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IMPLIED VOLATILITY UNDER FRACTIONAL STOCHASTIC VOLATILITY IN BLACK-SCHOLES MODEL

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Abstract

In the original Black-Scholes model, the risky asset process is driven by a standard Brownian motion and the risk is quantified by a constant volatility parameter. The volatility that corresponds to actual market data for option prices in Black-Scholes model is called the implied volatility. Thus, if we may observe the market price of the option, then the implied volatility, that is, the volatility implied by the market price, can be determined by inverting the option formula.

A natural generalization is to model the constant volatility parameter by a stochastic process. There is precedent for the work where the risky asset process and the volatility-driving process are driven by standard Brownian motions. A typical situation is as follows: The risky asset process X is driven by a standard Brownian motion W and the volatility-driving process Y is driven by another standard Brownian motion $\hat{B}$ so that Y is a

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fast mean-reverting Ornstein-Uhlenbeck process, under the assumption that the standard Brownian motions W and $\hat{B}$ have constant correlation $\rho \in (-1, 1)$. For instance, Fouque et al. [7] consider such a situation, derive an approximation for option prices by a singular perturbation expansion and obtain the implied volatility by an approximating price. However, Fouque et al. [6], Lee [15] and Sircar and Papanicolaou [26] show the need for introducing also a slowly varying factor in the model for the stochastic volatility. The fast mean-reversion approximation is particularly suited for pricing long-dated options, whereas the slow mean-reversion approximation is particularly suited for pricing short-dated options.

Here we consider a Black-Scholes model where the risky asset process X is driven by a standard Brownian motion W and the volatility-driving process Y is driven by a fractional Brownian motion (fBm) B_H with arbitrary Hurst parameter $H \in (0, 1)$ so that Y is a mean-reverting fractional Ornstein-Uhlenbeck process (fOU process); we assume that W and B_H are independent, that is, volatility shocks are uncorrelated with asset-price shocks.

The rate of mean-reversion α of a mean-reverting fOU process Y is characterized in terms of $1/\varepsilon$ and δ with small positive parameters ε and δ according to fast scale and slow one, respectively. In each case, we obtain the corrected Black-Scholes price for European call option and hence asymptotic expansion for the implied volatility. In the case of fast scale, the corrected Black-Scholes price is derived by a singular perturbation analysis of the pricing partial differential equation as $\varepsilon \to 0$ and the asymptotic expansion for the implied volatility is obtained by a regular perturbation analysis as $\varepsilon \to 0$. On the other hand, in the case of slow scale, both the corrected Black-Scholes price and the asymptotic expansion for the implied volatility are derived by a regular perturbation analysis as $\delta \to 0$.

In order to obtain a pricing partial differential equation, we shall need to apply fractional Ito formula to the differential of the total value of the portfolio influenced by fBm B_H with arbitrary Hurst parameter $H \in (0, 1)$. For this purpose, we shall take the stochastic integral with respect to fBm B_H for algebraically integrable integrands in the sense of Hu [11] and hence obtain a concrete and computable expression for fractional Ito formula.

Our theorems correspond to an extension of the results in Fouque et al. [6, 7], Lee [15] and Sircar and Papanicolaou [26] to a Black-Scholes model with fOU process as volatility-driving process under the uncorrelated condition such that volatility shocks and asset-price shocks are independent.

1. Introduction

A one-dimensional *fractional Brownian motion* (*fBm*) with Hurst parameter $H \in (0, 1)$ is a Gaussian stochastic process with $B_H(0) = 0$ such that

$$E[B_H(t)] = 0, \quad E[B_H(t)B_H(s)] = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}$$

for all $s, t \in \mathbb{R}$. Here $E[\cdot]$ denotes the mathematical expectation with respect to the probability law μ_H for $B_H(\cdot)$.

The fBm $B_H(\cdot)$ is self-similar with self-similar index H, that is, for every $c > 0$, the process $\{B_H(ct); t \in \mathbb{R}\}$ is identical in distribution to $\{c^H B_H(t); t \in \mathbb{R}\}$. Since for $H \neq 1/2$, fBm $B_H(\cdot)$ is neither a Markov process, nor a semimartingale, usual stochastic calculus cannot be applied to the field of the network traffic analysis and mathematical finance. If $H = 1/2$, then $B_H(\cdot)$ is one-dimensional *standard Brownian motion* (*sBm*).

Let us consider the *Black-Scholes* (*BS*) model in a market with stochastic volatility driven by fBm $B_H(\cdot)$ with T, the time of maturity, where the price of a risk-less asset (a bank account or bond) $A(t)$ at time $t \in [0, T]$ and the price of a risky asset (a stock) $X(t)$ at time $t \in [0, T]$ are given by the following equations:

$$dA(t) = rA(t)dt, \quad A(0) = 1.$$

Here r represents the constant risk-less interest rate and hence $A(t) = e^{rt}$.

$$dX(t) = \mu X(t)dt + \sigma(t)X(t)dW(t), \tag{1.1}$$

$$\sigma(t) = f(Y(t)), \tag{1.2}$$

$$dY(t) = \alpha(m - Y(t))dt + \beta dB_H(t) \tag{1.3}$$

with constants $\mu(> r)$, $m > 0$, $\alpha > 0$ and $\beta > 0$, where f is positive suitably regular function. The factor $(Y(t))$ is called the *volatility-driving process*.

Assumption 1.1. We assume the following:

(i) $(W(t))$ is a one-dimensional standard Brownian motion (sBm).

(ii) $(B_H(t))$ is a one-dimensional fractional Brownian motion (fBm) with Hurst parameter H. Throughout this paper, let H be arbitrary in $(0, 1)$ and fixed.

(iii) $(W(t))$ and $(B_H(t))$ are independent.

(iv) $f : \mathbb{R} \to \mathbb{R}$ is continuous.

The process $(Y(t))$ is a mean-reverting *fractional Ornstein-Uhlenbeck process* (fOU process); α measures the characteristic speed of *mean-reversion* of $(Y(t))$.

In our case, there is one risky asset X and two random sources W and B_H. Namely, there are two sources of randomness instead of one as in the classical BS model, and hence the market is *incomplete* and martingale measures are not unique (see Hu [12] for the case, where $1/2 < H < 1$).

Remark 1.2 (European call option)**.** A *European call option* is a contract that gives the right (but not the obligation) to buy at time T (the maturity) a stock at price K (the strike or exercise price), which is fixed when the contract is signed. If $X(T) > K,$ then the option enables its owner to buy the asset at price K and then sell it immediately at price $X(T);$ the *payoff*, that is, the difference $X(T) - K$ between the two prices is realized gain. If $X(T) \leq K,$ then the gain is zero. For example, we can express the payoff $F(\omega)$ at time T of a European call option in BS model by

$$F(\omega) = h(X(T, \omega)) \text{ with } h(X) = (X - K)^+ := \max\{X - K, 0\}.$$

Remark 1.3 (Black-Scholes price C_{BS})**.** In the classical BS model, the risk is quantified by a constant volatility parameter σ. Denote by $P(t, X)$ the European call price with time $t \in [0, T]$ and the current stock price $X = X(t).$ Then $P(t, X)$ is found as the solution of the BS equation, that is, the partial differential equation (PDE) of the second order. We denote the price function P by $C_{BS};$ this will be given by (6.1) in Section 6. We denote C_{BS} by $C_{BS}(\sigma),$ emphasizing the dependence on σ. Further, we denote C_{BS} by $C_{BS}(t, X; K, T; \sigma),$ emphasizing the dependence on K, T and σ.

Remark 1.4 (Fast scale volatility factor)**.** When volatility shocks are uncorrelated with stock-price shocks, Narita [17, 18] extends that of Fouque et al. [7] to the case where the volatility-driving process can be driven by fBm $B_H(t)$ with arbitrary Hurst parameter $H \in (0, 1)$, rather than only sBm, and obtains the corrected price of European call option of the BS model. In this case, the rate of mean-reversion α of the volatility-driving process $(Y(t))$ is fast as follows:

$$\alpha = 1/\varepsilon \quad \text{and} \quad \beta = O(1/\varepsilon^H), \quad 0 < \varepsilon << 1.$$

In the case of fast scale, let P^ε denote the corrected price of a European call option. Then P^ε is expanded in power of $\sqrt{\varepsilon}$, such as

$$P^\varepsilon = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon}P_3 + \cdots$$

for small ε. Moreover,

$$P^\varepsilon \approx P_0 + \tilde{P}_1$$

for small ε. Here P_0 is the solution of the classical BS equation with *effective constant volatility* $\bar{\sigma}$, that is, $P_0 = C_{BS}(\bar{\sigma})$, where $\bar{\sigma}^2$ is the quadratic average of volatility with respect to the invariant distribution of the volatility-driving process $(Y(t))$; i.e., $\bar{\sigma}^2 = \langle f^2 \rangle$. The first correction $\tilde{P}_1$ is given by $\tilde{P}_1 = \sqrt{\varepsilon}P_1$ with $P_1 = P_1(t, X)$, and $\tilde{P}_1$ is the solution of PDE of the Black-Scholes type. These are obtained by a singular perturbation expansion with respect to ε.

We shall revisit the results above in Section 5 (Lemma 5.2 and Theorem 5.3) in order to compute the implied volatility.

Remark 1.5 (Implied volatility)**.** The *implied volatility* is the volatility parameter implied by the market price which can be determined by inverting the option formula. More precisely, given a time-*t* asset price X and observed option price, C_{obs}, the implied volatility is defined as the I that solves

$$C_{BS}(t, X; K, T; I) = C_{obs},$$

where C_{BS} is the Black-Scholes price. We refer to Lee [14, 15] and the references therein for the implied volatility.

Remark 1.6 (Slow scale volatility factor)**.** If the rate of mean-reversion is slow, then the volatility-driving process $(Y(t))$ can be described by

$$dY(t) = \delta\alpha(m - Y(t))dt + \delta^H \beta dB_H(t) \tag{1.4}$$

depending on a small parameter $\delta > 0.$ The motivation for the specification (1.4) is that $Y(t) = \tilde{Y}(\delta t),$ where

$$d\tilde{Y}(t) = \alpha(m - \tilde{Y}(t))dt + \beta d\tilde{B}_H(t)$$

for fBm $\tilde{B}_H(t)$ with Hurst parameter $H \in (0, 1).$ We notice that $\tilde{B}_H(\delta t)$ is identical in law with $\delta^H \tilde{B}_H(t).$ Thus, Y can be viewed as $\tilde{Y}$ modified to run on a time scale slower by a factor of δ. We also notice that (1.4) is formally obtained by (1.3) with α and β replaced by $\delta\alpha$ and $\delta^H\beta,$ respectively.

Fouque et al. [7] develop a theory of pricing, hedging and implied volatility under rapidly mean-reverting stochastic volatility in sBm environment. Following Sircar and Papanicolaou [26] (S-P, for short), Lee [15, Section 5] assumes that volatility varies slowly in time in sBm environment, where $Y(t)$ is described by (1.4) with $H = 1/2,$ reviews S-P's calculation of an asymptotic expansion for implied volatility and hence extends it to the next order.

Remark 1.7 (Slow variation versus rapid variation)**.** A given volatility process can be said to *vary rapidly* or *slowly*, depending on the time horizon in the application at hand. Lee [15, Section 5.5] comments on the differences, and suggests how these two lines of development can be reconciled as follows: If the goal is to price a sufficiently long-dated option, then the volatility process will appear to vary rapidly; it will have many fluctuations over the long lifetime of the option. If, however, the goal is to price a sufficiently short-dated option, then that same volatility process will not have much time to vary, and thus it will appear to vary slowly, relative to the option's lifetime. It follows that the fast-mean-reversion model in Fouque et al. [7] is best suited to long-dated options, and the slow variation model to short-dated options.

(i) The first purpose of this paper is to obtain the asymptotics for the implied volatility when the volatility-driving process $(Y(t))$ is fast-mean-reverting and described by (1.3) with arbitrary Hurst parameter $H \in (0, 1).$ By a regular perturbation expansion, we shall obtain as follows: Up to an error of order $O(\varepsilon)$, where $\varepsilon = 1/\alpha$, the implied volatility I is expanded around effective constant volatility $\overline{\sigma}$ such that

$$I = \overline{\sigma} + \sqrt{\varepsilon} I_1 + O(\varepsilon).$$

Here $\sqrt{\varepsilon} I_1 = -V_2/\overline{\sigma}$ with a small coefficient V_2 (Theorem 6.1).

(ii) The second purpose of this paper is to obtain the corrected price of European call option of the BS model when the volatility-driving process $(Y(t))$ is slow-mean-reverting and described by (1.4) with arbitrary Hurst parameter $H \in (0, 1)$. By a regular perturbation expansion with respect to δ, we shall obtain the corrected price P^{δ} as follows: For δ small enough,

$$P^{\delta} \approx P_0 + \sqrt{\delta} P_1 + \delta P_2.$$

Here $P_0(t, X, y)$ is the Black-Scholes price of the claim at the volatility level $I_0 = f(y)$, i.e., $P_0 = C_{BS}(I_0)$, under the assumption that $f(y) > 0$. The first correction $P_1(t, X, y)$ and the second correction $P_2(t, X, y)$ are the solutions of PDEs of the Black-Scholes type. These are given in terms of $f(y)$, $\gamma(y)$, $\frac{\partial C_{BS}}{\partial \sigma}(I_0)$ and $\frac{\partial^2 C_{BS}}{\partial \sigma^2}(I_0)$, i.e., the volatility-driving function, the market price of risk, the Vega and the DVegaDvol in the Greeks, in that order (Theorem 7.2).

(iii) The final purpose of this paper is to obtain the asymptotics for the implied volatility when the volatility-driving process $(Y(t))$ is slow-mean-reverting and described by (1.4) with arbitrary Hurst parameter $H \in (0, 1)$. By a regular perturbation expansion with respect to δ, we shall obtain the implied volatility I^{δ} as follows: For δ small enough,

$$I \approx I_0 + \sqrt{\delta} I_1 + \delta I_2.$$

Here $I_0 = f(y) > 0$, $I_1 = -(T - t)\alpha^{\frac{1}{2} - H} \beta\gamma(y) f'(y)$ with $\gamma(y)$ the market price of risk. Further, I_2 is given in terms of $f(y)$, $\gamma(y)$, the Vega $\frac{\partial C_{BS}}{\partial \sigma}(I_0)$ and the DVegaDVol $\frac{\partial^2 C_{BS}}{\partial \sigma^2}(I_0)$ in the Greeks (Theorem 8.1).

The model of fast mean-reverting stochastic volatility under sBm environment is investigated by many authors; we refer to Fouque et al. [5-8] and the references therein. A typical assumption in their works is that volatility shocks are correlated with asset-price shocks. Namely, instead of (1.3), the volatility-driving process $(Y(t))$ is given by the following equation:

$$dY(t) = \alpha(m - Y(t))dt + \beta d\hat{B}(t),$$

$$\hat{B}(t) = \rho W(t) + \sqrt{1 - \rho^2} B(t),$$

where $(W(t))$ and $(B(t))$ are independent sBms, and ρ is the correlation between price and volatility shocks with $\rho \in (-1, 1)$. We notice that sBms $(W(t))$ and $(B(t))$ are uncorrelated but sBms $(W(t))$ and $(\hat{B}(t))$ are correlated; $d\langle W, \hat{B}\rangle(t) = \rho dt$. In this case, Fouque et al. [7] obtain the corrected price of the European call option and hence show the asymptotics of the implied volatility as given by an affine function form as follows:

$$I = a\left[\frac{\log\left(\frac{\text{strike price}}{\text{asset price}}\right)}{\text{time to maturity}}\right] + b + O(1/\alpha).$$

Here the parameters a and b are estimated as the slope and intercept of the linefit. That is, if C_{obs} is the stochastic volatility call option price with payoff function $h(X) = (X - K)^+$, then I defined by

$$C_{BS}(I) = C_{obs},$$

where C_{BS} is the Black-Scholes formula, is given by

$$I = a\left[\frac{\log\left(\frac{K}{X}\right)}{(T - t)}\right] + b + O(1/\alpha).$$

Moreover, Fouque et al. [6] generalize the one presented in Fouque et al. [5] where only the fast scale factor is considered, and then introduce the multiscale stochastic volatility model with fast and slow scale volatility factors. Fouque et al. [6] combine a singular perturbation expansion with respect to the fast scale with a regular perturbation with respect to the slow scale, obtain a leading order term which is the Black-Scholes price with an effective constant volatility, and hence show a simple and accurate parametrization of the implied volatility surface. Their model is as follows:

$$dX(t) = \mu X(t)dt + f(Y(t), Z(t))X(t)dW^{(0)}(t), \tag{1.5}$$

$$dY(t) = \frac{1}{\varepsilon}(m - Y(t))dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}dW^{(1)}(t), \tag{1.6}$$

$$dZ(t) = \delta c(Z(t))dt + \sqrt{\delta}g(Z(t))dW^{(2)}(t), \tag{1.7}$$

depending on small parameters ε and δ; $0 < \varepsilon, \ \delta << 1$. Here m and ν are positive constants which relate to the invariant distribution of $(Y(t))$, such as $Y(t) \sim N(m, \nu^2)$ in the long-run distribution. The function $f(y, z)$ is a smooth positive function that is bounded and bounded away from zero. The volatility process $\sigma(t)$ is driven by two diffusion processes $Y(t)$ and $Z(t)$:

$$\sigma(t) = f(Y(t), Z(t)).$$

The coefficients $c(z)$ and $g(z)$ are smooth and at most linearly growing at infinity. In (1.5), (1.6) and (1.7), $W^{(0)}$ and $W^{(1)}$ are standard Brownian motions such that they have constant correlation $\rho_1 \in (-1, 1)$, i.e., $d\langle W^{(0)}, W^{(1)}\rangle(t) = \rho_1 dt$, and $W^{(2)}$ is another Brownian motion. Here a general correlation structure between the three standard Brownian motions $W^{(0)}$, $W^{(1)}$ and $W^{(2)}$ is given as follows:

$$\begin{pmatrix} W^{(0)}(t) \\ W^{(1)}(t) \\ W^{(2)}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \tilde{\rho}_{12} & \sqrt{1-\rho_2^2-\tilde{\rho}_{12}^2} \end{pmatrix} \mathbf{W}(t), \tag{1.8}$$

where $\mathbf{W}(t)$ is a standard three-dimensional Brownian motion, and where the constant coefficients ρ_1, ρ_2 and $\tilde{\rho}_{12}$ satisfy $|\rho_1| < 1$ and $\rho_2^2 + \tilde{\rho}_{12}^2 < 1$. Observe that with this parametrization, the covariation between $W^{(1)}(t)$ and $W^{(2)}(t)$ is given by $t\rho_{12}$, where $\rho_1 := \rho_1\rho_2 + \tilde{\rho}_{12}\sqrt{1-\rho_1^2}$. However, only the two parameters ρ_1 and ρ_2 will play an explicit role in the correction derived from the asymptotic analysis in Fouque et al. [6].

Remark 1.8 (Multiscale and perturbation)**.** Note that the slow factor in the volatility model corresponds to a small perturbation and the resulting regular perturbation scenario has been considered in many different settings. The fast factor on the other hand leads to a singular perturbation situation and gives rise to a diffusion homogenization problem that is not so widely applied. The model described by (1.5), (1.6) and (1.7) is the correction to the model given in Fouque et al. [7], where $\rho_2 = \rho_{12} = 0.$ Fouque et al. [6] obtain the asymptotics of the price $P^{\varepsilon, \delta}$ of a European call option with strike K, maturity T and payoff h, which is produced by the model described by (1.5), (1.6) and (1.7), and hence find an expansion for the corresponding implied volatility such that

$$I = I_0 + I_1^{\varepsilon} + I_1^{\delta} + \cdots.$$

Here $I_0 = \bar{\sigma}(z)$ and $\bar{\sigma}^2(z) = \langle f^2(\cdot, z)\rangle,$ i.e., the quadratic average of volatility with respect to the invariant distribution of the volatility-driving process $(Y(t)),$ which depends on the slow factor z, and I_1^{ε} (respectively, I_1^{δ}) is proportional to $\sqrt{\varepsilon}$ (respectively, δ).

In (1.1)-(1.4), we assume that sBm $(W(t))$ and fBm $(B_H(t))$ are independent; each of them governs risky asset price $(X(t))$ and mean-reverting fOU process $(Y(t)),$ respectively.

Our model described by (1.1), (1.2) and (1.3) (resp. (1.4)) corresponds to the multiscale model described by (1.5), (1.6) and (1.7) with $f(Y(t), Z(t))$ replaced by $f(Y(t)),$ where $Y(t)$ stands for the fast (resp. slow) mean-reverting fOU process driven by fBm $B_H(t)$ with arbitrary Hurst parameter $H \in (0, 1).$

In particular, our result in the slow scale model described by (1.1), (1.2) and (1.4) extends that in Lee [15, Section 5] to a slowly varying volatility model driven by fBm $B_H(t)$ with arbitrary Hurst parameter $H \in (0, 1).$

In order to proceed to asymptotic analysis for corrected price and implied volatility, we shall prepare for stochastic integral (Section 2), Ito formula and fOU process (Section 3), pricing PDE (Section 4) and fast scale (Section 5).

2. Stochastic Integral

Here we begin to introduce stochastic integration theory. For given $H \in (1/2, 1)$, define $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ by

$$\phi(s, t) := H(2H-1)|s-t|^{2H-2}, \quad s, t \in \mathbb{R}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable such that

$$|f|_\phi^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s, t) ds dt < \infty.$$

Then the stochastic integral with respect to fBm B_H is well defined to be a Gaussian random variable. It follows from Gripenberg and Norros [9] and Nualart [23] that for any deterministic integrand $f \in L^2(\mathbb{R}, \mathbb{R}) \cap L^1(\mathbb{R}, \mathbb{R})$,

$$E\left[\int_0^\infty f(t) dB_H(t)\right] = 0,$$

$$E\left[\left(\int_0^\infty f(t) dB_H(t)\right)^2\right] = \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s, t) ds dt.$$

In this paper, we follow the stochastic integration theory with respect to fBm B_H by Hu [11, Chapters 6-7]; Hu [11] extends the integral above to the general integrands under arbitrary Hurst parameter $H \in (0, 1)$.

Remark 2.1 (Pathwise integral)**.** There are several definitions of stochastic integrals for general integrands with respect to fBm B_H. One of them is the fractional *pathwise integral* which is taken by the limit of the usual Riemann sum as defined using pointwise products. However, this integral does not have expectation zero. Further, Rogers [25] shows that *arbitrage* is possible when the risky asset has a log-normal price driven by a fBm if stochastic integrals are defined using pointwise product.

Remark 2.2 (Wick-Ito integral)**.** In the white noise approach, the *Wick product* is used instead of the ordinary product in the Riemann sums in order to define the stochastic integrals. The Wick product for F and G is written by $F \lozenge G$; here commutative law, associative law and distributive law hold. If at least one of F and

G is deterministic, e.g., $F = a_0 \in \mathbb{R}$, then the Wick product coincides with the ordinary product in the deterministic case. Such an integral is called *Wick-Ito integral* or *fractional Ito-integral*. By the method of the Wick-Ito integral, Duncan et al. [3] and Hu and Øksendal [13] obtain *no-arbitrage* property and fractional Black-Scholes formula for pricing of European call option in financial market where the risky asset is driven by fBm B_H with Hurst parameter $H \in (1/2, 1)$. Elliott and Van Der Hoek [4] extend the preceding results to the case of arbitrary Hurst parameter $H \in (0, 1)$. We can refer to Holden et al. [10] for Wick calculus.

In BS model, a risky asset is often formulated by a *geometric Brownian motion* (*gBm*) which is a solution of linear stochastic differential equation (SDE). For an application of Wick calculus to option pricing, for instance, we can refer to Necula [22] and Narita [19]; here risky asset is formulated by a fractional gBm which is a solution of SDE driven by fBm B_H with Hurst parameter $H \in (1/2, 1)$. For the details of an application of the Wick calculus to SDEs, we can also refer to Biagini et al. [1, 2], Holden et al. [10], Narita [20, 21] and the references therein.

Remark 2.3 (Fractional calculus)**.** Another definition of stochastic integrals with respect to fBm B_H for general integrands is given by *fractional calculus* for arbitrary Hurst parameter $H \in (0, 1)$. In this case, the stochastic integration theory is based on both the left- and right-sided Riemann-Liouville *fractional integral* and the left- and right-sided Riemann-Liouville *fractional derivative*. A risky asset in BS model can be formulated by a fractional gBm which is a solution of SDE driven by fBm B_H under fractional calculus. We can refer to Mishura [16], Nualart [23] and the references therein for the existence of pathwise solutions and the uniqueness in law for SDEs driven by fBm B_H.

In general, quadratic variations of stochastic integrals with respect to fBm B_H for general integrands have abstract and complicated expression, and hence there is difficulty in application of Ito formula.

Remark 2.4 (Stochastic integral in the sense of Hu [11])**.** In this paper, we take the stochastic integral in the sense of Hu [11, Chapters 6-7] (Hu integral, for short). This is the stochastic integral with respect to B_H $(H \in (0, 1))$ for *algebraically integrable* integrands; the integration theory is developed by using Wiener chaos expansion and an idea of creation operator from quantum field theory. If $1/2 < H < 1$, then the Hu integral coincides with the Wick-Ito integral in the sense of Duncan et al. [3]. If $0 < H < 1$, then the Hu integral for deterministic integrands coincides with the stochastic integral of variation in the sense of Hu [11, Definition 6.11] and Nualart-Pardoux [24].

The Hu integral with respect to fBm B_H $(H \in (0, 1))$ has expectation zero and can be concretely evaluated in the case of deterministic integrands. This enables us to apply Ito formula to linear SDE of the form

$$dX(t) = a(t)X(t)dt + b(t)X(t)dB_H(t) \ (H \in (0, 1))$$

with deterministic coefficients $a(t)$ and $b(t)$. Therefore, we can obtain an explicit formula for the solution $X(t)$ of the SDE above and hence derive a pricing PDE.

By reason of above mentioned background, in this paper, we adopt the Hu-integral, derive an applicable Ito formula, compute financial derivatives and hence obtain the corrected Black-Scholes price.

For consideration of implied volatility, we shall need understanding of derivation of the corrected price formula for European call option in a market with fOU process $(Y(t))$ as the fast- and slow-mean-reverting volatility factor. Hence, we shall make preparations for fOU process and asymptotics for option pricing in the following Sections 3 and 4, respectively. We can refer to Fouque et al. [7, Chapters 2 and 5] and Narita [17, Sections 8 and 9] for the details of option pricing in a market driven by sBm and fBm, respectively.

3. Ito Formula and fOU Process

Let $\Theta_H = \Theta_H([0, T])$ be the Hilbert space as defined in Hu [11, Chapter 5]; Θ_H is the space of integrands associated with the induced transformation of representation for fBm $B_H(t)$. Let $f(s)$ be given over $[0, T]$. Let $0 \le s \le t \le T$. Then, considering the functions $f_t(s)$ restricted to $[0, t]$, that is, $f_t(s) = f(s)\chi_{[0,t]}(s)$, we shall use $\Theta_{H,t}$ to denote $\Theta_H([0, t])$, where the norm $\| f_t \|_{\Theta_{H,t}}$ is well defined. According to Hu [11, pp. 102-103], we summarize expression for $\| f_t \|_{\Theta_{H,t}}$ as follows:

(i) Let $H > 1/2$. Then

$$\| f_t \|^2_{\Theta_{H,t}} = H(2H-1)\int_0^t\int_0^t | v-u |^{2H-2} f(u) f(v) du dv.$$

(ii) Let $0 < H < 1/2$ and let f be continuously differentiable. Then

$$\| f_t \|^2_{\Theta_{H,t}} = Hf(0)\int_0^t v^{2H-1} f(v) dv + Hf(t)\int_0^t | t-v |^{2H-1} f(v) dv$$

$$+ H\int_0^t\int_0^t | v-u |^{2H-1}\operatorname{sign}(v-u) f'(u) f(v) du dv.$$

For, example, if $f \equiv 1$, then $\| f_t \|_{\Theta_{H,t}} = t^H$, and hence $\frac{d}{dt}\| f_t \|_{\Theta_{H,t}} = Ht^{H-1}$.

Hu [11, p. 103] shows Ito formula for general deterministic f and Hurst parameter $H \in (0, 1)$ as follows:

Theorem 3.1 (Ito formula)**.** *Let* $0 < H < 1$ *and let* $f \in \Theta_{H,T} \cap L^2([0, T])$ *be a deterministic function. Denote* $f_t(s) = f(s)_{\chi[0,t]}(s)$, $0 \le s \le t \le T$. *Suppose that* $f_t \in \Theta_{H,t}$ *and* $\| f_t \|_{\Theta_{H,t}}$ *is continuously differentiable as a function of* $t \in [0, T]$. *Denote*

$$X(t) = X(0) + \int_0^t g(s) ds + \int_0^t f(s) dB_H(s), \quad 0 \le t \le T, \tag{3.1}$$

where $X(0)$ *is a constant, g is deterministic with* $\int_0^T | g(s) | ds < \infty$. *Let F be an entire function of order less than* 2. *Namely,*

$$M_f(r) := \sup_{|z|=r} | f(z) | < Ce^{A_r K} \text{ for all } r,$$

where K is a positive number less than 2 *and C is a constant. Then*

$$F(t, X(t)) = F(0, X(0)) + \int_0^t \frac{\partial F}{\partial s}(s, X(s)) ds + \int_0^t \frac{\partial F}{\partial x}(s, X(s)) dX(s)$$

$$+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X(s))\left[\frac{d}{ds}\| f_s \|^2_{\Theta_{H,s}}\right] ds, \quad 0 \le t \le T. \tag{3.2}$$

Here the stochastic integral in (3.2) *is in the sense of the Hu integral.*

Equation (3.2) is rewritten by the stochastic differentials as follows:

$$dF(t,\ X(t)) = \frac{\partial F}{\partial t}(t,\ X(t))dt + \frac{\partial F}{\partial x}(t,\ X(t))dX(t)$$

$$+ \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,\ X(t))\left[\frac{d}{dt}\| f_t \|^2_{\Theta_{H,t}}\right]dt.$$

Let $a(t)$ and $b(t)$ be bounded measurable functions of $t \in [0, T]$. Let $(B_H(t), 0 \le t \le T)$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Then we first consider the fractional gBm governed by the following linear SDE:

$$dX(t) = a(t)X(t)dt + b(t)X(t)dB_H(t), \quad 0 \le t \le T,$$

$$X(0) = x \in \mathbb{R}. \tag{3.3}$$

Let $H > 1/2$. Then, Biagini et al. [1], Hu and Øksendal [13] and Narita [20, 21] obtain the existence and uniqueness of the solution of (3.3) when the stochastic integral is in the sense of the Wick-Ito integral. We notice that if $H > 1/2$, then the Wick-Ito integral coincides with the Hu integral.

The following theorem is due to Hu [11, Chapter 14].

Theorem 3.2 (Expression for solution of linear SDE)**.** *Let* $0 < H < 1$. *Let* $a(t)$ *and* $b(t)$ *be continuous functions of* $t \in [0, T]$. *If* $0 < H < 1/2$, *then we assume that* $(b(t), 0 \le t \le T)$ *is continuously differentiable. Then equation* (3.3) *has a unique solution such that*

$$X(t) = x\exp\left\{\int_0^t a(s)ds + \int_0^t b(s)dB_H(s) - \frac{1}{2}\| b_t \|^2_{\Theta_{H,t}}\right\}, \tag{3.4}$$

where $b_t(u) = b(u)_{\chi[0,t]}(u),\ 0 \le u \le t \le T$.

We next consider the solution of SDE (1.3) with the initial state $Y(0) = y_0 \in \mathbb{R}$. Define $x(t)$ by

$$x(t) = \int_0^t e^{\alpha s}(\alpha m)ds + \int_0^t e^{\alpha s}\beta dB_H(s) + y_0.$$

Apply Theorem 3.1 to the process $x(t)$ and the function $F(t, x) = e^{-\alpha t}x$. Then, we obtain that the process $Y(t) := e^{-\alpha t}x(t)$ is the pathwise unique solution of (1.3) with the following lemma (Narita [17, Lemmas 8.1 and 8.8]):

Lemma 3.3 (Property of fractional OU process)**.** *Let* $0 < H < 1$. *Let* $Y(t)$ *be the fractional OU process given by SDE* (1.3) *with the initial state* $Y(0) = y_0 \in \mathbb{R}$. *Then* $Y(t)$ *is the pathwise unique solution of* (1.3) *with the following form*:

$$Y(t) = m + e^{-\alpha t}(y_0 - m) + \beta e^{-\alpha t}\int_0^t e^{\alpha s} dB_H(s). \tag{3.5}$$

Further, $Y(t)$ *is a Gaussian stochastic process and has the long-run distribution which is the normal distribution* $N(m, \nu_H^2)$ *with mean m and variance* ν_H^2 *such that the density is given by*

$$n(y) = \frac{1}{\sqrt{2\pi\nu_H^2}} \exp\left(-\frac{(y-m)^2}{2\nu_H^2}\right), \tag{3.6}$$

where

$$\nu_H^2 = \beta^2 H\left(\frac{1}{\alpha}\right)^{2H} \Gamma(2H), \tag{3.7}$$

and $\Gamma(\cdot)$ *is the Gamma function, i.e.,* $\Gamma(x) = \int_0^\infty e^{-\xi}\xi^{x-1}d\xi$.

4. Pricing PDE

Under Assumption 1.1, we consider the BS model described by (1.1), (1.2) and (1.3). In this case, there are one risky asset *X* and two random sources *W* and B_H. Namely, there are two sources of randomness instead of one as in the classical BS model. When constructing a portfolio, the derivatives cannot be perfectly hedged with just the underlying asset. Instead, we also need a benchmark derivative called *G*. A risk-less portfolio Π is formed, containing the quantity $-\Delta_X$ of the underlying asset *X*, the quantity $-\Delta_G$ of another traded asset *G* (benchmark option) and the priced derivative, whose value we denote by $P(t, X, y)$. The total value of the portfolio is

$$\Pi = P - \Delta_X X - \Delta_G G. \tag{4.1}$$

The differential of the portfolio value is needed to construct a risk-less and no-arbitrage, satisfying

$$d\Pi = dP - \Delta_X dX - \Delta_G dG. \tag{4.2}$$

Then, the classical Ito formula and the fractional one are applied to dP and dG, respectively, to obtain the stochastic differential $d\Pi$. Collecting the dX and dY, we get

$$d\Pi = \left[\frac{\partial P}{\partial X} - \Delta_G \frac{\partial G}{\partial X} - \Delta_X\right] dX + \left[\frac{\partial P}{\partial y} - \Delta_G \frac{\partial G}{\partial y}\right] dY + LPdt - \Delta_G LGdt.$$

Here L is the operator defined by

$$L = \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 X^2 \frac{\partial^2}{\partial X^2} + \frac{1}{2} e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]\frac{\partial^2}{\partial y^2}, \tag{4.3}$$

where

$$g_t(s) = g(s)\chi_{[0,t]}(s), \quad g(s) = e^{\alpha s}\beta, \quad 0 \le s \le t \le T.$$

We can refer to Remark 4.2 for concrete expression for (4.3).

We want this portfolio to be risk-less by eliminating the coefficients in front of dX and dY. This yields a linear equation with respect to Δ_G and Δ_X, which is solved as follows:

$$\Delta_G = \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial G}{\partial y}\right)^{-1}, \tag{4.4}$$

$$\Delta_X = \frac{\partial P}{\partial X} - \frac{\partial G}{\partial X}\left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial G}{\partial y}\right)^{-1}. \tag{4.5}$$

Thus, if the portfolio is well-balanced according to (4.4) and (4.5), then the risk is eliminated. Moreover, we want $\Pi(t)$ to be risk-less with instantaneous interest rate r, and hence the avoidance of arbitrage is the following condition:

$$d\Pi = r\Pi dt = r(P - \Delta_X X - \Delta_G G)dt. \tag{4.6}$$

In (4.6), if the risk is eliminated, then

$$d\Pi = LPdt - \Delta_G LGdt.$$

Substitute expressions (4.4) and (4.5) for Δ_G and Δ_X into equation (4.6). Further, collect all P terms on the left-hand side and all G terms on the right-hand side. Then we get the following equation:

$$(\tilde{L}P)\left(\frac{\partial P}{\partial y}\right)^{-1} = (\tilde{L}G)\left(\frac{\partial G}{\partial y}\right)^{-1}, \tag{4.7}$$

where $\tilde{L} = L + r\left(X\frac{\partial}{\partial X} - \cdot\right)$ with the operator L as defined by (4.3), i.e.,

$$\tilde{L} = \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2X^2\frac{\partial^2}{\partial X^2} + \frac{1}{2}e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]\frac{\partial^2}{\partial y^2} + r\left(X\frac{\partial}{\partial X} - \cdot\right).$$

In (4.7), the left-hand side is a function of P only and the right-hand side is a function of G only, and hence both sides of this equation are equal to some function depending only on t, X and y. Thus, we write both sides as $-k(t, X, y)$, where k is the real-world drift term less the market price of risk. This results in the following PDE:

$$(\tilde{L}P)\left(\frac{\partial P}{\partial y}\right)^{-1} = -k(t, X, y). \tag{4.8}$$

Hereafter, without loss of generality, the arbitrary function k will be given as

$$k(t, X, y) = \alpha(m - y) - \beta\varphi(t, X, y; H)$$

with a function φ characterized by the market price of volatility risk and the Hurst parameter $H \in (0, 1)$ (see (4.13) and (4.14)).

Equation (4.8) yields the following PDE (Narita [17, Lemma 9.1]):

Lemma 4.1 (Pricing PDE)**.** *The equation governing P can be written as*

$$\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial X^2}f(y)^2X^2 + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] - rP + rX\frac{\partial P}{\partial X}$$

$$= -k(t, X, y)\frac{\partial P}{\partial y} \tag{4.9}$$

with a suitable function k of variables t, X and y. Here $g_t(u) = g(u)\chi_{[0,t]}(u)$, $g(u) = e^{\alpha u}\beta$, $0 \le u \le t \le T$, and $\| g_t \|_{\Theta_{H,t}}$ denotes the norm of the function g_t in a Hilbert space $\Theta_{H,t}$ as defined by Hu [11, *Chapter* 5]. *The terminal condition for P is the contract function $h(X)$, i.e., $P(T, X, y) = h(X(T))$; for example, $h(X(T)) = (X(T) - K)^+$ with T and K, the time of maturity and the strike price, respectively.*

Remark 4.2. If $g_t(u) = g(u)\chi_{[0,t]}(u)$ and $g(u) = e^{\alpha u}\beta$ for $0 \le u \le t \le T$, then the explicit form of

$$e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]$$

is given by Narita [17, Lemma 8.7] as follows:

(i) If $H > 1/2$, then

$$e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] = 2\beta^2 H\left\{e^{-\alpha t}t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1}B(\alpha t)\right\}. \tag{4.10}$$

(ii) If $0 < H < 1/2$, then

$$\begin{aligned} e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] &= H\beta^2 t^{2H-1}e^{-\alpha t} \\ &+ 2H^2\beta^2 t^{-1}\left(\frac{1}{\alpha}\right)^{2H} B(\alpha t) \\ &+ H\alpha\beta^2 t^{-1}\left(\frac{1}{\alpha}\right)^{2H}\left\{tB(\alpha t) - \left(\frac{1}{\alpha}\right)C(\alpha t)\right\} \\ &+ H\alpha\beta^2\left(\frac{1}{\alpha}\right)^{2H} B(\alpha t). \end{aligned} \tag{4.11}$$

Here

$$B(x) = \int_0^x z^{2H-1}e^{-z}dz, \quad C(x) = \int_0^x z^{2H}e^{-z}dz. \tag{4.12}$$

In Lemma 4.1, the function k cannot be determined by arbitrage theory alone. However, it is completely determined in terms of the traded benchmark asset G. We can say that the market knows the function k. In our model described by (1.1), (1.2) and (1.3), it is convenient to assume that

$$k(t, X, y) = \alpha(m - y) - \alpha^{\frac{1}{2}-H}\beta\gamma(t, X, y), \tag{4.13}$$

appealing to the Hurst parameter $H \in (0, 1)$. We notice that (4.13) is equal to the function as given in Fouque et al. [7], if $H = 1/2$ is formally substituted into (4.13). The function $\gamma(t, X, y)$ is called the *market price of risk*. Further, for simplicity, we assume that the function γ depends only on the variable y. Consequently, we take the following assumption:

Assumption 4.3. Let $0 < H < 1.$ Then the function k has the form

$$k(t, X, y) = \alpha(m - y) - \alpha^{\frac{1}{2}-H}\beta\gamma(y). \tag{4.14}$$

5. Fast Scale

We shall introduce the following assumption on the scaling to model fast mean-reversion in market volatility.

Assumption 5.1. Let $0 < H < 1.$ Then we assume the following:

(i) The rate of mean-reversion α or its inverse, the typical correlation time of $(Y(t)),$ is characterized by a small parameter ε such that

$$\varepsilon = \frac{1}{\alpha}.$$

(ii) Let ν_H^2 be given by (3.7), which controls the long-run size of the volatility fluctuations. Then we assume this quantity remains fixed as we consider smaller and smaller values of ε such that

$$\beta = \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\left(\frac{1}{\alpha}\right)^{-H} = \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\frac{1}{\varepsilon^H}.$$

Then we can obtain the pricing in terms of ε as follows (Narita [17, Lemma 10.3]):

Lemma 5.2 (Pricing PDE in terms of ε). *Let* $0 < H < 1$. *Then, under Assumptions* 1.1, 4.3 *and* 5.1, *for* ε *small enough, the pricing PDE* (4.9) *of Lemma* 4.1 *can be written in terms of* ε *as follows*:

$$\frac{\partial P^{\varepsilon}}{\partial t} + \frac{1}{2}\frac{\partial^2 P^{\varepsilon}}{\partial X^2} f(y)^2 X^2 + \frac{\nu_H^2}{\varepsilon}\frac{\partial^2 P^{\varepsilon}}{\partial y^2} + r\left(X\frac{\partial P^{\varepsilon}}{\partial X} - P^{\varepsilon}\right)$$

$$+ \left\{\frac{1}{\varepsilon}(m-y) - \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\frac{1}{\sqrt{\varepsilon}}\gamma(y)\right\}\frac{\partial P^{\varepsilon}}{\partial y} = 0 \tag{5.1}$$

for $t < T$ *with the terminal condition* $P^{\varepsilon}(T, X, y) = h(X)$, *where* $h(X)$ *stands for the nonnegative payoff function*.

Finally, we can obtain a corrected Black-Scholes price formula as given by Theorem 5.3; the fractional European call price can be expanded around the classical European call price and the explicit expression for the quantity in the corrected term can be given. This results from singular perturbation method.

In Section 6, we shall need to apply Theorem 5.3 in order to derive implied volatility in the case of fast scale. Therefore, in the following, we shall introduce the outline of the asymptotic analysis for singularly perturbed equation (5.1).

We write PDE (5.1) with the notation as follows:

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P^{\varepsilon} = 0, \tag{5.2}$$

where we define

$$\mathcal{L}_0 = \nu_H^2\frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \tag{5.3}$$

$$\mathcal{L}_1 = -\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\gamma(y)\frac{\partial}{\partial y}, \tag{5.4}$$

$$\mathcal{L}_2 = \mathcal{L}_{BS}(f(y)) = \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2 X^2\frac{\partial^2}{\partial X^2} + r\left(X\frac{\partial}{\partial X} - 1\right). \tag{5.5}$$

Here $\mathcal{L}_{BS}(\sigma)$ is the classical Black-Scholes operator with the deterministic volatility parameter σ, that is,

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 X^2\frac{\partial^2}{\partial X^2} + r\left(X\frac{\partial}{\partial X} - 1\right). \tag{5.6}$$

The method is to expand the solution P^{ε} in power of $\sqrt{\varepsilon}$,

$$P^{\varepsilon} = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon}P_3 + \cdots \tag{5.7}$$

for small ε, where $P_0, P_1, \ldots$ are functions of (t, X, y) to be determined by the terminal conditions

$$P_0(T, X, y) = h(X), \quad P_i(T, X, y) = 0 \text{ for } i \geq 1.$$

In the following, we let $\langle\cdot\rangle$ denote the averaging with respect to the invariant distribution $N(m, \nu_H^2)$ of the fractional OU process $(Y(t))$ (see Lemma 3.3 and (3.6)):

$$\langle g \rangle = \int_{-\infty}^{\infty} g(y)n(y)dy = \frac{1}{\sqrt{2\pi\nu_H^2}} \int_{-\infty}^{\infty} g(y)\exp\left(-\frac{(y-m)^2}{2\nu_H^2}\right)dy. \tag{5.8}$$

Notice that this averaged quantity does not depend on ε. Further, we let $\overline{\sigma}$ denote the *effective constant volatility* defined by

$$\overline{\sigma}^2 = \langle f^2 \rangle = \int_{-\infty}^{\infty} f(y)^2 n(y)dy, \tag{5.9}$$

which is the average with respect to the invariant distribution $N(m, \nu_H^2)$ of the process $(Y(t))$.

In the following, let $\psi(y)$ be the solution of the Poisson equation:

$$\mathcal{L}_0\psi = f(y)^2 - \overline{\sigma}^2. \tag{5.10}$$

Then, $\psi(y)$ will play an essential role in derivation of the first correction for the Black-Scholes price.

According to Fouque et al. [7], we denote the first correction by

$$\tilde{P}_1 = \sqrt{\varepsilon}P_1. \tag{5.11}$$

Then, by the method of singular perturbation, we can obtain that P_0 and P_1 are constants with respect to the variable y, i.e., $P_0 = P_0(t, X)$ and $P_1 = P_1(t, X)$. Further, we can obtain that P_0 and P_1 satisfy the following equations:

$$\mathcal{L}_{BS}(\overline{\sigma})P_0 = 0 \tag{5.12}$$

with terminal condition $P_0(T, X) = h(X)$, and

$$\mathcal{L}_{BS}(\overline{\sigma})\tilde{P}_1 = H(t, X) \tag{5.13}$$

with terminal condition $\tilde{P}_1(T, X) = 0$. Here

$$H(t, X) = V_2 X^2 \frac{\partial^2 P_0}{\partial X^2}, \tag{5.14}$$

where V_2 is a small coefficient, given in terms of $\alpha = 1/\varepsilon$ by

$$V_2 = -\frac{1}{\sqrt{\alpha}}\frac{1}{2}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\langle\gamma\psi'\rangle; \tag{5.15}$$

$\gamma(y)$ is the market price of risk appearing in Assumption 4.3 and $\psi(y)$ is the solution of (5.10).

Moreover, the solution of equation (5.13) is explicitly given by

$$\tilde{P}_1(t, X) = -(T - t)H(t, X). \tag{5.16}$$

Hence we obtain the following theorem (Narita [17, Theorem 11.1, Theorem 12.1]):

Theorem 5.3. *Let* $0 < H < 1$. *Suppose Assumptions* 1.1, 4.3 *and* 5.1. *Then, for* ε *small enough, the corrected Black-Scholes price is given by*

$$P \approx P_0 + \tilde{P}_1 = P_0 - (T - t)\left(V_2 X^2 \frac{\partial^2 P_0}{\partial X^2}\right), \tag{5.17}$$

where P_0 *is the solution of the classical BS equation with effective constant volatility* $\overline{\sigma}$ *as given by* (5.9), *i.e.,* $\mathcal{L}_{BS}(\overline{\sigma})P_0 = 0$ *with terminal condition* $P_0(T, X) = h(X)$ *for a payoff function* $h(X)$. *Further,* V_2 *is a small coefficient as given by* (5.15) *in terms of* $\alpha = 1/\varepsilon$. *The first correction* $\tilde{P}_1(t, X)(= \sqrt{\varepsilon}P_1(t, X))$ *is a solution of*

$$\mathcal{L}_{BS}(\overline{\sigma})\tilde{P}_1 = H(t, X),$$

and the terminal condition $\tilde{P}_1(T, X) = 0$, *where the source term* $H(t, x)$ *is defined by* (5.14). *Moreover, in equation* (5.15), *we find that*

$$\langle \gamma\psi' \rangle = \left(\frac{1}{\nu_H^2}\right)\int_{-\infty}^{\infty} \gamma(y)\left(\int_{-\infty}^{y} (f(z)^2 - \bar{\sigma}^2) n(z) dz\right) dy = -\frac{1}{\nu_H^2} \langle G(f^2 - \bar{\sigma}^2) \rangle,$$

where $G(y)$ *is the primitive function of* $\gamma(y)$, *that is,* $G(y) = \int \gamma(y) dy$. *Hence* V_2 *has the explicit expression of the following form*:

$$V_2 = \frac{1}{\sqrt{\alpha}} \frac{1}{2\nu_H} \left(\frac{1}{\sqrt{H\Gamma(2H)}}\right) \langle G(f^2 - \bar{\sigma}^2) \rangle. \tag{5.18}$$

6. Implied Volatility in the Case of Fast Scale

We will show how our corrected price as given by (5.17), and in particular, the parameter V_2 as given by (5.15) or (5.18), is easily related to observed prices or implied volatilities.

We recall the classical BS model. Namely, the prices of the risk-less asset $A(t)$ and the risky asset $X(t)$ at time $t \in [0, T]$ with the time of maturity *T*, are given as follows:

$$dA(t) = rA(t) dt, \quad A(0) = 1,$$

where *r* represents the constant risk-less interest rate.

$$dX(t) = \mu X(t) dt + \sigma X(t) dW(t),$$

where $W(t)$ is a standard Brownian motion, and $\mu \in \mathbb{R}$, $\sigma > 0$ are deterministic constants. The coefficient μ is a constant appreciation rate of the stock price and the coefficient σ, referred to as the (stock price) *volatility*, is interpreted as a measure of uncertainty about future stock price movement.

In a standard European call option with strike price *K* and the time of maturity *T*, the payoff function is given by

$$h(X) = \max\{X - K, 0\} := (X - K)^+.$$

The payoff at maturity date *T* is $h(X(T))$. If $X(T) \le K$, then the option is worthless, and if $X(T) > K$, then the holder of the call can buy the underlying asset for *K* (dollars) and sell it at market price, making a profit of $X(T) - K$.

The price C_t at time $t \in [0, T]$ of a European call option with strike price K and maturity date T in the classical BS market is given by the following formula:

$$C_t = C_{BS}(t, X(t)),$$

where

$$C_{BS} : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$$

is the Black-Scholes call option pricing formula

$$C_{BS}(t, X) = XN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{6.1}$$

with

$$d_1 = \frac{\log\left(\frac{X}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{6.2}$$

$$d_2 = \frac{\log\left(\frac{X}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} (= d_1 - \sigma\sqrt{T-t}), \tag{6.3}$$

and $N(\cdot)$ is the cumulative probability of the standard normal distribution, i.e.,

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{y^2}{2}\right) dy.$$

Using the relations

$$\frac{\partial d_{1,2}}{\partial X} = \frac{1}{X\sigma\sqrt{T-t}},$$

$$N'(d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}},$$

$$e^{-\frac{d_2^2}{2}} = e^{-\frac{d_1^2}{2}}\left(\frac{Xe^{r(T-t)}}{K}\right),$$

we can easily derive the following ‘Greeks’:

$$\textbf{Delta} \quad \frac{\partial C_{BS}}{\partial X} = N(d_1) > 0, \tag{6.4}$$

$$\textbf{Gamma} \qquad \frac{\partial^2 C_{BS}}{\partial X^2} = \frac{e^{-\frac{d_1^2}{2}}}{X\sigma\sqrt{2\pi(T-t)}} > 0, \tag{6.5}$$

$$\textbf{Vega} \qquad \frac{\partial C_{BS}}{\partial \sigma} = \frac{Xe^{-\frac{d_1^2}{2}}\sqrt{T-t}}{\sqrt{2\pi}} > 0. \tag{6.6}$$

Occasionally, it will be convenient to explicitly account for the dependence of the option's price on some or all of the parameters K, T, r and σ. For example, to stress the dependence of the BS price on the volatility σ, we write $C_{BS}(\sigma)$. We will also denote C_{BS} by $C_{BS}(t, X; K, T; \sigma)$ to emphasize the dependence on K, T and σ. Only the volatility σ, the standard deviation of the returns scaled by the square root of the time increment, need to be estimated from data, assuming that the interest rate r is known.

Given an observed European call price C_{obs} for a contract with strike price K and time of maturity T, the *implied volatility* I is defined to be the volatility parameter that must go into the BS formula (6.1) to match the observed price:

$$C_{BS}(t, X; K, T; I) = C_{obs}, \tag{6.7}$$

where C_{BS} is the BS price.

Then a unique nonnegative implied volatility $I > 0$ can be found given $C_{obs} > C_{BS}(t, X; K, T; 0)$. In fact, the BS call pricing formula $C_{BS}(\sigma) = C_{BS}(t, X; K, T; \sigma)$ is a continuous – indeed, differentiable – increasing function of σ by (6.6) with boundaries

$$\lim_{\sigma\to 0} C_{BS}(\sigma) = \begin{cases} X(t) - Ke^{-r(T-t)} & \text{if } X(t) > Ke^{-r(T-t)}, \\ 0 & \text{if } X(t) \le Ke^{-r(T-t)}, \end{cases}$$

$$\lim_{\sigma\to\infty} C_{BS}(\sigma) = X(t).$$

Hence the inverse function exists.

When studying a real market price, the implied volatility is not constant as assumed in the classical BS model, but varies. The result, when plotting the implied volatility against K or the ratio of the strike price to the current price, is called the *smile curve* or *volatility smile*. It is interesting to investigate if there are any smile effects in reality.

In the following, the implied volatility for a European call option will be calculated in terms of V_2 as given by (5.15) or (5.18). The implied volatility is computed by solving the relation between theoretical and observed prices given in (6.7) with respect to the implied volatility I. Theorem 5.3 shows that the approximating price is given by

$$P_0 + \tilde{P}_1.$$

Here $P_0 = C_{BS}$, the BS formula for a call option is given by (6.1), d_1 and d_2 are given by (6.2) and (6.3), respectively; in this case, the constant volatility σ is replaced by the effective volatility $\bar{\sigma}$ as given by (5.9). Namely, $d_1 = d_1(\bar{\sigma})$ and $d_2 = d_2(\bar{\sigma})$, depending on the deterministic volatility parameter $\bar{\sigma}$, and hence

$$\mathcal{L}_{BS}(\bar{\sigma})P_0 = 0 \text{ with } P_0 = C_{BS}(\bar{\sigma}).$$

Further, $\tilde{P}_1 = \sqrt{\varepsilon}P_1$, $\varepsilon = 1/\alpha$, and $\tilde{P}_1$ is the solution of

$$\mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_1 = H(t, X)$$

with the source function $H(t, X)$ as given by (5.14). In order to apply the approximate solution

$$P \approx P_0 + \tilde{P}_1 = P_0 - (T - t)\left(V_2 X^2 \frac{\partial^2 P_0}{\partial X^2}\right),$$

we need the second derivative of P_0 with respect to X, i.e., the Gamma (6.5). Then equation (5.14) gives

$$H(t, X) = V_2 X^2 \frac{\partial^2 P_0}{\partial X^2} = \left(\frac{Xe^{\frac{-d_1^2}{2}}}{\bar{\sigma}\sqrt{2\pi(T - t)}}\right)V_2.$$

This implies that expression (5.16) for $\tilde{P}_1(T, X)$ is written as

$$\tilde{P}_1(t, X) = -(T - t)H(t, x) = \frac{Xe^{\frac{-d_1^2}{2}}}{\bar{\sigma}\sqrt{2\pi}}(-V_2\sqrt{T - t}). \tag{6.8}$$

Taking the corrected pricing formula as observed price,

$$C_{obs} = P_0 + \tilde{P}_1(t, X) \tag{6.9}$$

in equation (6.7), the relation that determines the implied volatility is thus

$$C_{BS}(t, X; K, T; I) = P_0 + \tilde{P}_1(t, X). \tag{6.10}$$

Equation (6.10) can be solved by expanding I as

$$I = \overline{\sigma} + \sqrt{\varepsilon} I_1 + \varepsilon I_2 + \cdots, \tag{6.11}$$

and inserting this in the left-hand side of (6.10) (see Appendix):

$$\begin{aligned} &C_{BS}(t, X; K, T; \overline{\sigma}) + \sqrt{\varepsilon} I_1 \frac{\partial C_{BS}}{\partial \sigma}(t, X; K, T; \overline{\sigma}) + \cdots \\ &= P_0 + \tilde{P}_1(t, X) + \cdots. \end{aligned} \tag{6.12}$$

Here we recall (5.11), that is,

$$\tilde{P}_1(t, x) = \sqrt{\varepsilon} P_1(t, X), \quad \varepsilon = 1/\alpha.$$

Then (6.12) leads to

$$\sqrt{\varepsilon} I_1 = \tilde{P}_1(t, X) \left[\frac{\partial C_{BS}}{\partial \sigma}(t, X, K, T; \overline{\sigma}) \right]^{-1}. \tag{6.13}$$

In other words, up to an error of order $O(\varepsilon)$, where $\varepsilon = 1/\alpha$, the implied volatility is given by

$$I = \overline{\sigma} + \sqrt{\varepsilon} I_1 + O(\varepsilon),$$

that is,

$$I = \overline{\sigma} + \tilde{P}_1(t, X) \left[\frac{\partial C_{BS}}{\partial \sigma}(t, X, K, T; \overline{\sigma}) \right]^{-1} + O(1/\alpha). \tag{6.14}$$

Inserting expressions (6.6) for Vega at $\sigma = \overline{\sigma}$ and (6.8) for $\tilde{P}_1(t, X)$ into equation (6.14), we have

$$\begin{aligned} I &= \overline{\sigma} + \left[\frac{X e^{\frac{-d_1^2}{2}}}{\overline{\sigma}\sqrt{2\pi}} \left(-V_2 \sqrt{T - t} \right) \right] \times \left[\frac{X e^{-\frac{d_1^2}{2}} \sqrt{T - t}}{\sqrt{2\pi}} \right]^{-1} + O(1/\alpha) \\ &= \overline{\sigma} - \frac{V_2}{\overline{\sigma}} + O(1/\alpha). \end{aligned}$$

Therefore, we obtain the following theorem:

Theorem 6.1. *Suppose Assumptions* 1.1, 4.3 *and* 5.1. *Then the implied volatility* I *is given by*

$$I = \bar{\sigma} - \frac{V_2}{\bar{\sigma}} + O(1/\alpha). \tag{6.15}$$

Remark 6.2. The source term $H(t, X)$, that is, $V_2 X^2 (\partial^2 P_0 / \partial X^2)$ depends on the market price of volatility risk γ; see (5.14), (5.15) and (5.18). Adding equations (5.12) for P_0 and (5.13) for $\tilde{P}_1$, we obtain

$$\mathcal{L}_{BS}(\bar{\sigma})(P_0 + \tilde{P}_1) = V_2 X^2 \frac{\partial^2 P_0}{\partial X^2} \tag{6.16}$$

with the terminal condition $(P_0 + \tilde{P}_1)(T, X) = h(X)$. According to Fouque et al. [7, p. 140], introduce, for V_2 small enough, the *corrected effective volatility*

$$\tilde{\sigma} = \sqrt{\bar{\sigma}^2 - 2V_2}.$$

Then equation (6.16) can be written as

$$\mathcal{L}_{BS}(\tilde{\sigma})(P_0 + \tilde{P}_1) = -V_2 X^2 \frac{\partial^2 \tilde{P}_1}{\partial X^2} = O(\varepsilon).$$

Here by (5.15) and (5.11), we used that $V_2 = O(\sqrt{\varepsilon})$ and $\tilde{P}_1 = O(\sqrt{\varepsilon})$, respectively, and assumed sufficient smoothness in $\tilde{P}_1$ so that $\partial^2 \tilde{P}_1 / \partial X^2 = O(\sqrt{\varepsilon})$. Thus, the corrected price $P_0 + \tilde{P}_1$ has the same order of accuracy as the solution $\tilde{P}$ of

$$\mathcal{L}_{BS}(\tilde{\sigma})\tilde{P} = 0$$

with the same terminal condition $\tilde{P}(T, X) = h(X)$. By reason of this, the V_2 term is simply a volatility level correction.

7. Slow Scale

Let $H \in (0, 1)$ be arbitrary Hurst parameter and fixed. Let $0 < \delta << 1$. Then we consider the BS model described by (1.1), (1.2) and (1.4), where $(Y(t))$ is the slow scale volatility factor such that

$$dY(t) = \delta\alpha(m - Y(t))dt + \delta^H \beta dB_H(t) \tag{7.1}$$

depending on a small parameter $\delta > 0$. Observe that (7.1) (i.e., (1.4)) is obtained by (1.3) with the coefficients α and β replaced by $\delta\alpha$ and $\delta^H\beta$, respectively. Then, in order to derive the pricing PDE for this model, we have only to consider the pricing PDE (4.9) of Lemma 4.1, except that α and β are replaced by $\delta\alpha$ and $\delta^H\beta$, respectively. Then, in the multiplier of the partial derivative $\partial^2 P/\partial y^2$ appearing in (4.9), we shall compute the factor

$$e^{-2(\delta\alpha)t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right],$$

where $g_t(s) = \chi_{[0,t]}(s)g(s)$ and $g(s) = e^{(\delta\alpha)s}(\delta^H\beta)$; we shall apply Remark 4.2 with α and β replaced by $\delta\alpha$ and $\delta^H\beta$, respectively.

Recall the functions given by (4.12), i.e.,

$$B(x) = \int_0^x z^{2H-1}e^{-z}dz, \quad C(x) = \int_0^x z^{2H}e^{-z}dz.$$

Observe that for $t > 0$,

$$B(\delta\alpha t) \to 0, \quad C(\delta\alpha t) \to 0 \quad \text{as} \quad \delta \to 0.$$

Then, by (4.10) and (4.11), we have the following:

(i) If $H > 1/2$, then

$$\begin{aligned} e^{-2\delta\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] &= 2(\delta^{2H}\beta^2)H\left\{e^{-\delta\alpha t}t^{2H-1} + \left(\frac{1}{\delta\alpha}\right)^{2H-1}B(\delta\alpha t)\right\} \\ &= 2H\beta^2\left\{\frac{\delta^{2H}}{e^{\delta\alpha t}}t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1}\delta B(\delta\alpha t)\right\}. \end{aligned}$$

Notice that for $t > 0$,

$$\lim_{\delta\to 0}\frac{\delta^{2H}}{e^{\delta\alpha t}} = 0 \quad \text{and} \quad \lim_{\delta\to 0}\delta B(\delta\alpha t) = 0.$$

This yields that for $t > 0$,

$$\lim_{\delta \to 0} e^{-2\delta\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] = 0 \ (H > 1/2). \tag{7.2}$$

(ii) If $0 < H < 1/2$, then

$$\begin{aligned} e^{-2\delta\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] &= H(\delta^{2H}\beta^2)t^{2H-1}e^{-\delta\alpha t} \\ &\quad + 2H^2(\delta^{2H}\beta^2)t^{-1}\left(\frac{1}{\delta\alpha}\right)^{2H} B(\delta\alpha t) \\ &\quad + H(\delta\alpha)(\delta^{2H}\beta^2)t^{-1}\left(\frac{1}{\delta\alpha}\right)^{2H}\left\{tB(\delta\alpha t) - \left(\frac{1}{\delta\alpha}\right)C(\delta\alpha t)\right\} \\ &\quad + H(\delta\alpha)(\delta^{2H}\beta^2)\left(\frac{1}{\delta\alpha}\right)^{2H} B(\delta\alpha t) \\ &= H\beta^2 t^{2H-1}\left(\frac{\delta^{2H}}{e^{\delta\alpha t}}\right) + 2H^2\beta^2 t^{-1}\left(\frac{1}{\alpha}\right)^{2H} B(\delta\alpha t) \\ &\quad + H\beta^2 t^{-1}\left(\frac{1}{\alpha}\right)^{2H-1}\left\{(\delta t)B(\delta\alpha t) - \left(\frac{1}{\alpha}\right)C(\delta\alpha t)\right\} \\ &\quad + H\beta^2\left(\frac{1}{\alpha}\right)^{2H-1}\delta B(\delta\alpha t). \end{aligned}$$

This yields that for $t > 0$,

$$\lim_{\delta \to 0} e^{-2\delta\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] = 0 \ (0 < H < 1/2). \tag{7.3}$$

Equations (7.2) and (7.3) imply that in PDE (4.9) of Lemma 4.1, the multiplier of the partial derivative $\partial^2 P/\partial y^2$ tends to zero as $\delta \to 0$ in the case of slow scale.

In the case of slow scale, we shall need Assumption 4.3 with α and β replaced by $\delta\alpha$ and $\delta^H\beta$, respectively. Then we obtain the following theorem:

Lemma 7.1 (Pricing PDE in terms of δ)**.** *Let* $0 < H < 1$. *Suppose Assumption* 1.1 *and Assumption* 4.3 *with* α *and* β *replaced by* $\delta\alpha$ *and* $\delta^H\beta$, *respectively*, *that is*, *suppose that*

$$k(t,\ X,\ y) = \delta\alpha(m - y) - \sqrt{\delta}\alpha^{\frac{1}{2}-H}\beta\gamma(y).$$

Then, for δ *small enough, the pricing PDE* (4.9) *of Lemma* 4.1 *can be written in terms of* δ *as follows*:

$$\frac{\partial P^{\delta}}{\partial t} + \frac{1}{2}\frac{\partial^2 P^{\delta}}{\partial X^2} f(y)^2 X^2 + rX\frac{\partial P^{\delta}}{\partial X} - rP^{\delta}$$

$$+ \left(\delta\alpha(m - y) - \sqrt{\delta}\alpha^{\frac{1}{2}-H}\beta\gamma(y)\right)\frac{\partial P^{\delta}}{\partial y} = 0 \tag{7.4}$$

for $t < T$ *with the terminal condition* $P^{\delta}(T,\ X,\ y) = h(X)$, *where* $h(X)$ *stands for the nonnegative payoff function*.

For a moment, assume that $f(y) > 0$ $(y \in \mathbb{R})$ and set $I_0 = f(y)$. Then we observe that when $\delta = 0$, equation (7.4) reduces to the Black-Scholes PDE with volatility parameter I_0; the solution is $C_{BS}(I_0)$.

For equation (7.4), we will find the asymptotic solution of the form

$$P^{\delta} = P_0 + \sqrt{\delta}P_1 + \delta P_2 + \cdots \tag{7.5}$$

with the terminal conditions

$$P_0(T,\ X,\ y) = h(X), \quad P_i(T,\ X,\ y) = 0 \quad \text{for} \quad t \geq 1.$$

Substitution of this form into (7.4) yields

$$\left(\frac{\partial P_0}{\partial t} + \sqrt{\delta}\,\frac{\partial P_1}{\partial t} + \delta\,\frac{\partial P_2}{\partial t} + \cdots\right)$$

$$+ \left(\frac{1}{2}\,f(y)^2 X^2\,\frac{\partial^2 P_0}{\partial X^2} + \sqrt{\delta}\,\frac{1}{2}\,f(y)^2 X^2\,\frac{\partial^2 P_1}{\partial X^2} + \delta\,\frac{1}{2}\,f(y)^2 X^2\,\frac{\partial^2 P_2}{\partial X^2} + \cdots\right)$$

$$+ rX\left(\frac{\partial P_0}{\partial X} + \sqrt{\delta}\,\frac{\partial P_1}{\partial X} + \delta\,\frac{\partial P_2}{\partial X} + \cdots\right)$$

$$- r(P_0 + \sqrt{\delta}P_1 + \delta P_2 + \cdots)$$

$$+\,\delta\alpha(m-y)\left(\frac{\partial P_0}{\partial y}+\sqrt{\delta}\,\frac{\partial P_1}{\partial y}+\delta\,\frac{\partial P_2}{\partial y}+\cdots\right)$$

$$-\,\sqrt{\delta}\alpha^{\frac{1}{2}-H}\beta\gamma(y)\left(\frac{\partial P_0}{\partial y}+\sqrt{\delta}\,\frac{\partial P_1}{\partial y}+\delta\,\frac{\partial P_2}{\partial y}+\cdots\right)=0.$$

Let $\mathcal{L}_{BS}(\sigma)$ be the Black-Scholes operator with the deterministic volatility parameter σ, i.e.,

$$\mathcal{L}_{BS}(\sigma)=\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial X^2}+r\left(X\frac{\partial}{\partial X}-\cdot\right).$$

In the following Steps 1-4, we assume that $f(y)>0$ $(y\in\mathbb{R})$ and that $f'(y)\gamma(y)\neq 0$ $(y\in\mathbb{R})$. Define $I_0=f(y)$. Then, equating the coefficients of the powers of δ to 0, we get the following equations:

$$\mathcal{L}_{BS}(I_0)P_0=0, \tag{7.6}$$

$$\mathcal{L}_{BS}(I_0)P_1=\alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_0}{\partial y}, \tag{7.7}$$

$$\mathcal{L}_{BS}(I_0)P_2=-\left[\alpha(m-y)\frac{\partial P_0}{\partial y}-\alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_1}{\partial y}\right], \tag{7.8}$$

$$\ldots.$$

Step 1. First, consider (7.6), that is, the Black-Scholes PDE with the volatility parameter $I_0=f(y)>0$. Then we get the solution

$$P_0=C_{BS}(I_0).$$

Step 2. Next, consider (7.7). Then, since y is a parameter in (7.7), we can find a solution P_1 of the form

$$P_1=A(y)\frac{\partial P_0}{\partial y} \tag{7.9}$$

with a function $A(y)$, independent of $(t,\ X)$. Then, it is easy to see that

$$\mathcal{L}_{BS}(I_0)P_1=\mathcal{L}_{BS}(I_0)\left[A(y)\frac{\partial P_0}{\partial y}\right]=A(y)\mathcal{L}_{BS}(I_0)\left[\frac{\partial P_0}{\partial y}\right]. \tag{7.10}$$

Changing the order of differentiations with respect to y and X, we have

$$\mathcal{L}_{BS}(I_0)\left[\frac{\partial P_0}{\partial y}\right] = \frac{\partial}{\partial y}\left\{\frac{\partial P_0}{\partial t} + rX\,\frac{\partial P_0}{\partial X} - rP_0\right\} + \frac{1}{2}\,f(y)^2 X^2\,\frac{\partial^2}{\partial X^2}\left(\frac{\partial P_0}{\partial y}\right). \tag{7.11}$$

Here by derivative product rule, we have

$$\begin{aligned}\frac{\partial}{\partial y}\left\{\frac{1}{2}\,f(y)^2 X^2\,\frac{\partial^2 P_0}{\partial X^2}\right\} &= f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_0}{\partial X^2} + \frac{1}{2}\,f(y)^2\,\frac{\partial}{\partial y}\left(X^2\,\frac{\partial^2 P_0}{\partial X^2}\right)\\ &= f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_0}{\partial X^2} + \frac{1}{2}\,f(y)^2 X^2\,\frac{\partial^2}{\partial X^2}\left(\frac{\partial P_0}{\partial y}\right),\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{2}\,f(y)^2 X^2\,\frac{\partial^2}{\partial X^2}\left(\frac{\partial P_0}{\partial y}\right)\\ &= \frac{\partial}{\partial y}\left\{\frac{1}{2}\,f(y)^2 X^2\,\frac{\partial^2 P_0}{\partial X^2}\right\} - f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_0}{\partial X^2}.\end{aligned} \tag{7.12}$$

Substitute (7.12) into the right-hand side of (7.11). Then we get

$$\mathcal{L}_{BS}(I_0)\left[\frac{\partial P_0}{\partial y}\right] = \frac{\partial}{\partial y}\{\mathcal{L}_{BS}(I_0)P_0\} - f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_0}{\partial X^2}. \tag{7.13}$$

Here P_0 satisfies (7.6), i.e., $\mathcal{L}_{BS}(I_0)P_0 = 0$ with $I_0 = f(y)$, and hence (7.13) is equal to

$$\mathcal{L}_{BS}(I_0)\left[\frac{\partial P_0}{\partial y}\right] = -f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_0}{\partial X^2}. \tag{7.14}$$

Thus, (7.9), (7.10) and (7.14) yield

$$\mathcal{L}_{BS}(I_0)P_1 = A(y)\left[-f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_0}{\partial X^2}\right].$$

Combining this with (7.7), we get

$$A(y)\left[-f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_0}{\partial X^2}\right] = \alpha^{\frac{1}{2}-H}\beta\gamma(y)\,\frac{\partial P_0}{\partial y}. \tag{7.15}$$

Observe the Black-Scholes operator (5.6) and formula (6.1). Recall the Greeks (6.4), (6.5) and (6.6) for the Black-Scholes price $C_{BS}(t, X; \sigma)$ with respect to the deterministic volatility parameter σ. Then we shall need the following equations for the nomenclature of the Greeks:

$$\mathcal{L}_{BS}(\sigma)C_{BS} = 0, \tag{7.16}$$

$$\textbf{Gamma}\ \frac{\partial^2 C_{BS}}{\partial X^2} = \frac{n(d_1)}{X\sigma\sqrt{T-t}}, \tag{7.17}$$

$$\begin{aligned}\textbf{Vega}\ \frac{\partial C_{BS}}{\partial \sigma} &= X\sqrt{T-t}\,n(d_1)\\ &= (T-t)\sigma X^2 \frac{\partial^2 C_{BS}}{\partial X^2},\end{aligned} \tag{7.18}$$

$$\textbf{DVegaDVol}\ \frac{\partial^2 C_{BS}}{\partial \sigma^2} = X\sqrt{T-t}\,n(d_1)\left(\frac{d_1 d_2}{\sigma}\right). \tag{7.19}$$

‘DVegaDVol’ corresponds to the change in Vega resulting from a change in volatility. Here d_1 and $d_2 (i = 1, 2)$ are given by (6.2) and (6.3), respectively, and $n(d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}$. We notice that for $i = 1, 2,$ we can write d_i as $d_i(\sigma)$, emphasizing dependence on the deterministic volatility parameter σ.

When $P_0 = C_{BS}(\sigma)$ with $\sigma = I_0 = f(y) > 0,$ it holds that

$$\frac{\partial P_0}{\partial y} = \left(\frac{\partial C_{BS}}{\partial \sigma}\frac{\partial \sigma}{\partial y}\right)\bigg|_{\sigma=I_0} = \left(\frac{\partial C_{BS}}{\partial \sigma}\right)\bigg|_{\sigma=I_0} \cdot f'(y),$$

and hence equation (7.18) implies

$$\frac{\partial P_0}{\partial y} = (T-t)\,f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_0}{\partial X^2}. \tag{7.20}$$

It follows from (7.15) and (7.20) that

$$A(y)\left\{-\left(\frac{1}{T-t}\right)\frac{\partial P_0}{\partial y}\right\} = \alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_0}{\partial y}.$$

Hence

$$A(y) = -(T - t)\alpha^{\frac{1}{2}-H}\beta\gamma(y). \tag{7.21}$$

Therefore, by (7.9) and (7.21), we obtain

$$P_1 = A(y)\frac{\partial P_0}{\partial y} = -(T - t)\alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_0}{\partial y}. \tag{7.22}$$

Step 3. For PDE (7.8), we shall find a solution P_2 of the form

$$P_2 = B(y)\frac{\partial P_0}{\partial y} + C(y)\frac{\partial P_1}{\partial y} \tag{7.23}$$

with functions $B(y)$ and $C(y)$, independent of (t, X). We shall find the expressions for $B(y)$ and $C(y)$ in the following. We first notice

$$\mathcal{L}_{BS}(I_0)P_2 = \mathcal{L}_{BS}(I_0)\left[B(y)\frac{\partial P_0}{\partial y}\right] + \mathcal{L}_{BS}(I_0)\left[C(y)\frac{\partial P_1}{\partial y}\right]. \tag{7.24}$$

Here, consider (7.10) with $A(y)$ replaced by $B(y)$ and observe (7.14). Then we get

$$\mathcal{L}_{BS}(I_0)\left[B(y)\frac{\partial P_0}{\partial y}\right] = B(y)\left\{-f(y)f'(y)X^2\frac{\partial^2 P_0}{\partial X^2}\right\}. \tag{7.25}$$

Thus, (7.20) yields

$$\mathcal{L}_{BS}(I_0)\left[B(y)\frac{\partial P_0}{\partial y}\right] = B(y)\left(\left(-\frac{1}{T-t}\right)\frac{\partial P_0}{\partial y}\right). \tag{7.26}$$

We next notice that equations (7.11)-(7.13) hold with P_0 replaced by P_1 and hence

$$\mathcal{L}_{BS}(I_0)\left[C(y)\frac{\partial P_1}{\partial y}\right] = C(y)\mathcal{L}_{BS}(I_0)\left[\frac{\partial P_1}{\partial y}\right]$$

$$= C(y)\frac{\partial}{\partial y}\{\mathcal{L}_{BS}(I_0)P_1\} - f(y)f'(y)X^2\frac{\partial^2 P_1}{\partial X^2}. \tag{7.27}$$

By (7.7), P_1 satisfies that $\mathcal{L}_{BS}(I_0)P_1 = \alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_0}{\partial y}$, and hence equation (7.27) is equivalent to

$$\mathcal{L}_{BS}(I_0)\left[C(y)\frac{\partial P_1}{\partial y}\right]$$

$$= C(y)\left[\frac{\partial}{\partial y}\left\{\alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_0}{\partial y}\right\} - f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_1}{\partial X^2}\right]. \tag{7.28}$$

We shall find another expression for the right-hand side of (7.28) in the following. By (7.22), we first find

$$\frac{\partial}{\partial y}\left\{\alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_0}{\partial y}\right\} = -\left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}. \tag{7.29}$$

We next find expression for the remaining term

$$f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_1}{\partial X^2}.$$

Considering (7.22) and changing the order of differentiations, we have

$$X^2\,\frac{\partial^2 P_1}{\partial X^2} = -(T-t)\,\alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial}{\partial y}\left(X^2\,\frac{\partial^2 P_0}{\partial X^2}\right). \tag{7.30}$$

Here and hereafter, we assume that

$$f(y)\,f'(y) \neq 0 \ (y \in \mathbb{R}) \quad \text{and} \quad \gamma(y) \neq 0 \ (y \in \mathbb{R}).$$

Then (7.20) is equivalent to

$$X^2\,\frac{\partial^2 P_0}{\partial X^2} = \left(\frac{1}{T-t}\right)\left(\frac{1}{f(y)\,f'(y)}\right)\frac{\partial P_0}{\partial y},$$

and hence equation (7.30) can be rewritten as follows:

$$X^2\,\frac{\partial^2 P_1}{\partial X^2} = -\alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial}{\partial y}\left\{\left(\frac{1}{f(y)\,f'(y)}\right)\frac{\partial P_0}{\partial y}\right\}$$

$$= -\alpha^{\frac{1}{2}-H}\beta\gamma(y)\times\left\{\left(\frac{-(f(y)\,f'(y))'}{(f(y)\,f'(y))^2}\right)\frac{\partial P_0}{\partial y} + \left(\frac{1}{f(y)\,f'(y)}\right)\frac{\partial}{\partial y}\left(\frac{\partial P_0}{\partial y}\right)\right\}.$$

This implies

$$f(y)f'(y)X^2\frac{\partial^2 P_1}{\partial X^2}$$

$$= -\alpha^{\frac{1}{2}-H}\beta\gamma(y)\left\{-\left(\frac{(f(y)f'(y))'}{f(y)f'(y)}\right)\frac{\partial P_0}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial P_0}{\partial y}\right)\right\}$$

$$= \alpha^{\frac{1}{2}-H}\beta\gamma(y)\left(\frac{(f(y)f'(y))'}{f(y)f'(y)}\right)\frac{\partial P_0}{\partial y}-\gamma(y)\frac{\partial}{\partial y}\left(\alpha^{\frac{1}{2}-H}\beta\frac{\partial P_0}{\partial y}\right). \tag{7.31}$$

Here by (7.22), we have

$$\alpha^{\frac{1}{2}-H}\beta\frac{\partial P_0}{\partial y} = -\left(\frac{1}{T-t}\right)\frac{1}{\gamma(y)}P_1.$$

This is substituted into the last term of the right-hand side of (7.31) as follows:

$$f(y)f'(y)X^2\frac{\partial^2 P_1}{\partial X^2}$$

$$= \alpha^{\frac{1}{2}-H}\beta\gamma(y)\left(\frac{(f(y)f'(y))'}{f(y)f'(y)}\right)\frac{\partial P_0}{\partial y}+\gamma(y)\left(\frac{1}{T-t}\right)\frac{\partial}{\partial y}\left(\frac{1}{\gamma(y)}P_1\right)$$

$$= \alpha^{\frac{1}{2}-H}\beta\gamma(y)\left(\frac{(f(y)f'(y))'}{f(y)f'(y)}\right)\frac{\partial P_0}{\partial y}$$

$$+\gamma(y)\left(\frac{1}{T-t}\right)\left\{-\left(\frac{\gamma'(y)}{\gamma(y)^2}\right)P_1+\frac{1}{\gamma(y)}\frac{\partial P_1}{\partial y}\right\}$$

$$= \alpha^{\frac{1}{2}-H}\beta\gamma(y)\left(\frac{(f(y)f'(y))'}{f(y)f'(y)}\right)\frac{\partial P_0}{\partial y}$$

$$+\left(\frac{1}{T-t}\right)\left\{-\left(\frac{\gamma'(y)}{\gamma(y)}\right)P_1+\frac{\partial P_1}{\partial y}\right\}. \tag{7.32}$$

In the last equation of (7.32), we insert expression (7.22) for P_1 into the term $\left(\frac{\gamma'(y)}{\gamma(y)}\right)P_1$ and hence obtain the following equations:

$$
\begin{aligned}
&f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_1}{\partial X^2}\\
&=\alpha^{\frac{1}{2}-H}\beta\gamma(y)\left(\frac{(f(y)\,f'(y))'}{f(y)\,f'(y)}\right)\frac{\partial P_0}{\partial y}\\
&\quad+\left(\frac{1}{T-t}\right)\left\{-\left(\frac{\gamma'(y)}{\gamma(y)}\right)\left(-(T-t)\alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_0}{\partial y}\right)+\frac{\partial P_1}{\partial y}\right\}\\
&=\alpha^{\frac{1}{2}-H}\beta\gamma(y)\left(\frac{(f(y)\,f'(y))'}{f(y)\,f'(y)}\right)\frac{\partial P_0}{\partial y}+\alpha^{\frac{1}{2}-H}\beta\gamma'(y)\frac{\partial P_0}{\partial y}\\
&\quad+\left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}\\
&=\alpha^{\frac{1}{2}-H}\beta\left\{\gamma(y)\left(\frac{(f(y)\,f'(y))'}{f(y)\,f'(y)}\right)+\gamma'(y)\right\}\frac{\partial P_0}{\partial y}+\left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}\\
&=\alpha^{\frac{1}{2}-H}\beta\left\{\frac{\gamma(y)\cdot(f(y)\,f'(y))'+\gamma'(y)\cdot(f(y)\,f'(y))}{f(y)\,f'(y)}\right\}\frac{\partial P_0}{\partial y}+\left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&f(y)\,f'(y)\,X^2\,\frac{\partial^2 P_1}{\partial X^2}\\
&=\alpha^{\frac{1}{2}-H}\beta\left\{\frac{(\gamma(y)\cdot(f(y)\,f'(y)))'}{f(y)\,f'(y)}\right\}\frac{\partial P_0}{\partial y}+\left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}.
\end{aligned}
\tag{7.33}
$$

Thus, by (7.29) and (7.33), we can rewrite (7.28) as follows:

$$
\begin{aligned}
&\mathcal{L}_{BS}(I_0)\left[C(y)\frac{\partial P_1}{\partial y}\right]\\
&=C(y)\left[-\left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}\right.\\
&\quad\left.-\alpha^{\frac{1}{2}-H}\beta\left\{\frac{(\gamma(y)\cdot(f(y)\,f'(y)))'}{f(y)\,f'(y)}\right\}\frac{\partial P_0}{\partial y}-\left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}\right].
\end{aligned}
\tag{7.34}
$$

Step 4. By (7.26) and (7.34), equation (7.24) can be written as follows:

$$\begin{aligned}\mathcal{L}_{BS}(I_0)P_2 = B(y)&\left\{-\left(\frac{1}{T-t}\right)\frac{\partial P_0}{\partial y}\right\}\\ &+ C(y)\Bigg[-\left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}\\ &\qquad - \alpha^{\frac{1}{2}-H}\beta\left\{\frac{(\gamma(y)\cdot(\gamma(y)f'(y)))'}{f(y)f'(y)}\right\}\frac{\partial P_0}{\partial y} - \left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}\Bigg].\end{aligned}$$

Since P_2 satisfies PDE (7.8), the equation above yields

$$\begin{aligned}&B(y)\left\{-\left(\frac{1}{T-t}\right)\frac{\partial P_0}{\partial y}\right\}\\ &+ C(y)\Bigg[-\left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}\\ &\qquad - \alpha^{\frac{1}{2}-H}\beta\left\{\frac{(\gamma(y)\cdot(f(y)f'(y)))'}{f(y)f'(y)}\right\}\frac{\partial P_0}{\partial y} - \left(\frac{1}{T-t}\right)\frac{\partial P_1}{\partial y}\Bigg]\\ &= -\left[\alpha(m-y)\frac{\partial P_0}{\partial y} - \alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_1}{\partial y}\right]. \qquad (7.35)\end{aligned}$$

Observe (7.22) and substitute the equation

$$\frac{\partial P_1}{\partial y} = A'(y)\frac{\partial P_0}{\partial y} + A(y)\frac{\partial^2 P_0}{\partial y^2}$$

with $A(y) = -(T-t)\alpha^{\frac{1}{2}-H}\beta\gamma(y)$ into (7.35). Then, we get

$$\begin{aligned}&B(y)\left\{-\left(\frac{1}{T-t}\right)\frac{\partial P_0}{\partial y}\right\}\\ &+ C(y)\left[-\left(\frac{1}{T-t}\right)\left\{A'(y)\frac{\partial P_0}{\partial y} + A(y)\frac{\partial^2 P_0}{\partial y^2}\right\}\right.\end{aligned}$$

$$- \alpha^{\frac{1}{2}-H}\beta\left\{\frac{(\gamma(y)\cdot(f(y)f'(y)))'}{f(y)f'(y)}\right\}\frac{\partial P_0}{\partial y}$$

$$-\left(\frac{1}{T-t}\right)\left\{A'(y)\frac{\partial^2 P_0}{\partial y}+A(y)\frac{\partial^2 P_0}{\partial y^2}\right\}\Bigg]$$

$$= -\left[\alpha(m-y)\frac{\partial P_0}{\partial y}-\alpha^{\frac{1}{2}-H}\beta\gamma(y)\left\{A'(y)\frac{\partial P_0}{\partial y}+A(y)\frac{\partial^2 P_0}{\partial y^2}\right\}\right]. \tag{7.36}$$

Compare the both sides of (7.36). Then we choose $B(y)$ and $C(y)$ so that the coefficients in front of the partial derivatives $\frac{\partial P_0}{\partial y}$ and $\frac{\partial^2 P_0}{\partial y^2}$ on the left- and right-hand sides are equal. Namely, we find $B(y)$ and $C(y)$ by the following relation:

$$B(y)\left\{-\left(\frac{1}{T-t}\right)\right\}$$

$$+\,C(y)\left[-\left(\frac{2}{T-t}\right)A'(y)-\alpha^{\frac{1}{2}-H}\beta\left\{\frac{(\gamma(y)\cdot(f(y)f'(y)))'}{f(y)f'(y)}\right\}\right]$$

$$= -\left[\alpha(m-y)-\alpha^{\frac{1}{2}-H}\beta\gamma(y)A'(y)\right] \tag{7.37}$$

and

$$C(y)\left[-\left(\frac{2}{T-t}\right)A(y)\right]=\left[\alpha^{\frac{1}{2}-H}\beta\gamma(y)A(y)\right]. \tag{7.38}$$

Thus, solving equations (7.37) and (7.38) with respect to $B(y)$ and $C(y)$, we obtain

$$B(y)=(T-t)\alpha(m-y)+\frac{(T-t)^2}{2}\alpha^{1-2H}\beta^2\gamma(y)\left\{\frac{(\gamma(y)\cdot(f(y)f'(y)))'}{f(y)f'(y)}\right\}, \tag{7.39}$$

$$C(y)=-\left(\frac{T-t}{2}\right)\alpha^{\frac{1}{2}-H}\beta\gamma(y). \tag{7.40}$$

Theorem 7.2. *Let* $0 < H < 1$. *Assume that* $f(y)$ *is a positive and twice continuously differentiable function and* $\gamma(y)$ *is a differentiable function such that* $f'(y)\gamma(y) \neq 0$ $(y \in \mathbb{R})$. *Then, for* α *small enough, the corrected Black-Scholes price is given by*

$$P \approx P_0 + \sqrt{\delta}P_1 + \delta P_2. \tag{7.41}$$

Here P_0 *is the Black-Scholes price of the claim at the volatility level* $I_0 = f(y)$, *that is,*

$$\mathcal{L}_{BS}(I_0)P_0 = 0, \quad P_0(t, X, y) = C_{BS}(t, X; K, T; I_0)(= C_{BS}(I_0), \text{ for short})$$

with C_{BS} *as given by* (7.6) *with terminal condition* $P_0(T, X, y) = h(X)$ *for a payoff function* $h(X)$. *The first correction* $P_1(t, X, y)$ *is given by*

$$P_1 = -(T-t)\alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_0}{\partial y};$$

P_1 *is a solution of the equation*

$$\mathcal{L}_{BS}(I_0)P_1 = \alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_0}{\partial y}.$$

The second correction $P_2(t, X, y)$ *is given by*

$$P_2 = B(y)\frac{\partial P_0}{\partial y} + C(y)\frac{\partial P_1}{\partial y},$$

where $B(y)$ *and* $C(y)$ *are given by* (7.39) *and* (7.40), *respectively*; P_2 *is a solution of the equation*

$$\mathcal{L}_{BS}(I_0)P_2 = -\left[\alpha(m-y)\frac{\partial P_0}{\partial y} - \alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_1}{\partial y}\right].$$

In Theorem 7.2, the first correction P_1 is given in terms of $\frac{\partial P_0}{\partial y}$, and the second correction P_2 is given in terms of the derivatives $\frac{\partial P_0}{\partial y}$ and $\frac{\partial P_1}{\partial y}$. Recall that $P_0 = C_{BS}(\sigma)$ with $\sigma = I_0 = f(y) > 0$ and consider

$$\frac{\partial P_0}{\partial y} = \left(\frac{\partial C_{BS}}{\partial \sigma} \frac{\partial \sigma}{\partial y} \right) \bigg|_{\sigma = I_0}.$$

Then, by (7.18), (7.19) and (7.20), we can rewrite the derivatives $\frac{\partial P_0}{\partial y}$ and $\frac{\partial P_1}{\partial y}$ at $(t, X; K, T; I_0)$ in terms of Vega, DVegaDVol and Gamma of the Greeks. These are summarized in the following remark:

Remark 7.3.

$$\frac{\partial P_0}{\partial y} = f'(y) \frac{\partial C_{BS}}{\partial \sigma},$$

$$\frac{\partial P_0}{\partial y} = (T - t) f(y) f'(y) X^2 \frac{\partial^2 C_{BS}}{\partial X^2},$$

$$\frac{\partial P_1}{\partial y} = \frac{\partial}{\partial y} \left[-(T - t) \alpha^{\frac{1}{2} - H} \beta \gamma(y) \frac{\partial P_0}{\partial y} \right]$$

$$= -(T - t) \alpha^{\frac{1}{2} - H} \beta \left[\gamma'(y) \frac{\partial P_0}{\partial y} + \gamma(y) \frac{\partial^2 P_0}{\partial y^2} \right]$$

$$= -(T - t) \alpha^{\frac{1}{2} - H} \beta$$

$$\times \left[\gamma'(y) f'(y) \frac{\partial C_{BS}}{\partial \sigma} + \gamma(y) \left\{ f''(y) \frac{\partial C_{BS}}{\partial \sigma} + (f'(y))^2 \frac{\partial^2 C_{BS}}{\partial \sigma^2} \right\} \right],$$

$$\frac{\partial P_1}{\partial y} = -(T - t) \alpha^{\frac{1}{2} - H} \beta$$

$$\times \left[\gamma'(y) (T - t) f(y) f'(y) X^2 \frac{\partial^2 C_{BS}}{\partial X^2} \right.$$

$$+ \gamma(y) \left\{ (T - t) f(y) f''(y) X^2 \frac{\partial^2 C_{BS}}{\partial X^2} \right.$$

$$\left. \left. + (f'(y))^2 X \sqrt{T - t}\, n(d_1) \left(\frac{d_1 d_2}{\sigma} \right) \right\} \right].$$

Here $n(d)$ is the density function of the standard normal distribution $N(0, 1)$, and d_1 and d_2 are given by (6.2) and (6.2) at the volatility level $\sigma = I_0 = f(y) > 0$; $d_1 = d_1(I_0)$ and $d_2 = d_2(I_0)$.

8. Implied Volatility in the Case of Slow Scale

Taking the corrected pricing formula (7.41) as observed price,

$$C_{obs} = P_0 + \sqrt{\delta}P_1 + \delta P_2 \tag{8.1}$$

in equation (6.7), the relation that determines the implied volatility is thus

$$C_{BS}(t, X; K, T; I) = P_0 + \sqrt{\delta}P_1 + \delta P_2. \tag{8.2}$$

Equation (8.2) can be solved by expanding I as

$$I = I_0 + \sqrt{\delta}I_1 + \delta I_2 + \cdots. \tag{8.3}$$

In (8.3), the same notation I_0 as that in the preceding section appears. However, here we denote by I_0 the zeroth order-term to be determined later on. Inserting (8.3) into the left-hand side of (8.2), we get the following (see Appendix):

$$\begin{aligned} &C_{BS}(t, X; K, T; I_0) + \sqrt{\delta}I_1 \frac{\partial C_{BS}}{\partial \sigma}(t, X; K, T; I_0) \\ &+ \delta\left(I_2 \frac{\partial C_{BS}}{\partial \sigma}(t, X; K, T; I_0) + \frac{1}{2} I_1^2 \frac{\partial^2 C_{BS}}{\partial \sigma^2}(t, X; K, T; I_0)\right) + \cdots \\ = {}& P_0 + \sqrt{\delta}P_1 + \delta P_2 + \cdots. \end{aligned} \tag{8.4}$$

The terms of order $(\sqrt{\delta})^0, \sqrt{\delta}, \delta, \ldots$ will be studied. Equating terms of order $(\sqrt{\delta})^0$, we must have

$$C_{BS}(t, X; K, T; I_0) = P_0,$$

where P_0 is the solution of equation (7.6) associated with the Black-Scholes operator $\mathcal{L}_{BS}(\sigma)$ with the volatility level $\sigma = f(y) > 0$. Hence we find

$$I_0 = f(y). \tag{8.5}$$

Equating terms of order $\sqrt{\delta}$, we get

$$I_1 \frac{\partial C_{BS}}{\partial \sigma}(t, X; K, T; I_0) = P_1,$$

and hence

$$I_1 = P_1\left(\frac{\partial C_{BS}}{\partial \sigma}(t, X; K, T; I_0)\right)^{-1}.$$

By (7.22) and Remark 7.3, we observe

$$P_1 = -(T-t)\alpha^{\frac{1}{2}-H}\beta\gamma(y)\frac{\partial P_0}{\partial y}$$

$$= -(T-t)\alpha^{\frac{1}{2}-H}\beta\gamma(y)\left(f'(y)\frac{\partial C_{BS}}{\partial \sigma}\right).$$

This implies

$$I_1 = -(T-t)\alpha^{\frac{1}{2}-H}\beta\gamma(y)\left(f'(y)\frac{\partial C_{BS}}{\partial \sigma}(t, X; K, T; I_0)\right)$$

$$\times\left(\frac{\partial C_{BS}}{\partial \sigma}(t, X; K, T; I_0)\right)^{-1}$$

$$= -(T-t)\alpha^{\frac{1}{2}-H}\beta\gamma(y)f'(y). \tag{8.6}$$

Further, equating terms of order δ, we get

$$I_2 = \left(P_2 - \frac{1}{2}I_1^2\frac{\partial^2 C_{BS}}{\partial \sigma^2}(t, X; K, T; I_0)\right)\times\left(\frac{\partial C_{BS}}{\partial \sigma}(t, X; K, T; I_0)\right)^{-1}. \tag{8.7}$$

In (8.7), P_2 is represented by (7.23) in terms of $\frac{\partial P_0}{\partial y}$ and $\frac{\partial P_1}{\partial y}$; these partial derivatives are given by Remark 7.3. Moreover, I_1 is given by (8.6). Then substitution of these expressions into (8.7) and arrangement of the equation yield the following:

$$I_2 = B(y) f'(y)$$

$$+ C(y)\left[-(T-t)\alpha^{\frac{1}{2}-H}\beta\right.$$

$$\times\left\{\gamma'(y) f'(y)\right.$$

$$\left.\left.+ \gamma(y)\left(f''(y) + \frac{\frac{\partial^2 C_{BS}}{\partial\sigma^2}(t,\ X;\ K,\ T;\ I_0)}{\frac{\partial C_{BS}}{\partial\sigma}(t,\ X;\ K,\ T;\ I_0)}(f'(y))^2\right)\right\}\right]$$

$$-\frac{1}{2}(T-t)^2\alpha^{1-2H}\beta^2\gamma^2(y)(f'(y))^2\left(\frac{\frac{\partial^2 C_{BS}}{\partial\sigma^2}(t,\ X;\ K,\ T;\ I_0)}{\frac{\partial C_{BS}}{\partial\sigma}(t,\ X;\ K,\ T;\ I_0)}\right). \tag{8.8}$$

Here $B(y)$ and $C(y)$ are the functions as given by (7.39) and (7.40).

Theorem 8.1. *Assume the same condition as that in Theorem* 7.2. *Then the implied volatility I is given by*

$$I \approx I_0 + \sqrt{\delta}I_1 + \delta I_2 \tag{8.9}$$

as $\delta \to 0$. *Here* $I_0 = f(y)$ *and* $I_1 = -(T-t)\alpha^{\frac{1}{2}-H}\beta\gamma(y) f'(y)$. *Further,* I_2 *is represented by* (8.8) *with the coefficients* $B(y)$ *and* $C(y)$ *as given by* (7.39) *and* (7.40), *respectively*.

Appendix: Asymptotic Expansion

Let η be a small parameter such that $0 < \eta << 1$. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then we assume that g can be expanded as follows:

$$g(x_0 + \eta x_1 + \eta^2 x_2 + \cdots) = \sum_{n=0}^{\infty} g_n(x_0, x_1, x_2, ..., x_n)\eta^n$$

with

$$g_0(x_0) = g(x_0)$$

and

$$g_n(x_0, x_1, x_2, ..., x_n) = \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ 1\le k\le n}} \frac{x_{n_1} \cdot x_{n_2} \cdots x_{n_k}}{k!} \left[\frac{d^k}{ds^k} g(s)\right]_{s=x_0}.$$

The first coefficients can be calculated, for example,

$$g_0(x_0) = g(x_0),$$

$$g_1(x_0, x_1) = x_1 \cdot g'(x_0),$$

$$g_2(x_0, x_1, x_2) = x_2 \cdot g'(x_0) + \frac{x_1^2}{2} \cdot g''(x_0),$$

$$\ldots.$$

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