therefore, (MP) and $(\mathrm{MP})^+$ hold.

(LP1)

holds by (2) above.

(LP2)

This is Lemma 26.

(CP1) and $(\mathrm{CP1})^+$

This follows from Lemma 24 and the definition of $\sigma_{(\xi, \nu)}$.

(CP2)

This is Lemma 27.

(CP3) and $(\mathrm{CP3})^+$

Let $x \in J_\nu^X$, $i \in \omega$ and $y = h_{\nu, B_\nu}(i, x)$. Since C_ν is unbounded in ν, there is a $\lambda \in C_\nu$ such that $x, y \in J_\lambda^X$. By Lemma 25, $B_\lambda = B_\nu \cap J_\lambda^X$. So $y = h_{\lambda, B_\lambda}(i, x)$.

(DP1)

Holds by the definition of μ_ν.

(DF)

Let $\mu := \mu_\nu$, $k := n(\mu)$ and

$$\pi(n, \beta, \xi) := \text{ the uncollapse of } h_\mu^{k+n}[\omega \times (J_\beta^X \times \{\alpha_\mu^{**}, p_\mu^*, \xi^*\}^{<\omega})],$$

where

$\xi^* :=$ minimal such that $h_{\mu_\nu}^{k+n-1}(i, \xi^*) = \xi$ for an $i \in \omega$,

$p_\mu^* :=$ minimal such that $h_\mu^{k+n-1}(i, p_\mu^*) = p_\mu$ for some $i \in \omega$,

$\alpha_\mu^{**} :=$ minimal such that $h_\mu^{k+n-1}(i, \alpha_\mu^{**}) = \alpha_\mu^*$ for some $i \in \omega$.

Prove

$$| f_{(\beta, \xi, \mu)}^{1+n} | = \pi(n, \beta, \xi).$$

for all $n \in \omega$ by induction.

For $n = 0$, this holds by definition of $f_{(\beta, \xi, \mu)}^1 = f_{(\beta, \xi, \mu)}$. So assume that $| f_{(\beta, \xi, \mu)}^m | = \pi(m-1, \beta, \xi)$ is already proved for all $1 \leq m \leq n$. Then, by definition of $\tau(m, \mu)$,

$\alpha_{\tau(m, \mu)} =$ the $(k + m - 1)$ th projectum of μ.

Let $\pi(n, \beta, \xi) : I_{\overline{\mu}} \to I_\mu$. Then

$(*)$ $\xi(m, \mu) = \pi(n, \beta, \xi)\xi(m, \overline{\mu})$ for all $1 \leq m \leq n$:

Let $\pi := \pi(n, \beta, \xi)$, $\alpha := \pi^{-1}[\alpha_{\tau(m, \mu)} \cap rng(\pi)]$, $\rho := \pi(\alpha)$,

$r :=$ minimal such that $h_\mu^{k+m-2}(i, r) = p_\mu$ for an $i \in \omega$,

$\alpha' :=$ minimal such that $h_\mu^{k+m-2}(i, \alpha') = \alpha_\mu^*$ for an $i \in \omega$,

$p :=$ the $(k + m - 1)$ th parameter of μ

and

$$\pi(\bar{r}) = r, \quad \pi(\bar{p}) = p, \quad \pi(\bar{\alpha}') = \alpha'.$$

Let $\bar{\xi} := \xi(m, \bar{\mu})$. Then $\bar{p} = h_{\bar{\mu}}^{k+m-1}(i, \langle \bar{x}, \bar{\xi}, \bar{r}, \bar{\alpha}' \rangle)$ for a $\bar{x} \in J_{\alpha}^{X}$, because $\alpha = \alpha_{\tau(m, \bar{\mu})}$. So $p = h_{\mu}^{k+m-1}(i, \langle x, \xi, r, \alpha' \rangle)$, where $\pi(\bar{x}) = x$ and $\pi(\bar{\xi}) = \xi$. Thus $h_{\mu}^{k+m-1}[\omega \times (J_{\alpha_{\tau(m,\mu)}}^{X} \times \{\alpha', r, \xi\}^{<\omega})] = J_{\mu}^{X}$ by definition of p. So $\xi(m, \mu) \leq \xi$. Assume $\xi(m, \mu) < \xi$. Then

$$I_{\mu} \models (\exists \eta < \xi)(\exists i \in \omega)(\exists x \in J_{\rho}^{X})(\xi = h_{\mu}^{k+m-1}(i, \langle x, \eta, r, \alpha' \rangle)).$$

So

$$I_{\bar{\mu}} \models (\exists \eta < \bar{\xi})(\exists i \in \omega)(\exists x \in J_{\alpha}^{X})(\bar{\xi} = h_{\bar{\mu}}^{k+m-1}(i, \langle x, \eta, \bar{r}, \bar{\alpha}' \rangle)).$$

But this contradicts the definition of $\bar{\xi} = \xi(m, \bar{\mu})$.

So, for all $1 \leq m \leq n$,

$$\xi(m, \mu) \in rng(\pi(n, \beta, \xi)).$$

In addition, for all $\beta < \alpha_{\tau(m,\mu)}$,

$$d(f_{(\beta, \xi(m,\mu), \mu)}^{m}) < \alpha_{\tau(m,\mu)}.$$

Consider $\pi := \pi(m-1, \beta, \xi) = | f_{(\beta, \xi, \mu)}^{m} |$, where $\xi = \xi(m, \mu)$. Then $\pi : I_{\bar{\mu}} \to I_{\mu}$ is the uncollapse of $h_{\mu}^{k+m-1}[\omega \times (\beta \times \{\xi, \alpha', r\}^{<\omega})]$, where

$r :=$ minimal such that $h_{\mu}^{k+m-2}(i, r) = p_{\mu}$ for some $i \in \omega$,

$\alpha' :=$ minimal such that $h_{\mu}^{k+m-2}(i, \alpha') = \alpha_{\mu}^{*}$ for some $i \in \omega$.

And $h_{\bar{\mu}}^{k+m-1}[\omega \times (\beta \times \{\bar{\xi}, \bar{\alpha}', \bar{r}\}^{<\omega})] = J_{\bar{\mu}}^{X}$, where $\pi(\bar{\xi}) = \xi$, $\pi(\bar{\alpha}') = \alpha'$ and $\pi(\bar{r}) = r$. Assume $\alpha_{\tau(m, \mu)} \leq \bar{\mu} < \mu$. Then there were a function over $I_{\bar{\mu}}$ from $\beta < \alpha_{\tau(m,\mu)}$ onto $\alpha_{\tau(m,\mu)}$. This contradicts the fact that $\alpha_{\tau(m,\mu)}$ is a cardinal in I_{μ}. If $\bar{\mu} = \mu$, then $f_{(\beta, \bar{\xi}, \mu)}^{m} = id_{\mu}$. This contradicts the minimality of $\tau(m, \mu)$.

Since $\xi(m, \mu) \in rng(\pi(n, \beta, \xi))$, we can prove

$$rng(\pi(n, \beta, \xi)) \cap J^D_{\alpha_{\tau(m,\mu)}} \prec_1 \langle J^D_{\alpha_{\tau(m,\mu)}}, D_{\alpha_{\tau(m,\mu)}}, K^m_\mu \rangle$$

for all $1 \leq m \leq n$ as in Lemma 28.

We still must prove minimality. Let $f \Rightarrow \mu$ and $\beta \cup \{\xi\} \subseteq rng(f)$ such that

$$rng(f) \cap J^D_{\alpha_{\tau(m,\mu)}} \prec_1 \langle J^D_{\alpha_{\tau(m,\mu)}}, D_{\alpha_{\tau(m,\mu)}}, K^m_\mu \rangle,$$

$$\xi(m, \mu) \in rng(f)$$

holds for all $1 \leq m \leq n$. Show that f is Σ_{k+n}-elementary and that the first standard parameters including the $(k+n-1)$th are in $rng(f)$. That suffices because $\pi(n, \beta, \xi)$ is minimal.

Let p^{k+m}_μ be the $(k+m)$th standard parameter of μ.

Prove, by induction on $0 \leq m \leq n$,

f is Σ_{k+m}-elementary,

$p^1_\mu, \ldots, p^{k+m-1}_\mu \in rng(f)$.

For $m = 0$, this is clear because f $f \Rightarrow \mu$. So assume it to be proved for $m < n$ already. Then let $\alpha := \alpha_{\tau(m+1,\mu)}$ and $\overline{\alpha} = f^{-1}[\alpha \cap rng(f)]$. Consider $\pi := (f \restriction J^D_{\overline{\alpha}}) : \langle J^D_{\overline{\alpha}}, D_{\overline{\alpha}}, \overline{K} \rangle \to \langle J^D_\alpha, D_\alpha, K^{m+1}_\mu \rangle$. Construct a Σ_{k+m+1}-elementary extension $\tilde{\pi}$ of π. To do so, set

$$f_\mu = f^{m+1}_{(\beta, \xi(m+1,\mu), \mu)},$$

$$\mu(\beta) = d(f_\beta),$$

$$H = \bigcup\{f_\beta[rng(\pi) \cap J^D_{\mu(\beta)}] \mid \beta < \alpha\}.$$

Then $H \cap J^D_\alpha = rng(\pi)$. For $rng(\pi) \subseteq H \cap J^D_\alpha$ is clear because $f_\beta \restriction J^D_\beta = id \restriction J^D_\beta$. So let $y \in H \cap J^D_\alpha$, i.e., $y = f_\beta(x)$ for some $x \in rng(\pi)$ and a $\beta < \alpha$. Let $K^+ =$

$K^{m+1}_\mu - Lim(K^{m+1}_\mu)$ and $\beta(\eta) = \sup\{\beta \mid f^{m+1}_{(\beta, \xi(m+1, \eta), \eta)} \neq id_\eta\}$. Then

$$\langle J^D_\alpha, D_\alpha, K^{m+1}_\mu \rangle \models (\exists y)(\exists \eta \in K^+)(y = f^{m+1}_{(\beta, \xi(m+1, \eta), \eta)}(x) \in J^D_{\beta(\eta)}).$$

Since $rng(\pi) \prec_1 \langle J^D_\alpha, D_\alpha, K^{m+1}_\mu \rangle$, $y = f^{m+1}_{(\beta, \xi(m+1, \eta), \eta)}(x) \in rng(\pi)$ if $x \in rng(\pi)$ for such an η. But since $y = f^{m+1}_{(\beta, \xi(m+1, \eta), \eta)}(x) \in J^D_{\beta(\eta)}$, we get $f_\beta(x) = f^{m+1}_{(\beta, \xi(m+1, \eta), \eta)}(x) \in rng(\pi)$.

Show $H \prec_{k+m+1} I_\mu$. Since $f^{m+1}_{(\beta, \xi, \mu)} = \pi(m, \beta, \xi)$, $\alpha_{\tau(m+1, \mu)}$ is the $(k+m)$th projectum of μ. Like in $(*)$ above, we can show that the $(k+m)$th standard parameter p^{k+m}_μ of μ is in $rng(f_\beta)$. Now, let $I_\mu \models (\exists x)\varphi(x, y, p^1_\mu, ..., p^{k+m}_\mu)$, where φ is a Π_{k+m} formula and $y \in H \cap J^D_\alpha$. Since f_β is Σ_{k+m}-elementary, the following holds:

$$I_\mu \models (\exists x)\varphi(x, y, p^1_\mu, ..., p^{k+m}_\mu)$$

$$\Leftrightarrow (\exists \gamma \in K^{m+1}_\mu)(\exists x)(I_\gamma \models \varphi(x, y, p^1_\gamma, ..., p^{k+m}_\gamma)).$$

And since $rng(\pi) \prec_1 \langle J^D_\alpha, D_\alpha, K^{m+1}_\mu \rangle$,

$$rng(\pi) \models (\exists \gamma \in K^{m+1}_\mu)(\exists x)(I_\gamma \models \varphi(x, y, p^1_\gamma, ..., p^{k+m}_\gamma)).$$

Thus there is such an x in $rng(\pi)$ and therefore in H.

Let $\tilde{\pi}$ be the uncollapse of H. Then $\tilde{\pi}$ is Σ_{k+m}-elementary and, since $p^1_\mu, ..., p^{k+m}_\mu \in rng(f_\beta)$ for all $\beta < \alpha$, we have $p^1_\mu, ..., p^{k+m}_\mu \in rng(\pi) = H$. In addition, by the induction hypothesis, f is Σ_{k+m}-elementary and $p^1_\mu, ..., p^{k+m-1}_\mu \in rng(f)$. Again as in $(*)$ above, we can show that $p^{k+m}_\mu \in rng(f)$ using $\xi(m+1, \mu) \in rng(f)$. But since $\tilde{\pi}$ and f are the same on the $(k+m)$th projectum, we get $\tilde{\pi} = f$.

(SP) follows from $| f^{1+n}_{(\beta, \xi, \mu)} | = \pi(n, \beta, \xi)$, because for all $\nu \sqsubset \tau \sqsubseteq \mu_\nu$ such that $\tau \in S^+$ (resp. $\tau = \nu$) the following holds:

$$p_\tau \in rng(\pi(n, \beta, \xi)) \Leftrightarrow \xi_\tau \in rng(\pi(n, \beta, \xi)).$$

This may again be shown as $(*)$.

(DP2)

It is like $(*)$ in (DF).

(DP3)

(a) is clear.

(b) was already proved with (DF)$^+$. □

Theorem 30. *Let* $\langle X_\nu \mid \nu \in S^X \rangle$ *be such that*

(1) $L[X] \models S^X = \{\beta(\nu) \mid \nu \text{ singular}\}$

(2) $L[X]$ *is amenable*

(3) $L[X]$ *has condensation*

(4) $L[X]$ *has coherence*.

Then there is a sequence $C = \langle C_\nu \mid \nu \in \hat{S} \rangle$ *such that*

(1) $L[C] = L[X]$,

(2) $L[C]$ *has condensation*,

(3) C_ν *is club in* J_ν^C *w.r.t. the canonical well-ordering* $<_\nu$ *of* J_ν^C,

(4) $opt(\langle C_\nu, <_\nu \rangle) > \omega \Rightarrow C_\nu \subseteq \nu$,

(5) $\mu \in Lim(C_\nu) \Rightarrow C_\mu = C_\nu \cap \mu$,

(6) $opt(C_\nu) < \nu$.

Proof. First, construct from $L[X]$ a standard morass as in Theorem 29. Then construct an inner model $L[C]$ from it as in [6]. □

Acknowledgements

The present paper is a part of my dissertation [5]. I thank Dieter Donder for being my adviser, Hugh Woodin for an invitation to Berkeley, where part of the work was done, and the DFG-Graduiertenkolleg ‘Sprache, Information, Logik’ in Munich for their support.

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