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RECONSTRUCTION FROM IRREGULAR FOURIER SAMPLES AND GAUSSIAN SPECTRAL MOLLIFIERS

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Abstract

It is well known that a bandlimited function can be reconstructed in theory from a discrete set of its Fourier samples, provided that the samples are dense enough. This fact is a direct consequence of extensive studies on Fourier frames for $L^2([a, b])$. However, when the sample points do not form a lattice, there is no practical scheme (to our knowledge) for the reconstruction of *f*. In this paper, we propose a fast and easy to implement technique, for reconstructing a compactly supported function f from finitely irregular samples of $\hat{f}$. The scheme is based on the cubic-spline interploation and Gaussian spectral mollifiers. The scheme allows us to eliminate the Gibbs oscillations in many cases.

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1. Introduction

Let $f(x)$ be a compactly supported function in $L^2(\mathbb{R})$. An important question one often encounters, both in the study of mathematics itself and in applications, is: Given $\hat{f}(\xi)$, or some sample points of $\hat{f}(\xi)$, how can we reconstruct $f(x)$? Of course, if $\hat{f}(\xi)$ is known for every ξ, then $f(x)$ is readily obtained by the inverse Fourier transform

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \xi} d\xi.$$

However, it is not necessary that we know $\hat{f}(\xi)$ for all values ξ, given that $f(x)$ is compactly supported. Suppose that $\operatorname{supp}(f) \subseteq [a, b]$. Far more useful for applications is the Fourier series inversion formula

$$f(x) = \Delta \sum_{\lambda \in \Lambda} \hat{f}(\lambda) e^{2\pi i \lambda x}, \tag{1.1}$$

where $\Lambda = \{n\Delta : n \in \mathbb{Z}\}$ is a lattice in $\mathbb{R}$, with $0 < \Delta \le (b - a)^{-1}$. The inversion formula (1.1) follows from the well known fact that $\{e^{2\pi i \lambda x} : \lambda \in \Lambda\}$ for the above Λ forms a *tight frame* for $L^2([a, b])$ with tight frame bound $\frac{1}{\Delta}$.

The tight frame reconstruction requires $f(\xi)$ be known on a regular set, i.e., a lattice in this case. Unfortunately this is a luxury we may not have in some applications, such as in MRI (magnetic resonance imaging). In MRI the Fourier transform $\rho(\xi)$, where ρ is the density function of the scanned image (such as a planar section of patient's head), is sampled along several paths, with each path being a curve such as a spiral, a circle or a line. These sampled points do not contain a regular set (a lattice). So the image reconstruction in MRI must begin with irregular Fourier samples. Currently, samples are taken along many paths, resulting in a sufficiently dense set of samples. These sample points allow for a reasonable interpolation of $\hat{\rho}$ on a lattice, and therefore a reconstruction of ρ. The drawback is that it takes time to obtain many samples.

Another challenge we face is the Gibbs oscillation. Since we can only use finitely many sample points, the Gibbs oscillation is inevitable, even when we do have a regular set of samples of $\hat{f}$. Any reasonable reconstruction scheme therefore must address the problem of Gibbs oscillation.

In this paper, we propose a scheme for reconstruction of $f(x)$ from irregular samples of $\hat{f}$. A key ingredient is a technique called *discrete singular convolution*, first introduced by Wei [12]. This technique allows us to virtually eliminate the Gibbs oscillation in many cases. Our scheme is still in its early stage, and there are areas that need to be refined. The most important improvement would most likely come from choosing the right bases. In our study, we have experimented with Haar bases. But it is clear that better results can be expected from other types of bases, since the speed of decay in the Fourier transform plays an important role in our scheme. We shall discuss possible improvements for future work later. Nevertheless, our results have clearly demonstrated its promise. We hope it will serve our modest goal, that is, a valuable first step in addressing an important theoretical and practical challenge.

2. Irregular Fourier Frames and Reconstruction

Since we are concerned only with the reconstruction of compactly supported functions $f(x)$, we may without loss of generality assume that $f \in L^2([0, 1])$. It is well known that $\{e^{2\pi inx} : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([0, 1])$, which gives us the standard Fourier series expansion for $f(x)$:

$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi inx}.$$

However, reconstructing $f(x)$ – at least in theory – can be done often without an orthonormal basis. In this paper we focus on reconstructing $f(x)$ using Fourier frames. A set of elements $\{\mathbf{v}_j\}$ in a Hilbert space $\mathbf{H}$ is called a *frame* with frame bounds $A, B > 0$ if for every $\mathbf{u} \in \mathbf{H}$ we have

$$A\|\mathbf{u}\|^2 \leq \sum_j |\langle \mathbf{u}, \mathbf{v}_j \rangle|^2 \leq B\|\mathbf{u}\|^2. \tag{2.1}$$

If $A = B$, then $\mathbf{v}_j$ is called a *tight frame*. Fourier frames for $L^2([0, 1])$ are frames of the form $\{e^{2\pi i\lambda_j x}\}$. The following is a well known result of Beurling (cf. [15]).

Theorem 2.1. *Suppose that* $\Lambda = \{\lambda_j\} \subset \mathbb{R}$ *is uniformly separated and has* $D^-(\Lambda) > 1$, *where* $D^-(\Lambda)$ *denote the lower Beurling densities given by*

$$D^-(\Lambda) = \liminf_{n\to\infty} \inf_{a\in\mathbb{R}} \frac{1}{2n} \#(\Lambda \cap [-n + a, n + a]).$$

Then $\{e^{2\pi i\lambda x} : \lambda \in \Lambda\}$ *is a frame for* $L^2([0, 1])$.

Nevertheless, if $1 < D^-(\Lambda) \le D^+(\Lambda) < \infty$ but Λ is irregular, then there is no simple formula to reconstruct $f(x)$ from the Fourier samples $\{\hat{f}(\lambda)\}_{\lambda\in\Lambda}$. Moreover, in practice, we can only take finitely many data, posing an additional challenge to the reconstruction.

Here we propose a reconstruction scheme based on cubic-spline interpolation and a technique along the spirit of the discrete singular convolution (DSC), a technique introduced by Wei. The DSC technique in our case is essentially a Gaussian spectral mollifier aimed to enhance the decay in Fourier transforms of bandlimited functions, and it enables us to reconstruct singularities much better than without it. We shall discuss the DSC technique in more details later in the paper.

Let $\{\phi_n(x)\}_{n\in\mathbb{Z}}$ be an orthonormal basis for $L^2([0, 1])$. Then

$$f(x) = \sum_{n\in\mathbb{Z}} \langle f, \phi_n(x)\rangle \phi_n(x).$$

Notice that $\langle f, \phi_n(x)\rangle = \langle \hat{f}, \hat{\phi}_n(x)\rangle$. Since samples of $\hat{f}$ are given, it is natural that we estimate $\langle f, \phi_n(x)\rangle$ by estimating $\langle \hat{f}, \hat{\phi}_n(x)\rangle$. Now, given $\{\phi_n\}$ we can often compute $\hat{\phi}_n(\xi)$ explicitly. Furthermore, $\hat{f}$ is analytic and $\lim_{\xi\to\infty} \hat{f}(\xi) = 0$. Suppose we have $\hat{f}(\lambda)$, where $\lambda \in \Lambda$. We now interpolate $\hat{f}(\xi)\hat{\phi}_n(\xi)$ using $\{\hat{f}(\lambda)\hat{\phi}_n(\lambda) : \lambda \in \Lambda\}$ by cubic splines. Let $G_n(\xi)$ be the resulting cubic spline interpolation. Then we obtain an estimation of $\langle f, \phi_n(x)\rangle$ by $\int_{\mathbb{R}} G_n(\xi)d\xi$, and therefore a reconstruction of f by

$$f_{rec}(x) = \sum_{n \in \mathbb{Z}} c_n \phi_n(x), \quad \text{where } c_n := \int_{\mathbb{R}} G_n(\xi) d\xi. \tag{2.2}$$

If $f(x)$ is known to be real, then we use

$$f_{rec}(x) = \mathrm{Re}\left(\sum_{n \in \mathbb{Z}} c_n \phi_n(x)\right), \quad \text{where } c_n := \int_{\mathbb{R}} G_n(\xi) d\xi. \tag{2.3}$$

In practice, we are given only a finite set of data, i.e., we know $\{\hat{f}(\lambda) : \lambda \in \Lambda\}$ with Λ being finite. We can also choose only a finite basis $\{\phi_n\}_{n=0}^{N-1}$. But the technique will be the same. We use cubic spline to interpolate $\{\hat{f}(\lambda)\hat{\phi}_n(\lambda) : \lambda \in \Lambda\}$. Let $G_n(\xi)$ be the resulting interpolation. (Here we need to set $G_n(\xi) = 0$ for sufficiently large ξ). Then the reconstructed $f(x)$ will be

$$f_{rec}(x) = \sum_{n=0}^{N-1} c_n \phi_n(x), \quad \text{where } c_n := \int_{\mathbb{R}} G_n(\xi) d\xi, \tag{2.4}$$

or if $f(x)$ is real,

$$f_{rec}(x) = \mathrm{Re}\left(\sum_{n=0}^{N-1} c_n \phi_n(x)\right), \quad \text{where } c_n = \int_{\mathbb{R}} G_n(\xi) d\xi. \tag{2.5}$$

Our experiments indicate that by taking a simple basis such as the Haar basis $\phi_n(x) = \chi_{\left[\frac{n}{N}, \frac{n+1}{N}\right)}(x)$, this reconstruction scheme works very well if $f(x) \in C_0([0, 1])$, i.e., $f(x)$ is continuous in $\mathbb{R}$ and $\mathrm{supp}(f) \subseteq [0, 1]$. However, the Gibbs oscillation poses a big problem if *f* is discontinuous in $\mathbb{R}$, making a “straight out-of-the-box” application of (2.4) or (2.5) less useful. By incorporating the DSC technique with a Gaussian spectral mollifier, we have either eliminated or severely curbed the Gibbs oscillation in our reconstructions.

3. DSC Technique and Spectral Mollifiers

Let $\psi(x)$ be a bandlimited function with $\mathrm{supp}(\hat{\psi}) \subseteq \left[-\frac{1}{2} \cdot \frac{1}{2}\right]$. Then the Shannon Sampling Theorem states that for any $0 < \Delta \leq 1$ we have

$$\psi(x) = \Delta \sum_{n \in \mathbb{Z}} \psi(n\Delta) S(\Delta^{-1} x - n), \tag{3.1}$$

where $S(x) = \sin(\pi x)/\pi x$ is the sync function. While this sampling theorem is the cornerstone in signal processing, it possesses some intrinsic difficulties in applications. The sampling formula with only finitely many samples among $\psi(n\Delta)$, which is the case in all applications, is subject to the Gibbs oscillation in the Fourier domain. But Gibbs oscillation is only one of the challenges. It is known that the sampling formular (3.1) yields only an $O(\Delta)$ approximation for a general C^∞ function ψ. This fact explains why (3.1) has little use in applications such as numerical PDE, in which higher order of approximations are desired.

These problems are overcome in the work of Wei and his coauthors, see [12] and [14] as well as the references therein. The key ingredient in their work is to mollify the sync function with a suitable Gaussian. Let

$$S_a(x) = S(x)e^{-ax^2} = \frac{\sin(\pi x)}{\pi x} e^{-ax^2}, \; a > 0.$$

The sampling formula (3.1) is then modified by using $S_a(x)$ in place of $S(x)$, giving the approximation/reconstruction of $\psi(x)$ by

$$\psi^*(x) = \Delta \sum_{n \in \mathbb{Z}} \psi(n\Delta) S(\Delta^{-1} x - n), \tag{3.2}$$

where a is chosen to be proportional to be roughly Δ, $a = c\Delta$. This simple approximation proves to be surprisingly powerful: Gibbs oscillation is eliminated completely in many applications, and extraordinarily high accuracy is achieved in many numerical PDE solutions. Wei calls this scheme the *discrete singular convolution* (DSC) scheme. The DSC scheme is very robust, as the constant c can be taken over a large interval without having apparent impact on the outcome of the tasks. Despite the high performance many numerical analysts remained skeptical of the validity of the DSC scheme, as there was no mathematical proof that it works. Fortunately this is no longer the case, as one of us (Wang [11]) recently has given a rigorous proof that (3.2) yields an approximation of ψ, that is, $o(\Delta^N)$ for any N.

A major reason for the improved performance is that by adding a Gaussian factor one overcomes the slow decaying property of the sync function. This is also why the DSC scheme proves useful not just when the sync function basis is used. Numerical experiments have indicated similar improvements in performance when other types of basis functions are used. To reconstruct a function from its Fourier samples we incorporate the DSC scheme to the Fourier transform of Haar bases.

We incorporate the DSC technique in our reconstruction by adding a Gaussian mollifier to the spline interpolation $G_n(\xi)$. It should be pointed out that curbing Gibbs oscillation using spectral mollifiers has been used by many others, see e.g. Tadmor and Tanner [10] in all kind of ways. The way we use the mollifiers is different from all the existing methods, including those by Wei and those by Tadmor and Tanner. Nevertheless all these methods embody essentially the same goal of controlling the decay of the Fourier transforms. Now let $f(x) \in L^2([0, 1])$ and $\{\hat{f}(\lambda_j) : 0 \le j < M\}$ be given, where $\Lambda = \{\lambda_j\}$ is inside the interval $[-K, K]$. Let $\phi_n(x) = \chi_{I_n}(x)$ where $I_n = \left[\frac{n}{N}, \frac{n+1}{N}\right)$, $0 \le n < N$. We reconstruct $f(x)$ from the Fourier samples $\{\hat{f}(\lambda_j)\}$ in the form

$$f_{rec}(x) = \sum_{n=0}^{N-1} c_n \phi_n(x).$$

Note that N is the "resolution" of the reconstructed function f_{rec}, and the selection of which depends on several factors and the actual application. (In MRI the reconstruction resolution is typically 128×128). We require that $M \ge N$. Now, instead of taking $c_n = \int_{\mathbb{R}} G_n(\xi) d\xi$, where G_n is the cubic spline interpolation of $\hat{f}(\xi)\hat{\phi}_n(\xi)$ using the sample points $\{\hat{f}(\lambda_j)\hat{\phi}_n(\lambda_j)\}$, we apply the Gaussian mollifier by setting $c_n = \int_{\mathbb{R}} G_{n,a}(\xi) d\xi$ for a suitable $a > 0$, where $G_{n,a}$ is the cubic spline interpolation of $\hat{f}(\xi)\hat{\phi}_n(\xi)e^{-a\xi^2}$ using the sample points $\{\hat{f}(\lambda_j)\hat{\phi}_n(\lambda_j)e^{-a\lambda_j^2}\}$. The Gaussian factor helps making the integrand decay faster. The end result is improved reconstructions, as shown in the next section.

We should remark that there remains much to be done. The main direction for future work will be to choose other bases such as perhaps wavelet bases or spline type of bases. The drawback of Haar bases is that their Fourier transforms decay slowly. Smoother bases will address this problem. However, we opted for Haar bases because their Fourier transforms can easily be computed, a property that most of the other type of bases do not have. It would be a challenge to find bases that combine both properties.

Finally, we remark that the scheme is very fast and easy to implement. On a Pentium III PC we typically obtain our results in real time with N up to $N = 3000$.

4. Examples

For all of examples in this section, 900 irregular Fourier samples $\hat{f}(\xi_i)$ are used, where ξ_i are randomly chosen from $[-450, 450]$, such that $\xi_i < \xi_{i+1}$, $0.8 < \xi_{i+1} - \xi_i < 1.2$. The resolutions, i.e., the number of elements in our Haar bases, are set at $N = 512$. The value $a = 0.0001$ is used for the Gaussian mollifier in all examples. We experimented with different values, and the scheme works fine for all $a \in [0.00005, 0.0001]$. Furthermore, it does not seem to be affected much by the resolution N. However, we have no theoretical explanation why it should be so.

Example 1. $f(x) = \sin 4\pi x,\ x \in [0, 1]$. Figure 4.1 is the reconstruction of $f(x)$ from the cubic spline interpolation without the Gaussian mollifier. Figure 4.2 is the reconstruction of $f(x)$ from the cubic spline interpolation with the Gaussian mollifier e^{-ax^2}. As one can see, there is no discernable difference between the two reconstructions and the original signal. This is due to the fact that $f(x)$ is continuous in $\mathbb{R}$.

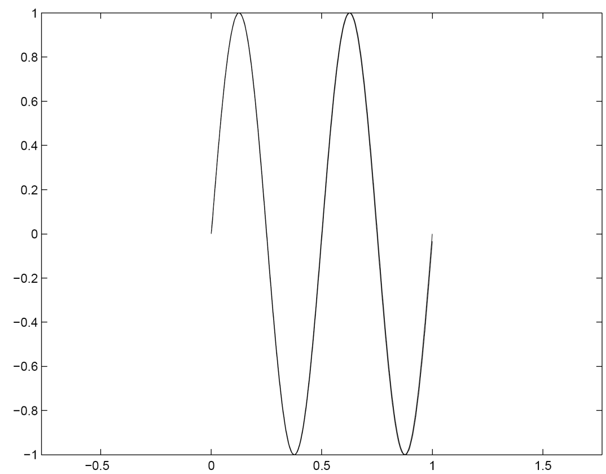

Figure 4.1

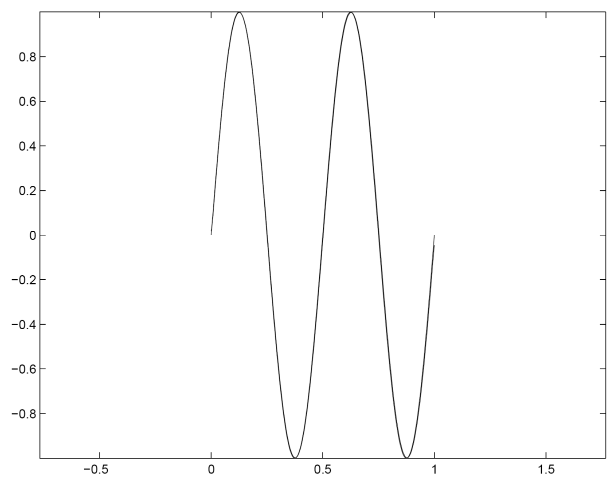

Figure 4.2

Example 2. Let $f(x) = x,\ x \in [0, 1]$. Note that this function has a jump at $x = 1$. Figure 4.3 is the reconstruction of $f(x)$ from the cubic spline interpolation without the Gaussian mollifier. One can see the Gibbs oscillation, particularly at $x = 0$. Figure 4.4 is the reconstruction of $f(x)$ from the cubic spline interpolation with the Gaussian mollifier e^{-ax^2}. There is no discernable difference between the original function and the reconstruction. The Gibbs oscillation is completely eliminated.

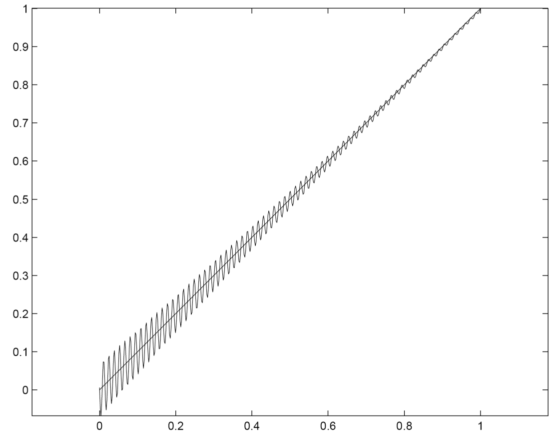

Figure 4.3

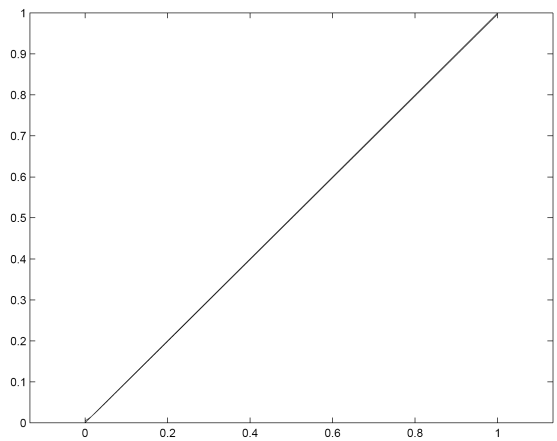

Figure 4.4

Example 3. $f(x) = \chi_{\left[\frac{1}{4}, \frac{3}{4}\right]}$, $x \in [0, 1]$. Figure 4.5 is the reconstruction of $f(x)$ from the Fourier expansion. 900 terms are used and 512 points are plotted. Figure 4.6 is the reconstruction of $f(x)$ from the cubic spline interpolation without a Gaussian mollifier. In both reconstructions there are pronounced Gibbs oscillations. Figure 4.7 is the reconstruction of $f(x)$ from the cubic spline interpolation with a Gaussian mollifier. Gibbs oscillation is eliminated. The tradeoff are “smoother” discontinuities. Despite this tradeoff we view this reconstruction to be far superior to the previous two.

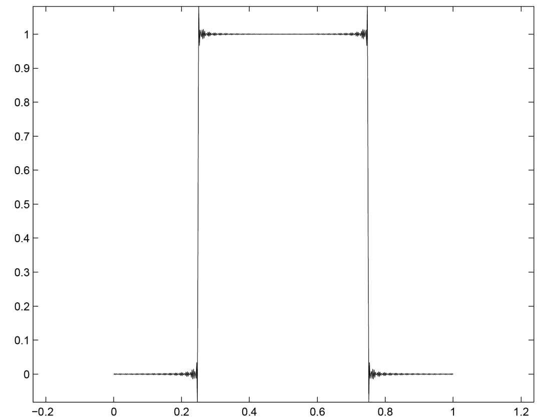

Figure 4.5

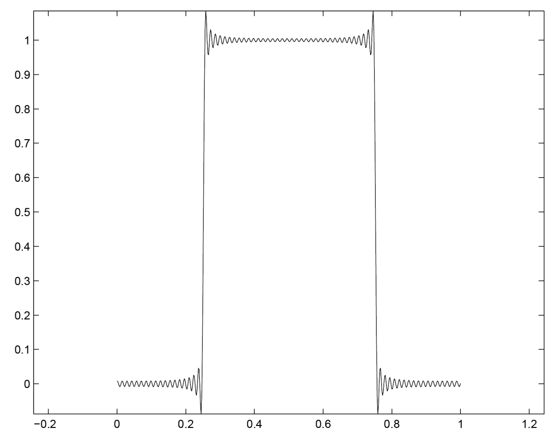

Figure 4.6

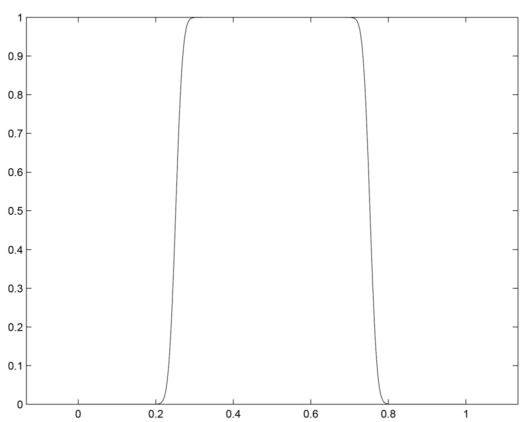

Figure 4.7

Example 4. This example $f(x)$ is created by using step functions and $\sin \pi x$. Figure 4.8 is a comparison between the original graph and the reconstruction of $f(x)$ from the cubic spline interpolation without the Gaussian mollifier. It is not a bad reconstruction but Gibbs oscillation is clearly present. Figure 4.9 is a comparison between the original graph and the reconstruction of $f(x)$ from the cubic spline interpolation with a Gaussian mollifier. It is clearly far superior, with Gibbs oscillation completely eliminated.

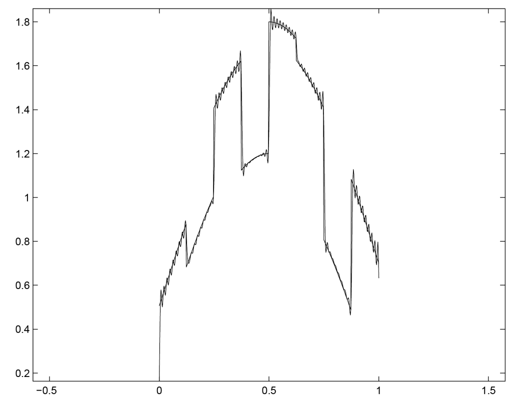

Figure 4.8

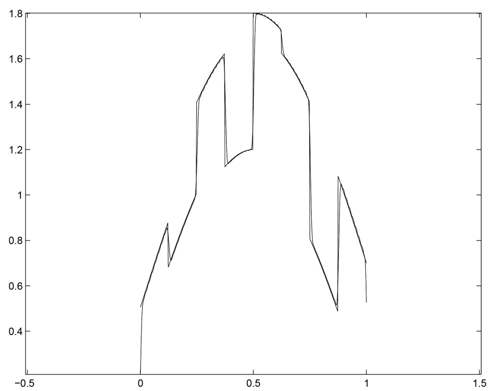

Figure 4.9

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