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THREE DIMENSIONAL Q-ALGEBRAS

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Abstract

We describe the structure of a 3-dimensional commutative Banach algebra $\mathcal{B}$ with identity by the classification of $\mathcal{B}$ through the number of the elements of the maximal ideal space $M_{\mathcal{B}}$. It is proved that $\mathcal{B}$ is isomorphic to a Q-algebra of a bidisc algebra. As an application, we study BQ-algebras and CQ-algebras which are generalizations of Q-algebras. A BQ-algebra (resp. CQ-algebra) is defined to be a commutative Banach algebra $\mathcal{B}$ with identity such that there exists a bounded (resp. contractive) isomorphism from a Q-algebra to $\mathcal{B}$.

1. Introduction

Let A be a uniform algebra on a compact Hausdorff space X. If I is a closed

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ideal of A, then the quotient algebra A/I is a commutative Banach algebra. If there exists an isometric isomorphism from A/I to $\mathcal{B}$, then $\mathcal{B}$ is called a *Q-algebra* of A. If $\mathcal{B}$ is a Q-algebra of A for some A, then $\mathcal{B}$ is called a *Q-algebra*. Bonsall and Duncan called Q-algebra $\mathcal{B}$ as an *IQ-algebra* (cf. [2, p. 270]). Given a Hilbert space H, we denote by $B(H)$ the set of all bounded linear operators on H. Cole (cf. [1, p. 216], [2, p. 272], [3, p. 98], [4, p. 31]) proved that there exists a Hilbert space H and a closed subalgebra $\mathcal{B}$ of $B(H)$ such that A/I is isometrically isomorphic to $\mathcal{B}$. Which commutative operator subalgebra $\mathcal{B}$ of $B(H)$ with identity is a Q-algebra? Drury [5] and Nakazi [9, Corollary 2] proved that a 2-dimensional commutative operator subalgebra of $B(\mathbb{C}^2)$ with identity is a Q-algebra. By the example of Holbrook [7], it follows that a 4-dimensional commutative operator subalgebra of $B(\mathbb{C}^4)$ with identity is not necessarily a Q-algebra. Suppose $\mathcal{B}$ is a 3-dimensional commutative operator subalgebra of $B(\mathbb{C}^3)$ with identity. Is $\mathcal{B}$ a Q-algebra? This is an important question. But it is too difficult for us to solve it. We consider BQ-algebras and CQ-algebras as the following. If there exists a bounded (resp. contractive) isomorphism from A/I to $\mathcal{B}$, then $\mathcal{B}$ is called a *BQ-algebra* (resp. *CQ-algebra*) of A. Since we consider a finite dimensional algebra in this paper, every such isomorphism is bounded. If $\mathcal{B}$ is a BQ-algebra (resp. CQ-algebra) of A for some A, then $\mathcal{B}$ is called a *BQ-algebra* (resp. *CQ-algebra*). Hence Q-algebra $\Rightarrow$ CQ-algebra $\Rightarrow$ BQ-algebra. Nakazi [9, Proposition 1] proved that a 2-dimensional commutative Banach algebra $\mathcal{B}$ with identity is a BQ-algebra. He proved that $\mathcal{B}$ is spanned by 1 and g, where $g^2 = 0$ or $g^2 = g$. We denote by $\mathbb{T}$ and $\mathbb{D}$ the unit circle and the open unit disc in the complex plane, respectively. Then $\mathcal{B}$ is a BQ-algebra of the disc algebra $A(\mathbb{T})$. To see this, take $A = A(\mathbb{T})$ and $I = \{f \in A(\mathbb{T}) : f(0) = f'(0) = 0\}$ or $I = \{f \in A(\mathbb{T}) : f(a) = f(b) = 0\}$ for distinct points a and b in $\mathbb{D}$.

Problem 1. Suppose $\mathcal{B}$ is a 3-dimensional commutative Banach algebra with identity. Prove that $\mathcal{B}$ is a BQ-algebra.

Definition 1.1. We denote by $M_{\mathcal{B}}$ the spectrum of $\mathcal{B}$, the space of all multiplicative linear functionals on $\mathcal{B}$, and denote by $\sharp M_{\mathcal{B}}$, the number of the elements of $M_{\mathcal{B}}$.

In Section 2, we will solve Problem 1. We will describe the structure of $\mathcal{B}$ by the classification of $\mathcal{B}$ through $\sharp M_{\mathcal{B}}$, and prove that a 3-dimensional commutative Banach algebra with identity is a *BQ*-algebra of $A(\mathbb{T}^2)$.

Problem 2. Describe all 3-dimensional *BQ*-subalgebras of $B(\mathbb{C}^3)$.

In Section 3, we will solve Problem 2. Is a *BQ*-algebra always a *CQ*-algebra? This is an important question. But it is too difficult for us to solve it. We consider an operator S_f^{μ} and a *CQ*-algebra $\{S_f^{\mu} : f \in A\}$ as the following (cf. [3, p. 98]).

Definition 1.2. Let μ be a probability measure on a compact Hausdorff space *X* and let *A* be a uniform algebra on *X*. Let $H^2(\mu)$ be the closure of *A* in $L^2(\mu)$ and let $H^2(\mu) \cap I^{\perp}$ be the annihilator of *I* in $H^2(\mu)$. Let *P* be the orthogonal projection from $H^2(\mu)$ onto $H^2(\mu) \cap I^{\perp}$. For any $f \in A$, we define S_f^{μ} as the operator on $H^2(\mu) \cap I^{\perp}$ such that $S_f^{\mu}\psi = P(f\psi)$, $(\psi \in H^2(\mu) \cap I^{\perp})$.

Then $S_{f+k}^{\mu} = S_f^{\mu}$ for *k* in *I* and $\| S_f^{\mu} \| \le \| f + I \|$. $S^{\mu} : A/I \to B(H^2(\mu) \cap I^{\perp})$ is a contractive isomorphism which sends $f + I \to S_f^{\mu}$ for each *f* in *A*. Hence $\{S_f^{\mu} : f \in A\}$ is a *CQ*-algebra. The kernel of S^{μ} contains *I*. If $\| S_f^{\mu} \| = \| f + I \|$, $(f \in A)$, then $\ker S^{\mu} = I$, and $\{S_f^{\mu} : f \in A\}$ is a *Q*-algebra.

Problem 3. Suppose $\mathcal{B}$ is a 3-dimensional *BQ*-subalgebra of $B(\mathbb{C}^3)$. Prove that $\mathcal{B}$ is not necessarily a *CQ*-algebra $\{S_f^{\mu} : f \in A\}$ for some uniform algebra *A*.

In Section 4, we will solve Problem 3 in the case when $\sharp M_{\mathcal{B}} = 1$ or 2. It is too difficult for us to prove it when $\sharp M_{\mathcal{B}} = 3$. We will study the structure of a *CQ*-algebra $\{S_f^{\mu} : f \in A\}$. By Theorems 4.3 and 4.5, if $\sharp M_{A/I} = 1, 2,$ then a set of all 3-dimensional *CQ*-algebras $\{S_f^{\mu} : f \in A\}$ is a proper subset of a set of all 3-dimensional *BQ*-subalgebras of $B(\mathbb{C}^3)$. If $\sharp M_{A/I} = 3$, then we have Remark B, but we do not know whether a set of all 3-dimensional *CQ*-algebras $\{S_f^{\mu} : f \in A\}$ is a proper subset of a set of all 3-dimensional *BQ*-subalgebras of $B(\mathbb{C}^3)$.

As an application of the results in Sections 2, 3 and 4, we will give some examples in Sections 5 and 6. If S^{μ} is isometric, then $\{S_f^{\mu} : f \in A\}$ is a Q-algebra. Hence we will consider whether S^{μ} is isometric for a concrete CQ-algebra $\{S_f^{\mu} : f \in A\}$ in Sections 5 and 6. In Section 5, for the disc algebra $A(\mathbb{T})$, and for $d\mu = d\theta/2\pi$ or $d\mu = rdrd\theta/\pi$, we will describe a 3-dimensional CQ-algebra $\{S_f^{\mu} : f \in A(\mathbb{T})\}$. By Sarason's theorem (cf. [3, p. 125], [12]), if $d\mu = d\theta/2\pi$, then $S^{\mu} : A(\mathbb{T})/I \to B(H^2(\mu) \cap I^{\perp})$ is an isometric isomorphism, and hence $\{S_f^{\mu} : f \in A(\mathbb{T})\}$ is a Q-algebra of $A(\mathbb{T})$. In Section 6, for the bidisc algebra $A(\mathbb{T}^2)$ and for $d\mu = d\theta_1 d\theta_2/(2\pi)^2$, we will describe a 3-dimensional CQ-algebra $\{S_f^{\mu} : f \in A(\mathbb{T}^2)\}$.

2. Banach Algebras and *BQ*-algebras

In this section, we solve Problem 1. Let $\mathcal{B}$ be a 3-dimensional commutative Banach algebra. We classify all $\mathcal{B}$ by the number $\sharp M_{\mathcal{B}} = 1, 2, 3$ of elements in $M_{\mathcal{B}}$ and establish the structure of $\mathcal{B}$ by the following Propositions 2.1, 2.2 and 2.3. These give the solution of Problem 1 as Theorem 2.8. Let ϕ be in $M_{\mathcal{B}}$. Then $\phi(fg) = \phi(f)\phi(g)$, $(f, g \in \mathcal{B})$. A 1st point derivation at ϕ is a linear functional D^1 on $\mathcal{B}$ which satisfies

$$D^1(fg) = D^1(f)\phi(g) + \phi(f)D^1(g), \quad (f, g \in \mathcal{B}).$$

A 2nd point derivation at ϕ is a linear functional D^2 on $\mathcal{B}$ which satisfies

$$D^2(fg) = D^2(f)\phi(g) + 2D^1(f)D^1(g) + \phi(f)D^2(g), \quad (f, g \in \mathcal{B}).$$

Proposition 2.1. *Let* $\mathcal{B}$ *be a* 3-*dimensional commutative Banach algebra with identity*. *Then the following conditions* (1) *and* (2) *are equivalent*:

(1) $\sharp M_{\mathcal{B}} = 1$, *that is,* $M_{\mathcal{B}} = \{\phi\}$ *for some* ϕ.

(2) (a) *or* (b) *below holds*:

(a) $\mathcal{B} = span\{1, g, h\}$ *for some g, h satisfying* $g^2 = h^2 = 0$.

(b) $\mathcal{B} = span\{1, g, g^2\}$ *for some g satisfying* $g^3 = 0$.

If (a) *holds, then* $gh = 0$ *and there exist two nontrivial* 1*st point derivations* D_1^1 *and* D_2^1 *at* ϕ *such that for all f in* $\mathcal{B}$,

$$f = \phi(f) + D_1^1(f)g + D_2^1(f)h.$$

If (b) *holds, then there exists a nontrivial* 1*st point derivation* D^1 *and a nontrivial* 2*nd point derivation* D^2 *at* ϕ *such that for all f in* $\mathcal{B}$,

$$f = \phi(f) + D^1(f)g + \frac{D^2(f)}{2}g^2 \quad (f \in \mathcal{B}).$$

Proposition 2.2. *Let* $\mathcal{B}$ *be a* 3-*dimensional commutative Banach algebra with identity. Then the following conditions are equivalent*:

(1) $\sharp M_{\mathcal{B}} = 2$, *that is,* $M_{\mathcal{B}} = \{\phi, \theta\}$ *for some* ϕ *and* θ.

(2) $\sharp M_{\mathcal{B}} = 2$, *that is,* $M_{\mathcal{B}} = \{\phi, \theta\}$ *for some* ϕ *and* θ, *and there exists a nontrivial* 1*st point derivation* D^1 *at* ϕ *or at* θ.

(3) *There exist g, h in* $\mathcal{B}$ *such that* $\mathcal{B} = span\{1, g, h\}$, *where* $g^2 - g = h^2 = gh = 0$.

If there exists a nontrivial 1*st point derivation* D^1 *at* ϕ, *then for all f in* $\mathcal{B}$,

$$f = \phi(f)(1 - g) + \theta(f)g + D^1(f)h.$$

Proposition 2.3. *Let* $\mathcal{B}$ *be a* 3-*dimensional commutative Banach algebra with identity. Then the following conditions are equivalent*:

(1) $\sharp M_{\mathcal{B}} = 3$, *that is,* $M_{\mathcal{B}} = \{\phi_1, \phi_2, \phi_3\}$ *for some* ϕ_1, ϕ_2, ϕ_3.

(2) *There exist g, h in* $\mathcal{B}$ *such that* $\mathcal{B} = span\{1, g, h\}$, *where* $g^2 - g = h^2 - h = gh = 0$.

Under these conditions, for all f in $\mathcal{B}$,

$$f = \phi_1(f)g + \phi_2(f)h + \phi_3(f)(1 - g - h).$$

Definition 2.4. If ϕ is an element of $M_{\mathcal{B}}$, then we denote by $\mathcal{B}_\phi$ the maximal ideal which is the kernel of ϕ.

Lemma 2.5. *Let* $\mathcal{B}$ *be a commutative Banach algebra with identity. Suppose g and h are in* $\mathcal{B}$, $g \neq 0$ *and* $gh = 0$. *Then there exists a* ϕ *in* $M_{\mathcal{B}}$ *such that* $\phi(h) = 0$.

Proof. It is sufficient to prove that if h is in $\mathcal{B}$ and $0 \in \sigma(h)$, then there exists a ϕ in $M_{\mathcal{B}}$ such that $\phi(h) = 0$. Since h is not invertible, $h\mathcal{B}$ is a proper ideal. Hence there exists a ϕ in $M_{\mathcal{B}}$ such that $\mathcal{B}_\phi$ contains $h\mathcal{B}$. Since $1 \in \mathcal{B}$, $h \in \mathcal{B}_\phi$. □

Lemma 2.6. *Let* $\mathcal{B} = span\{1, g, h\}$ *be a* 3-*dimensional commutative Banach algebra. If* $g^2 = h^2 = 0$, *then* $gh = 0$.

Proof. Since $gh = c_1 g + c_2 h$ for some $c_1, c_2 \in \mathbb{C}$, $g^2 h = c_1 g^2 + c_2 gh$ and $gh^2 = c_1 gh + c_2 h^2$. Hence if $g^2 = h^2 = 0$, then $c_1 gh = c_2 gh = 0$. This implies $gh = 0$. □

Proof of Proposition 2.1. (2) $\Rightarrow$ (1) Suppose (a) holds. For every ϕ in $M_{\mathcal{B}}$, $\phi(g) = \phi(h) = 0$, because $0 = \phi(g^2) = \phi(g)^2$ and $0 = \phi(h^2) = \phi(h)^2$. Hence $\sharp M_{\mathcal{B}} = 1$.

Suppose (b) holds. For every ϕ in $M_{\mathcal{B}}$, $\phi(g) = \phi(g^2) = 0$. Hence $\sharp M_{\mathcal{B}} = 1$.

(1) $\Rightarrow$ (2) Since $\dim \mathcal{B}_\phi = 2$, $\mathcal{B}_\phi = span\{g, h\}$ for some g and h.

First, we prove that if $g^2 \neq 0$ or $h^2 \neq 0$, then (2)(b) holds. Since (2)(a) is a symmetric condition with respect to g and h, it is sufficient to prove that if $g^2 \neq 0$, then (2)(b) holds. Since $g^2 \in \mathcal{B}$, $g^2 = c_3 g + c_4 h$ for some $c_3, c_4 \in \mathbb{C}$. Suppose $c_4 = 0$. Then $g(c_3 - g) = 0$. By Lemma 2.5, $c_3 = 0$, and hence $g^2 = 0$. This contradiction implies that $c^4 \neq 0$. Therefore $h \in span\{g, g^2\}$. By Lemma 2.5,

$\dim span\{1, g, g^2\} = 3$ and $g^3 = 0$. We have proved the equivalence of (1) and (2). Next, we prove the latter half. If (a) holds, then for any $f \in \mathcal{B}$, there exist uniquely complex numbers f_0, f_1, f_2 such that $f = f_0 + f_1 g + f_2 h$, $(f \in \mathcal{B})$. Hence $\phi(f) = f_0$. If we define c_1 and c_2 by $c_1(f) = f_1$ and $c_2(f) = f_2$, then $c_1, c_2 \in \mathcal{B}^*$, where $\mathcal{B}^*$ denotes the set of all bounded linear functionals on $\mathcal{B}$. Then it is sufficient to prove that $c_1 = D_1^1, c_2 = D_2^1$. By Lemma 2.6, $gh = 0$. Hence, for $F = F_0 + F_1 g + F_2 h$ and $G = G_0 + G_1 g + G_2 h$,

$$c_1(FG) = F_1 G_0 + F_0 G_1 = c_1(F)\phi(G) + \phi(F)c_1(G),$$

$$c_2(FG) = F_2 G_0 + F_0 G_2 = c_2(F)\phi(G) + \phi(F)c_2(G).$$

Thus $c_1 = D_1^1$ and $c_2 = D_2^1$.

If (b) holds, then for any $f \in \mathcal{B}$, there exist uniquely complex numbers f_0, f_1, f_2 such that

$$f = f_0 + f_1 g + f_2 g^2 \qquad (f \in \mathcal{B}).$$

Hence $\phi(f) = f_0$. If we define δ_1 and δ_2 by $\delta_1(f) = f_1$ and $\delta_2(f) = 2f_2$, then $\delta_1, \delta_2 \in \mathcal{B}^*$. It is sufficient to show that $\delta_1 = D^1$ and $\delta_2 = D^2$. For $F = F_0 + F_1 g + F_2 g^2$ and $G = G_0 + G_1 g + G_2 g^2$,

$$\delta_1(FG) = F_1 G_0 + F_0 G_1 = \delta_1(F)\phi(G) + \phi(F)\delta_1(G),$$

$$\delta_2(FG) = 2(F_2 G_0 + F_1 G_1 + F_0 G_2)$$

$$= \delta_2(F)\phi(G) + 2\delta_1(F)\delta_1(G) + \phi(F)\delta_2(G).$$

Thus $\delta_1 = D^1$ and $\delta_2 = D^2$. □

Proof of Proposition 2.2. (2) ⇒ (1) Trivial.

(1) ⇒ (3) First, we prove that there exists a nontrivial 1st point derivation D^1 at ϕ or at θ. Since $\mathcal{B}_\phi \cap \mathcal{B}_\theta$ is 1-dimensional, there exists a nonzero h in $\mathcal{B}_\phi \cap \mathcal{B}_\theta$ such that $h^2 = \alpha h$, for some $\alpha \in \mathbb{C}$. Hence $h(\alpha - h) = 0$. By Lemma 2.5, $\alpha = 0$.

Hence $h^2 = 0$. Since $\dim \mathcal{B}_\phi = 2$, $\mathcal{B}_\phi = span\{g, h\}$ for some g. Since $h^2 = 0$, $(\mathcal{B}_\phi)^2 = span\{g^2, gh\}$. If $gh = 0$, then $(\mathcal{B}_\phi)^2 = span\{g^2\}$, and hence $(\mathcal{B}_\phi)^2 \neq \mathcal{B}_\phi$. Therefore if $gh = 0$, then there exists a nontrivial 1st point derivation D^1 at ϕ (cf. [6, p. 22]). Suppose $gh \neq 0$. Since h is in $\mathcal{B}_\phi \cap \mathcal{B}_\theta$, gh is in $\mathcal{B}_\phi \cap \mathcal{B}_\theta$. Since $gh \neq 0$. there exists a nonzero $\gamma \in \mathbb{C}$ such that $gh = \gamma h$. Hence $h(\gamma - g) = 0$. By Lemma 2.5, $\phi(\gamma - g) = 0$ or $\theta(\gamma - g) = 0$. Since $\phi(g) = 0$ and $\gamma \neq 0$, this implies that $\theta(\gamma - g) = 0$. Since $1, g, h$ are linearly independent, $\gamma - g, h$ are linearly independent, and hence $\mathcal{B}_\theta = span\{\gamma - g, h\}$. Since $h^2 = h(\gamma - g) = 0$, $(\mathcal{B}_\theta)^2 = span\{(\gamma - g)^2\}$. Hence $(\mathcal{B}_\theta)^2 \neq \mathcal{B}_\theta$. Therefore, if $gh \neq 0$, then there exists a nontrivial 1st point derivation D^1 at θ (cf. [6, p. 22]).

Next, we prove that there exist g and h in $\mathcal{B}$ such that $g^2 - g = h^2 = gh = 0$. Suppose $M_{\mathcal{B}} = \{\phi, \theta\}$ and $\mathcal{B}^* = span\{\phi, \theta, D^1\}$, where D^1 is the 1st point derivation at ϕ. There exist g and h in $\mathcal{B}$ such that $D^1(1) = 0$, $(\phi(g), \theta(g), D^1(g)) = (0, 1, 0)$, $(\phi(h), \theta(h), D^1(h)) = (0, 0, 1)$.

Since $(\phi(g^2), \theta(g^2), D^1(g^2)) = (0, 1, 0)$, $\phi(g^2 - g) = \theta(g^2 - g) = D^1(g^2 - g) = 0$, and so $g^2 = g$. Since $\phi(h^2) = \theta(h^2) = D^1(h^2) = 0$, $h^2 = 0$. Since $\phi(gh) = \theta(gh) = D^1(gh) = 0$, $gh = 0$.

(3) $\Rightarrow$ (2) Suppose $\mathcal{B} = span\{1, g, h\}$, where $g^2 - g = h^2 = gh = 0$. If $\sharp M_{\mathcal{B}} = 1$, that is, $M_{\mathcal{B}} = \{\phi\}$, then $\phi(g) = 0$ or $\phi(g) = 1$ because $g^2 = g$. If $\phi(g) = 0$, then $(1 - g)\mathcal{B} = span\{1 - g, h\}$ is a maximal ideal, and hence $(1 - g)\mathcal{B} = \mathcal{B}_\phi$, because $M_{\mathcal{B}} = \{\phi\}$. Hence $1 - \phi(g) = \phi(1 - g) = 0$. This contradicts that $\phi(g) = 0$. If $\phi(g) = 1$, then $g\mathcal{B} = span\{g\} \not\subseteq \mathcal{B}_\phi$. This contradicts that $M_{\mathcal{B}} = \{\phi\}$. Thus $\sharp M_{\mathcal{B}} \geq 2$. If $\sharp M_{\mathcal{B}} = 3$, then $h = 0$ and it contradicts that $\dim \mathcal{B} = 3$. Thus $M_{\mathcal{B}} = \{\phi, \theta\}$. It is easy to see that there exists $\delta \in \mathcal{B}^*$ such that $\mathcal{B}^* = span\{\phi, \theta, \delta\}$, where $(\delta(1), \delta(g), \delta(h)) = (0, 0, 1)$. Since $h^2 = 0$, it follows that

$\phi(h) = \theta(h) = 0$. We may assume that $\phi(g) = 0$ or $\theta(g) = 0$. For, if $\phi(g) \neq 0$ and $\theta(g) \neq 0$, then $\phi(g) = \theta(g) = 1$. This contradicts that $g \neq 1$. If $\phi(g) = 0$, then $\theta(g) = 1$, because $\delta(g) = 0$. We will show that δ is the 1st point derivation at ϕ or at θ. If F and G are in $\mathcal{B}$, then $F = \alpha + \beta g + \gamma h$ and $G = a + bg + ch$ for some complex numbers α, β, γ, a, b, c. Then $FG = \alpha a + (\alpha b + \beta a + \beta b)g + (\alpha c + \gamma a)h$. If $\phi(g) = 1 - \theta(g) = 0$, then $(\phi(1), \theta(1), \delta(1)) = (1, 1, 0)$, $(\phi(g), \theta(g), \delta(g)) = (0, 1, 0)$, $(\phi(h), \theta(h), \delta(h)) = (0, 0, 1)$. Hence $\delta(FG) = \alpha c + \gamma a = \phi(F)\delta(G) + \delta(F)\phi(G)$. This implies that δ is the 1st point derivation at ϕ. If $1 - \phi(g) = \theta(g) = 0$, then $(\phi(1), \theta(1), \delta(1)) = (1, 1, 0)$, $(\phi(g), \theta(g), \delta(g)) = (1, 0, 0)$, $(\phi(h), \theta(h), \delta(h)) = (0, 0, 1)$. Hence $\delta(FG) = \alpha c + \gamma a = \theta(F)\delta(G) + \delta(F)\theta(G)$. This implies that δ is the 1st point derivation at θ, and hence (2) follows.

Therefore (1), (2) and (3) are equivalent. Under these conditions, suppose D is a nontrivial 1st point derivation at ϕ. Since $span\{1, g, h\} = span\{1 - g, g, h\}$, for all f in $\mathcal{B}$, there exist complex numbers $c_0(f)$, $c_1(f)$, $c_2(f)$ uniquely such that $f = c_0(f)(1 - g) + c_1(f)g + c_2(f)h$. Hence $\phi(f) = c_0(f)$, $\theta(f) = c_1(f)$. Let $\delta(f) = c_2(f)$. Then $\delta \in \mathcal{B}^*$ and $f = \phi(f)(1 - g) + \delta(f)h + \theta(f)g$. □

Proof of Proposition 2.3. (1) $\Rightarrow$ (2) If $M_{\mathcal{B}} = \{\phi_1, \phi_2, \phi_3\}$, then there exist g and h in $\mathcal{B}$ such that

$$\phi_1(g) = 1, \quad \phi_2(g) = \phi_3(g) = 0, \quad \phi_2(h) = 1, \quad \phi_1(h) = \phi_3(h) = 0.$$

Then $g^2 = g$, $h^2 = h$ and $gh = 0$, and so $\mathcal{B} = span\{1, g, h\}$.

(2) $\Rightarrow$ (1) If $\mathcal{B} = span\{1, g, h\}$, where $g^2 - g = h^2 - h = gh = 0$, then $span\{g, h\}$, $span\{1 - g, h\}$ and $span\{g, 1 - h\}$ are three distinct maximal ideals of $\mathcal{B}$. □

Lemma 2.7. *Let* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be two* 3-*dimensional commutative Banach algebras with identity. If* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *satisfy one of the conditions* (1) ~ (4), *then* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are algebraically isomorphic*:

(1) $\sharp M_{\mathcal{B}_1} = \sharp M_{\mathcal{B}_2} = 1$ *and both* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *have two different nontrivial* 1*st point derivations*.

(2) $\sharp M_{\mathcal{B}_1} = \sharp M_{\mathcal{B}_2} = 1$ *and both* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *have a nontrivial* 1*st point derivation and a nontrivial* 2*nd point derivation*.

(3) $\sharp M_{\mathcal{B}_1} = \sharp M_{\mathcal{B}_2} = 2$ *and both* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *have a nontrivial* 1*st point derivation*.

(4) $\sharp M_{\mathcal{B}_1} = \sharp M_{\mathcal{B}_2} = 3$.

Proof. We prove only (1) because other cases are similar. Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ satisfy (1). By Proposition 2.1, $\mathcal{B}_j = span\{1, g_j, h_j\}$, where $g_j^2 = h_j^2 = 0$ $(j = 1, 2)$. Now, it is clear to define an algebraic isomorphism from $\mathcal{B}_1$ to $\mathcal{B}_2$. □

Theorem 2.8. *A* 3-*dimensional commutative Banach algebra with identity is a* *BQ*-*algebra*.

Proof. We prove that if $\mathcal{B}$ is a 3-dimensional commutative Banach algebra, then $\mathcal{B}$ is algebraically isomorphic to some *Q*-algebra of the bidisc algebra $A(\mathbb{T}^2)$.

If $\sharp M_{\mathcal{B}} = 1$ and $\mathcal{B}$ has two different 1st point derivations, then $\mathcal{B}$ is algebraically isomorphic to some quotient algebra of $A(\mathbb{T}^2)$. In fact, by Proposition 2.1 and Lemma 2.7, if $I = \{f(z, w) \in A(\mathbb{T}^2) : f(0, 0) = f_z(0, 0) = f_w(0, 0) = 0\}$, then $A(\mathbb{T}^2)/I$ is a *BQ*-algebra which is algebraically isomorphic to $\mathcal{B}$, because $(z + I)^2 = (w + I)^2 = 0 + I$.

If $\sharp M_{\mathcal{B}} = 1$ and $\mathcal{B}$ has a 1st point derivation and a 2nd point derivation, then $\mathcal{B}$ is algebraically isomorphic to some quotient algebra of the disc algebra $A(\mathbb{T})$. In fact, by Proposition 2.1 and Lemma 2.7, if $I = \{f(z) \in A(\mathbb{T}) : f(0) = f'(0) = f''(0) = 0\}$, then $A(\mathbb{T})/I$ is a *BQ*-algebra which is algebraically isomorphic to $\mathcal{B}$, because $(z + I)^3 = 0 + I$. Let $wA(\mathbb{T}^2) = \{wf(z, w) : f(z, w) \in A(\mathbb{T}^2)\}$. Then $A(\mathbb{T}^2)/(I + wA(\mathbb{T}^2))$ is also a *BQ*-algebra which is algebraically isomorphic to $\mathcal{B}$. If $\sharp M_{\mathcal{B}} = 2$, then $\mathcal{B}$ is algebraically isomorphic to some quotient algebra of $A(\mathbb{T})$. In fact, by Proposition 2.2 and Lemma 2.7, if $I = \{f(z) \in A(\mathbb{T}) : f(0) = f'(0) =$

$f(a) = 0\}$ for nonzero point a in the open unit disc, then $A(\mathbb{T})/I$ is a BQ-algebra which is algebraically isomorphic to $\mathcal{B}$, because $(z^2/a^2 + I)^2 - (z^2/a^2 + I) = (z(z-a) + I)^2 = (z^2/a^2 + I)(z(z-a) + I) = 0 + I$. Then $A(\mathbb{T}^2)/(I + wA(\mathbb{T}^2))$ is also a BQ-algebra which is algebraically isomorphic to $\mathcal{B}$.

If $\sharp M_{\mathcal{B}} = 3$, then $\mathcal{B}$ is algebraically isomorphic to some quotient algebra of $A(\mathbb{T})$. In fact, by Proposition 2.3 and Lemma 2.7, if $I = \{f(z) \in A(\mathbb{T}) : f(0) = f(a) = f(b) = 0\}$ for nonzero distinct points a, b in the open unit disc, then $A(\mathbb{T})/I$ is a BQ-algebra which is algebraically isomorphic to $\mathcal{B}$, because $(z(z-a)/b(b-a) + I)^2 - (z(z-a)/b(b-a) + I) = (z(z-b)/a(a-b) + I)^2 - (z(z-b)/a(a-b) + I) = (z(z-a)/b(b-a) + I)(z(z-b)/a(a-b) + I) = 0 + I$. Then $A(\mathbb{T}^2)/(I + wA(\mathbb{T}^2))$ is also a BQ-algebra which is algebraically isomorphic to $\mathcal{B}$. □

Remark A. Let $\mathcal{B}$ be a commutative Banach algebra. Let $\phi \in M_{\mathcal{B}}$. Let D^1 be a nontrivial 1st point derivation at ϕ and let D^2 be a nontrivial 2nd point derivation at ϕ. Let α, β be complex numbers satisfying $\alpha \neq 0$. Let $D_0^1 = \alpha D^1$ and let $D_0^2 = \alpha^2 D^2 + \beta D^1$. Then D_0^1 is a nontrivial 1st point derivation at ϕ, and D_0^2 is a nontrivial 2nd point derivation at ϕ.

Proof. Let $f, g \in A$. Since D^1 is a 1st point derivation at ϕ, it follows that

$$\begin{aligned} D_0^1(fg) &= \alpha D^1(fg) \\ &= \alpha\{D^1(f)\phi(g) + \phi(f)D^1(g)\} \\ &= \alpha D^1(f)\phi(g) + \phi(f)\alpha D^1(g) \\ &= D_0^1(f)\phi(g) + \phi(f)D_0^1(g). \end{aligned}$$

Hence D_0^1 is a nontrivial 1st point derivation at ϕ. Since D^2 is a nontrivial 2nd point derivation at ϕ, it follows that

$$
\begin{aligned}
&D_0^2(fg)\\
&= \alpha^2 D^2(fg) + \beta D^1(fg)\\
&= \alpha^2\{D^2(f)\phi(g) + 2D^1(f)D^1(g) + \phi(f)D^2(g)\} + \beta\{D^1(f)\phi(g) + \phi(f)D^1(g)\}\\
&= \{\alpha^2 D^2(f) + \beta D^1(f)\}\phi(g) + 2\alpha^2 D^1(f)D^1(g) + \phi(f)\{\alpha^2 D^2(g) + \beta D^1(g)\}\\
&= D_0^2(f)\phi(g) + 2D_0^1(f)D_0^1(g) + \phi(f)D_0^2(g).
\end{aligned}
$$

Hence D_0^2 is a nontrivial 2nd point derivation at ϕ. □

3. *BQ*-subalgebras of $B(\mathbb{C}^3)$

In this section, we solve Problem 2. In the following corollaries, 3-dimensional *BQ*-subalgebras of $B(\mathbb{C}^3)$ with identity are represented in 3×3 matrix algebras. In Sections 4, 5 and 6, we will consider 3-dimensional 3×3 matrix algebras with respect to 3-dimensional *CQ*-algebras and *Q*-algebras. We will consider Problems 2 and 3. The results in this section follow immediately from the results in Section 2. Since all commuting matrices are simultaneously triangularizable by a unitary matrix, the following Corollary 3.1 (resp. Corollaries 3.2, 3.3) follows from Proposition 2.1 (resp. Corollaries 2.2, 2.3). The matrix in Corollary 3.3 is similar to one of McCullough and Paulsen [8, Proposition 2.2].

Corollary 3.1. *Let B be a* 3*-dimensional commutative* 3×3 *matrix algebra with identity. Suppose* $\sharp M_{\mathcal{B}} = 1$, *that is,* $M_{\mathcal{B}} = \{\phi\}$ *for some* ϕ. *Then* (a) *or* (b) *below holds. The equality means the unitary equivalence*:

(a) *There exist two nontrivial* 1*st point derivations* D_1^1 *and* D_2^1 *at* ϕ *such that for all f in* $\mathcal{B}$,

$$
f = \phi(f)\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + D_1^1(f)\begin{pmatrix} 0 & 0 & 0\\ x & 0 & 0\\ y & 0 & 0\end{pmatrix} + D_2^1(f)\begin{pmatrix} 0 & 0 & 0\\ w & 0 & 0\\ z & 0 & 0\end{pmatrix},
$$

where $xz - yw \neq 0$.

(b) *There exists a nontrivial* 1*st point derivation* D^1 *and a nontrivial* 2*nd point derivation* D^2 *at* ϕ *such that for all f in* $\mathcal{B}$,

$$f = \phi(f)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + D^1(f)\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ z & y & 0 \end{pmatrix} + \frac{D^2(f)}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ xy & 0 & 0 \end{pmatrix},$$

where $xy \neq 0$.

Corollary 3.2. *Let* $\mathcal{B}$ *be a* 3-*dimensional commutative* 3×3 *matrix algebra with identity. Suppose* $\sharp M_{\mathcal{B}} = 2$, *that is,* $M_{\mathcal{B}} = \{\phi, \theta\}$ *for some* ϕ *and* θ. *Then there exists a nontrivial* 1*st point derivation* D^1 *at* ϕ *or at* θ.

If there exists a nontrivial 1*st point derivation* D^1 *at* ϕ, *then for all f in* $\mathcal{B}$,

$$f = \phi(f)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y_1 & -y_2 & 0 \end{pmatrix} + D^1(f)\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ -xy_2 & 0 & 0 \end{pmatrix} + \theta(f)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_1 & y_2 & 1 \end{pmatrix},$$

where $x \neq 0$. *The equality means the unitary equivalence.*

Corollary 3.3. *Let* $\mathcal{B}$ *be a* 3-*dimensional commutative* 3×3 *matrix algebra with identity. Suppose* $\sharp M_{\mathcal{B}} = 3$, *that is,* $M_{\mathcal{B}} = \{\phi_1, \phi_2, \phi_3\}$ *for some* ϕ_1, ϕ_2, ϕ_3.

Then for all f in $\mathcal{B}$,

$$f = \phi_1(f)\begin{pmatrix} 1 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix} + \phi_2(f)\begin{pmatrix} 0 & 0 & 0 \\ -x & 1 & 0 \\ -xz & 0 & 0 \end{pmatrix} + \phi_3(f)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ xz - y & -z & 1 \end{pmatrix},$$

where x, y *and* z *are any complex numbers. The equality means the unitary equivalence.*

4. CQ-subalgebras of $B(\mathbb{C}^3)$

Let A be a uniform algebra on a compact Hausdorff space X. If I is a closed ideal of A, then the quotient algebra A/I becomes a commutative Banach algebra with identity. For a probability measure μ on X, we define the abstract Hardy space

$H^2(\mu)$, the orthogonal projection P and the contractive operator S_f^μ as Definition 1.2 in Introduction. First, we consider the case when $\sharp M_{A/I} = 1$. Lemma 4.1 and Lemma 4.2 are special cases of Corollary 3.1. By Theorem 4.3, if $\sharp M_{A/I} = 1$, then the set of all CQ-algebras $\{S_f^\mu : f \in A\}$ is a proper subset of the set of all BQ-subalgebras of $B(\mathbb{C}^3)$ in Corollary 3.1.

Lemma 4.1. *Let A be a uniform algebra on a compact Hausdorff space X. Let* ϕ *be an element of* M_A. *Let* $I = \{f \in A : \phi(f) = D_1^1(f) = D_2^1(f) = 0\}$, *where* D_1^1 *and* D_2^1 *are* 1*st point derivations at* ϕ. *Let* μ *be a probability measure on X such that* $\dim H^2(\mu) \cap I^\perp = 3$. *Let* k_1, k_2, k_3 *be reproducing kernels in* $H^2(\mu) \cap I^\perp$ *satisfying*

$$\phi(f) = \langle f, k_1 \rangle, \quad D_1^1(f) = \langle f, k_2 \rangle, \quad D_2^1(f) = \langle f, k_3 \rangle, \quad (f \in A).$$

Let $\{\psi_1, \psi_2, \psi_3\}$ *be an orthonormal basis of* $H^2(\mu) \cap I^\perp$ *which is made from* $\{k_1, k_2, k_3\}$ *by the Gram-Schmidt method. Then for this basis,*

$$S_f^\mu = \phi(f)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + D_1^1(f)\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix} + D_2^1(f)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix},$$

where

$$x = \frac{\| k_1 \|}{\sqrt{\| k_2 \|^2 - |\langle k_2, \psi_1 \rangle|^2}}, \quad y = \frac{-\langle \psi_2, k_3 \rangle x}{\sqrt{\| k_3 \|^2 - |\langle k_3, \psi_1 \rangle|^2 - |\langle k_3, \psi_2 \rangle|^2}},$$

$$z = \frac{\| k_1 \|}{\sqrt{\| k_3 \|^2 - |\langle k_3, \psi_1 \rangle|^2 - |\langle k_3, \psi_2 \rangle|^2}}.$$

Lemma 4.2. *Let A be a uniform algebra on a compact Hausdorff space X. Let* ϕ *be an element of* M_A. *Let* $I = \{f \in A : \phi(f) = D^1(f) = D^2(f) = 0\}$, *where* D^1 *is the* 1*st point derivation, and* D^2 *is the* 2*nd point derivation at* ϕ. *Let* μ *be a probability measure on X such that* $\dim H^2(\mu) \cap I^\perp = 3$. *Let* k_1, k_2, k_3 *be reproducing kernels in* $\dim H^2(\mu) \cap I^\perp$ *satisfying*

$$\phi(f) = \langle f, k_1 \rangle, \quad D^1(f) = \langle f, k_2 \rangle, \quad D^2(f) = \langle f, k_3 \rangle, \quad (f \in A).$$

Let $\{\psi_1, \psi_2, \psi_3\}$ *be an orthonormal basis of* $H^2(\mu) \cap I^{\perp}$ *which is made from* $\{k_1, k_2, k_3\}$ *by the Gram-Schmidt method. Then for this basis,*

$$S_f^{\mu} = \phi(f)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + D^1(f)\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ z & y & 0 \end{pmatrix} + \frac{D^2(f)}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ xy & 0 & 0 \end{pmatrix},$$

where

$$x = \frac{\| k_1 \|}{\sqrt{\| k_2 \|^2 - |\langle k_2, \psi_1 \rangle|^2}}, \quad y = \frac{2\sqrt{\| k_2 \|^2 - |\langle k_2, \psi_1 \rangle|^2}}{\sqrt{\| k_3 \|^2 - |\langle k_3, \psi_1 \rangle|^2 - |\langle k_3, \psi_2 \rangle|^2}},$$

$$z = \frac{2\langle \psi_1, k_2 \rangle - \langle \psi_2, k_3 \rangle x}{\sqrt{\| k_3 \|^2 - |\langle k_3, \psi_1 \rangle|^2 - |\langle k_3, \psi_2 \rangle|^2}}.$$

Theorem 4.3. *Let A be a uniform algebra on a compact Hausdorff space X. Let I be an ideal of A such that* $\dim A/I = 3$ *and* $\sharp M_{A/I} = 1$. *Let* μ *be a probability measure on X. Then the set of all* 3*-dimensional CQ-algebras* $\{S_f^{\mu} : f \in A\}$ *is a proper subset of the set of all BQ-subalgebras of* $B(\mathbb{C}^3)$ *in Corollary* 3.1.

Proof. By Corollary 3.1, if $\sharp M_{A/I} = 1$, then (a) or (b) of Corollary 3.1 holds. Suppose (a) holds. Then there exist two 1st point derivations D_1^1 and D_2^1 at ϕ such that

$$f = \phi(f)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + D_1^1(f)\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix} + D_2^1(f)\begin{pmatrix} 0 & 0 & 0 \\ w & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \quad (f \in \mathcal{B}),$$

where x, y, z, w are arbitrary complex numbers. On the other hand, by Lemma 4.1, if this is a matrix of some S_f^{μ}, then $w = 0$.

Suppose (b) of Corollary 3.1 holds. Then there exist a nontrivial 1st point derivation D^1 and a nontrivial 2nd point derivation D^2 at ϕ such that

$$f = \phi(f)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + D^1(f)\begin{pmatrix}0 & 0 & 0\\ x & 0 & 0\\ z & y & 0\end{pmatrix} + \frac{D^2(f)}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ xy & 0 & 0\end{pmatrix} \quad (f \in \mathcal{B}),$$

where x, y, z are arbitrary complex numbers. On the other hand, by Lemma 4.2, if this is a matrix of some S_f^μ, then $x > 0$ and $y \geq 0$. □

Second, we consider the case when $\sharp M_{A/I} = 2$. Lemma 4.4 corresponds to Corollary 3.2. By Theorem 4.5, if $\sharp M_{A/I} = 2$, then the set of all CQ-algebras $\{S_f^\mu : f \in A\}$ is a proper subset of the set of all BQ-subalgebras of $B(\mathbb{C}^3)$ in Corollary 3.2.

Lemma 4.4. *Let A be a uniform algebra on a compact Hausdorff space X. Let* ϕ, θ *be distinct elements of* M_A. *Let* $I = \{f \in A : \phi(f) = D^1(f) = \theta(f) = 0\}$, *where* D^1 *is the* 1*st point derivation at* ϕ. *Let* μ *be a probability measure on X such that* $\dim H^2(\mu) \cap I^\perp = 3$. *Let* k_1, k_2, k_3 *be reproducing kernels in* $H^2(\mu) \cap I^\perp$ *satisfying*

$$\phi(f) = \langle f, k_1\rangle, \quad D^1(f) = \langle f, k_2\rangle, \quad \theta(f) = \langle f, k_3\rangle \quad (f \in A).$$

Let $\{\psi_1, \psi_2, \psi_3\}$ *be an orthonormal basis of* $H^2(\mu) \cap I^\perp$ *which is made from* $\{k_1, k_2, k_3\}$ *by the Gram-Schmidt method. Then for this basis,*

$$S_f^\mu = \phi(f)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -y_1 & -y_2 & 0\end{pmatrix} + D^1(f)\begin{pmatrix}0 & 0 & 0\\ x & 0 & 0\\ -xy_2 & 0 & 0\end{pmatrix} + \theta(f)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ y_1 & y_2 & 1\end{pmatrix},$$

where

$$x = \frac{\| k_1 \|}{\sqrt{\| k_2 \|^2 - |\langle k_2, \psi_1\rangle|^2}},$$

$$y_j = \frac{\langle \psi_j, k_3\rangle}{\sqrt{\| k_3 \|^2 - |\langle k_3, \psi_1\rangle|^2 - |\langle k_3, \psi_2\rangle|^2}} \quad (j = 1, 2).$$

Theorem 4.5. *Let A be a uniform algebra on a compact Hausdorff space X. Let I be an ideal of A such that* $\dim A/I = 3$ *and* $\sharp M_{A/I} = 2$. *Let* μ *be a probability measure on X. Then the set of all* 3*-dimensional CQ-algebras* $\{S_f^\mu : f \in A\}$ *is a proper subset of the set of all BQ-subalgebras of* $B(\mathbb{C}^3)$ *in Corollary* 3.2.

Proof. By Corollary 3.2, there exists a 1st point derivation D^1 at ϕ such that

$$f = \phi(f)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y_1 & -y_2 & 0 \end{pmatrix} + D^1(f)\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ -xy_2 & 0 & 0 \end{pmatrix} + \theta(f)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_1 & y_2 & 1 \end{pmatrix},$$

where the equality means the unitary equivalence, and x, y_1, y_2 are arbitrary complex numbers. On the other hand, by Lemma 4.4, if this is a matrix of some S_f^μ, then $x > 0$. □

Third, we consider the case when $\sharp M_{A/I} = 3$ (cf. [11, Proposition 2.3]).

Remark B. Let A be a uniform algebra on a compact Hausdorff space X. *Let* ϕ_1, ϕ_2, ϕ_3 be distinct elements of M_A. Let $I = \{f \in A : \phi_1(f) = \phi_2(f) = \phi_3(f) = 0\}$. Let μ be a probability measure on X such that $\dim H^2(\mu) \cap I^\perp = 3$. Let k_1, k_2, k_3 be reproducing kernels in $H^2(\mu) \cap I^\perp$ satisfying

$$\phi_1(f) = \langle f, k_1 \rangle, \quad \phi_2(f) = \langle f, k_2 \rangle, \quad \phi_3(f) = \langle f, k_3 \rangle, \quad (f \in A).$$

Let $\{\psi_1, \psi_2, \psi_3\}$ be an orthonormal basis of $H^2(\mu) \cap I^\perp$ which is made from $\{k_1, k_2, k_3\}$ by the Gram-Schmidt method. Then for this basis,

$$S_f^\mu = \phi_1(f)\begin{pmatrix} 1 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix} + \phi_2(f)\begin{pmatrix} 0 & 0 & 0 \\ -x & 1 & 0 \\ -xz & z & 0 \end{pmatrix} + \phi_3(f)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ xz - y & -z & 1 \end{pmatrix},$$

where

$$x = \frac{-\langle k_1, k_2 \rangle}{\sqrt{\| k_1 \|^2 \| k_2 \|^2 - |\langle k_1, k_2 \rangle|^2}}, \quad y = \frac{-\langle \psi_1, k_3 \rangle - \langle \psi_2, k_3 \rangle x}{\sqrt{\| k_3 \|^2 - |\langle k_3, \psi_1 \rangle|^2 - |\langle k_3, \psi_2 \rangle|^2}},$$

$$z = \frac{-\langle \psi_2, k_3 \rangle}{\sqrt{\| k_3 \|^2 - |\langle k_3, \psi_1 \rangle|^2 - |\langle k_3, \psi_2 \rangle|^2}}.$$

5. *CQ*-algebras of the Disc Algebra

Let $\mathcal{B}$ be a 3-dimensional commutative Banach algebra with identity. By Theorem 2.8, $\mathcal{B}$ is always a *BQ*-algebra of the bidisc algebra $A(\mathbb{T}^2)$. In Section 4, we considered *CQ*-algebras for a uniform algebra. Unfortunately, those were very complicated. By the proof of Theorem 2.8, if there do not exist elements $g, h \in \mathcal{B}$ such that $\mathcal{B} = span\{1, g, h\}$ and $g^2 = h^2 = gh = 0$, then $\mathcal{B}$ is a *BQ*-algebra of the disc algebra $A(\mathbb{T})$. Then we can give the simple examples of a *CQ*-algebra for $A(\mathbb{T})$. In this section, for $d\mu = d\theta/2\pi$ or $d\mu = rdrd\theta/\pi$, we define $H^2(\mu)$, the orthogonal projection P and the operator S_f^μ as Definition 1.2 in Introduction. We will describe a 3-dimensional *CQ*-algebra $\{S_f^\mu : f \in A(\mathbb{T})\}$. By Sarason's theorem (cf. [3, p. 125], [12]), if $d\mu = d\theta/2\pi$, then $S^\mu : A(\mathbb{T})/I \to B(H^2(\mu) \cap I^\perp)$ is isometric, and hence $\{S_f^\mu : f \in A(\mathbb{T})\}$ is a *Q*-algebra of $A(\mathbb{T})$.

Let $a \in \mathbb{D}$ and let $\phi \in M_{A(\mathbb{T})}$ satisfy $\phi(f) = f(a)$, $(f \in A(\mathbb{T}))$. Let D^1 be a 1st point derivation at ϕ and let D^2 be a 2nd point derivation at ϕ. Then the following facts can be proved using induction:

(1) $D^1(f) = f'(a)D^1(z)$ $(f \in A(\mathbb{T}))$. Hence, $D^1(f)$ is a scalar multiple of $f'(a)$ (cf. [2, p. 87]).

(2) $D^2(f) = f'(a)D^2(z) + f''(a)\{D^1(z)\}^2$ $(f \in A(\mathbb{T}))$.

First, we consider the case when $\sharp M_{A(\mathbb{T})/I} = 1$. By the above statement (1), $D^1(f)$ is a scalar multiple of $f'(a)$. Hence, if $A = A(\mathbb{T})$, then there is not an example of Lemma 4.1. There is an example of Lemma 4.2 as the following.

Example A. Let $A = A(\mathbb{T})$ and let $d\mu = d\theta/2\pi$. Let $a \in \mathbb{D}$ and let $I = \{f \in A(\mathbb{T}) : f(a) = f'(a) = f''(a) = 0\}$. By Sarason's theorem (cf. [3, p. 125], [12]), $\| S_f^\mu \| = \| f + I \|$. Hence $\{S_f^\mu : f \in A(\mathbb{T})\}$ is a *Q*-algebra. Let $k_1 = 1/(1 - \bar{a}z)$, $k_2 = z/(1 - \bar{a}z)^2$ and $k_3 = 2z^2/(1 - \bar{a}z)^3$. Then $f(a) = \langle f, k_1 \rangle$, $f'(a) = \langle f, k_2 \rangle$,

$f''(a) = \langle f, k_3 \rangle$. By Lemma 4.2,

$$b_{21} = 1 - |a|^2, \qquad \frac{b_{32}}{b_{21}} = 1, \qquad \frac{b_{31}}{b_{21}} = -\bar{a},$$

and for the orthonormal basis $\{\psi_1, \psi_2, \psi_3\}$,

$$S_f^\mu = f(a)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + f'(a)\begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & b_{32} & 0 \end{pmatrix} + \frac{f''(a)}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{21}b_{32} & 0 & 0 \end{pmatrix}.$$

Example B. Let $A = A(\mathbb{T})$ and let $d\mu = r dr d\theta/\pi$. Let $a \in \mathbb{D}$ and let $I = \{f \in A(\mathbb{T}) : f(a) = f'(a) = f''(a) = 0\}$. Then $\| S_f^\mu \| \le \| f + I \|$. Hence $\{S_f^\mu : f \in A(\mathbb{T})\}$ is a CQ-algebra. Let $k_1 = 1/(1 - \bar{a}z)^2$, $k_2 = 2z/(1 - \bar{a}z)^3$ and $k_3 = 6z^2/(1 - \bar{a}z)^4$. Then $f(a) = \langle f, k_1 \rangle$, $f'(a) = \langle f, k_2 \rangle$, $f''(a) = \langle f, k_3 \rangle$. By Lemma 4.2,

$$b_{21} = \frac{1 - |a|^2}{\sqrt{2}}, \qquad \frac{b_{32}}{b_{21}} = \frac{2}{\sqrt{3}}, \qquad \frac{b_{31}}{b_{21}} = -\frac{\sqrt{2}}{\sqrt{3}}\bar{a},$$

and for the orthonormal basis $\{\psi_1, \psi_2, \psi_3\}$,

$$S_f^\mu = f(a)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + f'(a)\begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & b_{32} & 0 \end{pmatrix} + \frac{f''(a)}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{21}b_{32} & 0 & 0 \end{pmatrix}.$$

Corollary 5.1. *Let $A = A(\mathbb{T})$ and let $d\mu = r dr d\theta/\pi$. Then there is an ideal I of A such that $\dim A/I = 3$, $\sharp M_{A/I} = 1$, and an isomorphism $S^\mu : A/I \to B(H^2(\mu) \cap I^\perp)$ is not isometric and $S^\mu(f + I) = S_f^\mu$.*

Proof. By Examples A and B, if $f(z) = z - a$, then $\| S_f^{r dr d\theta/\pi} \| \ne \| S_f^{d\theta/2\pi} \|$, because $f(a) = f''(a) = 0$ and $f'(a) = 1$. By Sarason's theorem (cf. [3, p. 125], [12]), $\| S_f^{d\theta/2\pi} \| = \| f + I \|$. Hence $\| S_f^{r dr d\theta/\pi} \| \ne \| f + I \|$, and hence $S^\mu = S^{r dr d\theta/\pi}$ is not isometric. □

Example C. Let $A = A(\mathbb{T})$ and let $I = \{f \in A(\mathbb{T}) : f(0) = f'(0) = f''(0) = 0\}$. Let $\nu(r)$ be a probability measure on the interval $[0, 1]$ and let μ be a probability measure on the closed unit disc $\overline{\mathbb{D}}$ such that $d\mu = d\nu(r)d\theta/2\pi$. Then $\| S_f^\mu \| \le \| f + I \|$. Hence $\{S_f^\mu : f \in A(\mathbb{T})\}$ is a CQ-algebra. Let $k_1 = 1$, $k_2 = z/\int_0^1 r^2 d\nu(r)$ and $k_3 = 2z^2/\int_0^1 r^4 d\nu(r)$. Then $f(0) = \langle f, k_1 \rangle$, $f'(0) = \langle f, k_2 \rangle$, $f''(0) = \langle f, k_3 \rangle$. Since μ is a radial measure, it follows that k_1, k_2, k_3 are mutually orthogonal. Hence

$$\psi_1 = 1, \qquad \psi_2 = \frac{z}{\left\{\int_0^1 r^2 d\nu(r)\right\}^{1/2}}, \qquad \psi_3 = \frac{z^2}{\left\{\int_0^1 r^4 d\nu(r)\right\}^{1/2}}.$$

By Lemma 4.2, for the orthonormal basis $\{\psi_1, \psi_2, \psi_3\}$,

$$S_f^\mu = f(0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + f'(0)\begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ 0 & b_{32} & 0 \end{pmatrix} + \frac{f''(0)}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{21}b_{32} & 0 & 0 \end{pmatrix},$$

where

$$b_{21} = \frac{\| k_1 \|_{H^2(\mu)}}{\| k_2 \|_{H^2(\mu)}} = \left\{\int_0^1 r^2 d\nu(r)\right\}^{1/2} \le 1,$$

$$\frac{b_{32}}{b_{21}} = \frac{2\| k_2 \|_{H^2(\mu)}}{b_{21}\| k_3 \|_{H^2(\mu)}} = \frac{\left\{\int_0^1 r^4 d\nu(r)\right\}^{1/2}}{\int_0^1 r^2 d\nu(r)} \ge 1.$$

Hence $b_{32} = \max\{b_{32}, b_{21}\} = \| S_z^\mu \| \le 1$, and hence $b_{21} \le b_{32} \le 1$, where

$$S_z^\mu = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ 0 & b_{32} & 0 \end{pmatrix}.$$

Second, we consider the case when $\sharp M_{A(\mathbb{T})/I} = 2$. Since the proof of Example D is similar to one of Example A, the proof is omitted.

Example D. Let $A = A(\mathbb{T})$ and let $d\mu = d\theta/2\pi$. Let a, b be distinct points in $\mathbb{D}$ and let $I = \{f \in A(\mathbb{T}) : f(a) = f'(a) = f(b) = 0\}$. By Sarason's theorem (cf. [3, p. 125], [12]), $\| S_f^\mu \| = \| f + I \|$. Hence $\{S_f^\mu : f \in A(\mathbb{T})\}$ is a Q-algebra. Let $k_1 = 1/(1 - \bar{a}z)$, $k_2 = z/(1 - \bar{a}z)^2$ and $k_3 = 1/(1 - \bar{b}z)$. Then $f(a) = \langle f, k_1 \rangle$, $f'(a) = \langle f, k_2 \rangle$, $f(b) = \langle f, k_3 \rangle$. By Lemma 4.4, for some constant γ such that $|\gamma| = 1$,

$$\psi_1 = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}, \quad \psi_2 = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \cdot \frac{z-a}{1-\bar{a}z}, \quad \psi_3 = \gamma \frac{\sqrt{1-|b|^2}}{1-\bar{b}z}\left(\frac{z-a}{1-\bar{a}z}\right)^2,$$

$$x = 1-|a|^2, \quad y_1 = \left|\frac{1-\bar{a}b}{a-b}\right|^2 \frac{\sqrt{1-|a|^2}\sqrt{1-|b|^2}}{1-\bar{a}b}, \quad \frac{y_2}{y_1} = \frac{b-a}{1-\bar{a}b},$$

and for the orthonormal basis $\{\psi_1, \psi_2, \psi_3\}$,

$$S_f^\mu = f(a)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y_1 & -y_2 & 0 \end{pmatrix} + f'(a)\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ -xy_2 & 0 & 0 \end{pmatrix} + f(b)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_1 & y_2 & 1 \end{pmatrix}.$$

Example E. Let $A = A(\mathbb{T})$ and let $d\mu = r dr d\theta/\pi$. Let a, b be distinct points in $\mathbb{D}$ and let $I = \{f \in A(\mathbb{T}) : f(a) = f'(a) = f(b) = 0\}$. Then $\| S_f^\mu \| \le \| f + I \|$. Hence $\{S_f^\mu : f \in A(\mathbb{T})\}$ is a CQ-algebra. Let $k_1 = 1/(1 - \bar{a}z)^2$, $k_2 = 2z/(1 - \bar{a}z)^3$ and $k_3 = 1/(1 - \bar{b}z)^2$. Then $f(a) = \langle f, k_1 \rangle$, $f'(a) = \langle f, k_2 \rangle$, $f(b) = \langle f, k_3 \rangle$. By Lemma 4.4,

$$x = \frac{1-|a|^2}{\sqrt{2}}, \quad \frac{y_2}{y_1} = \sqrt{2}\,\frac{b-a}{1-\bar{a}b},$$

$$y_1 = \frac{1-a\bar{b}}{1-\bar{a}b} \cdot \frac{(1-|a|^2)(1-|b|^2)}{|a-b|^2} \cdot \frac{|1-\bar{a}b|}{\sqrt{3(1-|a|^2)(1-|b|^2)+|a-b|^2}},$$

and for the orthonormal basis $\{\psi_1, \psi_2, \psi_3\}$,

$$S_f^\mu = f(a)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y_1 & -y_2 & 0 \end{pmatrix} + f'(a)\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ -xy_2 & 0 & 0 \end{pmatrix} + f(b)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_1 & y_2 & 1 \end{pmatrix}.$$

Corollary 5.2. *Let* $A = A(\mathbb{T})$ *and let* $d\mu = rdrd\theta/\pi$*. Then there is an ideal* I *in* A *such that* $\dim A/I = 3$, $\sharp M_{A/I} = 2$, *and an isomorphism* $S^{\mu} : A/I \to B(H^2(\mu) \cap I^{\perp})$ *is not isometric, where* $S^{\mu}_f g = P(fg)$, $(g \in H^2(\mu) \cap I^{\perp})$ *and* $S^{\mu}(f + I) = S^{\mu}_f$.

Proof. By Examples D and E, if $f(z) = z - a$, then $\| S_f^{rdrd\theta/\pi} \| \neq \| S_f^{d\theta/2\pi} \|$, because $f(a) = f''(a) = 0$ and $f'(a) = 1$. By Sarason's theorem (cf. [3, p. 125], [12]), $\| S_f^{d\theta/2\pi} \| = \| f + I \|$. Hence $\| S_f^{rdrd\theta/\pi} \| \neq \| f + I \|$, and hence $S^{\mu} = S^{rdrd\theta/\pi}$ is not isometric. □

Third, we consider the case when $\sharp M_{A(\mathbb{T})/I} = 3$ (cf. [11, Proposition 2.3]).

Example F. Let $A = A(\mathbb{T})$ and let $d\mu = d\theta/2\pi$. Let a, b, c be distinct points in $\mathbb{D}$ and let $I = \{f \in A(\mathbb{T}) : f(a) = f(b) = f(c) = 0\}$. By Sarason's theorem (cf. [3, p. 125], [12]), $\| S^{\mu}_f \| = \| f + I \|$. Hence $\{S^{\mu}_f : f \in A(\mathbb{T})\}$ is a Q-algebra. Let $k_1 = 1/(1 - \bar{a}z)$, $k_2 = 1/(1 - \bar{b}z)$ and $k_3 = 1/(1 - \bar{c}z)$. Then $f(a) = \langle f, k_1 \rangle$, $f(b) = \langle f, k_2 \rangle$, $f(c) = \langle f, k_3 \rangle$. For some constant γ_j such that $|\gamma_j| = 1$,

$$\psi_1(z) = \frac{\sqrt{1 - |a|^2}}{1 - \bar{a}z}, \quad \psi_2(z) = \gamma_2 \frac{z - a}{1 - \bar{a}z} \cdot \frac{\sqrt{1 - |b|^2}}{1 - \bar{b}z},$$

$$\psi_3(z) = \gamma_3 \frac{z - a}{1 - \bar{a}z} \cdot \frac{z - b}{1 - \bar{b}z} \frac{\sqrt{1 - |c|^2}}{1 - \bar{c}z}.$$

By Remark B, for the orthonormal basis $\{\psi_1, \psi_2, \psi_3\}$,

$$S^{\mu}_f = \phi_1(f) \begin{pmatrix} 1 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix} + \phi_2(f) \begin{pmatrix} 0 & 0 & 0 \\ -x & 1 & 0 \\ -xz & z & 0 \end{pmatrix} + \phi_3(f) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ xz - y & -z & 1 \end{pmatrix},$$

where

$$x = \gamma_4 \frac{\sqrt{1 - |a|^2}\sqrt{1 - |b|^2}}{|a - b|}, \quad y = \gamma_5 \left| \frac{1 - a\bar{b}}{a - b} \right| \frac{\sqrt{1 - |a|^2}\sqrt{1 - |c|^2}}{|a - c|}$$

and

$$z = \gamma_6 \frac{\sqrt{1-|b|^2}\sqrt{1-|c|^2}}{|b-c|}.$$

Then

$$1+|y|^2\left|\frac{a-b}{1-\bar{b}a}\right|^2 = \left|\frac{1-\bar{a}c}{c-a}\right|^2, \quad 1+|z|^2 = \left|\frac{1-\bar{c}b}{b-c}\right|^2.$$

Example G. Let $\mathcal{D}^2$ be the Dirichlet space with the norm

$$\| f \|_{\mathcal{D}^2}^2 = \| f \|_{H^2}^2 + \int_{\mathbb{D}} | f'(re^{i\theta}) |^2 \, rdrd\theta/\pi.$$

Then there is not a probability measure μ satisfying $\| f \|_{\mathcal{D}^2} = \left(\int_{|z|\le 1} | f |^2 d\mu\right)^{1/2}$. Hence $\mathcal{D}^2$ is not an abstract Hardy space $H^2(\mu)$. Let $A = A(\mathbb{T})$ and let $I = \{f \in A(\mathbb{T}) : f(0) = f'(0) = 0\}$. Then $\| z + I \| = \inf\{\| z + f \| : f \in I\} = 1$. Let $\mathcal{H}_1 = H^2(d\theta/2\pi) \cap I^{\perp}$, $\mathcal{H}_2 = H^2(rdrd\theta/\pi) \cap I^{\perp}$ and $\mathcal{H}_3 = \mathcal{D}^2 \cap I^{\perp}$. Then we consider restriction of the shift operators $S_z^{\mathcal{H}_j}$ on $\mathcal{H}_j$ $(j = 1, 2, 3)$. With respect to the orthonormal basis $\{1, z, z^2\}$, $\{1, \sqrt{2}z, \sqrt{3}z^2\}$ and $\{1, z/\sqrt{2}, z^2/\sqrt{3}\}$,

$$S_z^{d\theta/2\pi} = S_z^{\mathcal{H}_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_z^{rdrd\theta/\pi} = S_z^{\mathcal{H}_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2}/\sqrt{3} & 0 \end{pmatrix},$$

$$S_z^{\mathcal{H}_3} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3}/\sqrt{2} & 0 \end{pmatrix}.$$

Since $\| S_z^{\mathcal{H}_1} \| = 1$ (or by the Sarason's theorem (cf. [3, p. 125], [12])), it follows that $\| S_z^{\mathcal{H}_1} \| = \| z + I \|$, and $\{S_f^{\mathcal{H}_1} : f \in A(\mathbb{T})\}$ is a *Q*-algebra. This is a special case of Example A. Since $\| S_z^{\mathcal{H}_2} \| = \sqrt{2}/\sqrt{3}$, it follows that $\| S_z^{\mathcal{H}_2} \| \le \| z + I \|$, and $\{S_f^{\mathcal{H}_2} : f \in A(\mathbb{T})\}$ is a *CQ*-algebra. This is a special case of Example B. Since

$\| S_z^{\mathcal{H}_3} \| = \sqrt{2}$, it follows that $\| S_z^{\mathcal{H}_3} \| > \| f + I \|$, and $\{S_f^{\mathcal{H}_3} : f \in A(\mathbb{T})\}$ is a *BQ*-algebra which is not a *CQ*-algebra of $A(\mathbb{T})$.

6. *CQ*-algebras of the Bidisc Algebra

Let $A(\mathbb{T}^2)$ be the bidisc algebra. Then we can give the simple examples of a *CQ*-algebra for $A(\mathbb{T}^2)$. In this section, for $d\mu = d\theta_1\, d\theta_2/2\pi$, we define $H^2(\mu)$, the orthogonal projection P and the contractive operator S_f^{μ} as Definition 1.2 in Introduction. We describe a 3-dimensional *CQ*-algebra $\{S_f^{\mu} : f \in A\}$. We consider the case when $M_{A/I}$ contains just 1 element. The proofs of Examples H and I are similar to one of Example A.

Example H. Let $A = A(\mathbb{T}^2)$ and let $d\mu = d\theta_1 d\theta_2/(2\pi)^2$. Let $(a, b) \in \mathbb{D}^2$ and let $I = \{f \in A(\mathbb{T}^2) : f(a, b) = f_z(a, b) = f_w(a, b) = 0\}$. Let

$$k_1 = \frac{1}{1-\bar{a}z} \cdot \frac{1}{1-\bar{b}w}, \quad k_2 = \frac{z}{(1-\bar{a}z)^2} \cdot \frac{1}{1-\bar{b}w}, \quad k_3 = \frac{1}{1-\bar{a}z} \cdot \frac{w}{(1-\bar{b}w)^2}.$$

Then $f(a, b) = \langle f, k_1 \rangle$, $f_z(a, b) = \langle f, k_2 \rangle$, $f_w(a, b) = \langle f, k_3 \rangle$. By Lemma 4.1,

$$\psi_1 = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \cdot \frac{\sqrt{1-|b|^2}}{1-\bar{b}z}, \quad \psi_2 = \frac{z-a}{1-\bar{a}z} \cdot \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \cdot \frac{\sqrt{1-|b|^2}}{1-\bar{b}w},$$

$$\psi_3 = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \cdot \frac{w-b}{1-\bar{b}w} \cdot \frac{\sqrt{1-|b|^2}}{1-\bar{b}w},$$

$$x = 1-|a|^2, \quad y = 0, \quad z = 1-|b|^2,$$

and for the orthonormal basis $\{\psi_1, \psi_2, \psi_3\}$,

$$S_f^{\mu} = f(a, b)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + f_z(a, b)\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix} + f_w(a, b)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix}.$$

Example I. Let $A = A(\mathbb{T}^2)$ and let $d\mu = d\theta_1 d\theta_2/(2\pi)^2$. Let $(a, b) \in \mathbb{D}^2$ and let $I = \{f \in A(\mathbb{T}^2) : f(a, b) = f_z(a, b) = f_{zz}(a, b) = 0\}$. Let

$$k_1 = \frac{1}{1-\bar{a}z} \cdot \frac{1}{1-\bar{b}w}, \quad k_2 = \frac{z}{(1-\bar{a}z)^2} \cdot \frac{1}{1-\bar{b}w}, \quad k_3 = \frac{2z^2}{(1-\bar{a}z)^3} \cdot \frac{1}{1-\bar{b}w}.$$

Then $f(a, b) = \langle f, k_1 \rangle$, $f_z(a, b) = \langle f, k_2 \rangle$, $f_{zz}(a, b) = \langle f, k_3 \rangle$. By Lemma 4.2,

$$\psi_1 = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \cdot \frac{\sqrt{1-|b|^2}}{1-\bar{b}z}, \quad \psi_2 = \frac{z-a}{1-\bar{a}z} \cdot \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \cdot \frac{\sqrt{1-|b|^2}}{1-\bar{b}w},$$

$$\psi_3 = \left(\frac{w-b}{1-\bar{b}w}\right)^2 \cdot \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \cdot \frac{\sqrt{1-|b|^2}}{1-\bar{b}w},$$

$$b_{21} = x = 1-|a|^2, \quad \frac{b_{32}}{b_{21}} = 1, \quad \frac{b_{31}}{b_{21}} = -\bar{a},$$

and for the orthonormal basis $\{\psi_1, \psi_2, \psi_3\}$,

$$S_f^{\mu} = f(a, b)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + f_z(a, b)\begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & b_{32} & 0 \end{pmatrix} + \frac{f_{zz}(a, b)}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{21}b_{32} & 0 & 0 \end{pmatrix}.$$

Corollary 6.1. *Let* $d\mu = d\theta_1 d\theta_2/(2\pi)^2$ *and let* $(a, b) \in \mathbf{D}^2$. *Let* $I = \{f \in A(\mathbb{T}^2)$ $: f(a, b) = f_z(a, b) = f_{zz}(a, b) = 0\}$. *Then* $S^{\mu} : A(\mathbb{T}^2)/I \to B(H^2(\mu) \cap I^{\perp})$ *is isometric*.

Proof. By Examples A and I,

$$\| S_f^{\mu} \| = \left\| f(a, b)S_1^{\mu} + f_z(a, b)S_{z-a}^{\mu} + \frac{f_{zz}(a, b)}{2} S_{(z-a)^2}^{\mu} \right\|$$
$$= \left\| f(a, b)S_1^{d\theta/2\pi} + f_z(a, b)S_{z-a}^{d\theta/2\pi} + \frac{f_{zz}(a, b)}{2} S_{(z-a)^2}^{d\theta/2\pi} \right\|$$
$$= \| S_g^{d\theta/2\pi} \|,$$

where

$$g(z) = f(a, b) + f_z(a, b)(z-a) + \frac{f_{zz}(a, b)}{2}(z-a)^2.$$

By Sarason's theorem (cf. [3, p. 125], [12]), $\| S_g^{d\theta/2\pi} \| = \| g + I_0 \|$, where $I_0 = \{f \in A(\mathbb{T}) : f(a) = f'(a) = f''(a) = 0\}$ and $S^{d\theta/2\pi} : A(\mathbb{T})/I_0 \to B(H^2(d\theta/2\pi) \cap I_0^{\perp})$ is isometric. Hence $\| S_f^{\mu} \| = \| g + I_0 \|$. By the calculation, it follows that

$\| g + I \| = \| g + I_0 \|$. Since $f(z, w) - g(z) \in I$, it follows that $\| S_f^\mu \| = \| g + I \| = \| f + I \|$. This implies that S^μ is isometric. □

Corollary 6.2. *Let* $A = A(\mathbb{T}^2)$ *and let* $d\mu = d\theta_1 d\theta_2 / (2\pi)^2$. *Then there is an ideal* I *of* A *such that* $\dim A/I = 3$ *and* $S^\mu : A/I \to B(H^2(\mu) \cap I^\perp)$ *is not isometric.*

Proof. If the following condition (1) implies (2) for any distinct points $\tau_1, \ldots, \tau_n \in M_A$ and complex numbers $w_1, \ldots, w_n$, then we say that A and $I = \bigcap_{j=1}^n \ker \tau_j$ satisfy the Pick property. In general, it is proved by the calculation that (2) implies (1).

(1) $[(1 - w_i \overline{w_j}) k_{ji}]_{i, j=1}^n \geq 0$, where $k_{ij} = \langle k_i, k_j \rangle_\mu$, and $\tau_j(f) = \langle f, k_j \rangle_\mu$, $(f \in A)$.

(2) There exists $f \in A$ such that $\tau_j(f) = w_j$ $(1 \leq j \leq n)$ and $\| f + I \| \leq 1$.

Then it is known that $S^\mu : A/I \to B(H^2(\mu) \cap I^\perp)$ is isometric if and only if A and $I = \bigcap_{j=1}^3 \ker \tau_j$ satisfy the Pick property (cf. [11, Proposition 4.6.]). By the definition of the Pick property, if A and $I = \bigcap_{j=1}^3 \ker \tau_j$ satisfy the Pick property, then A and $J = \bigcap_{j=1}^2 \ker \tau_j$ satisfy the Pick property, because if $[(1 - w_i \overline{w_j}) k_{ji}]_{i, j=1}^2 \geq 0$ and $w_3 = 0$, then $[(1 - w_i \overline{w_j}) k_{ji}]_{i, j=1}^3 \geq 0$. Hence, if $S^\mu : A/I \to B(H^2(\mu) \cap I^\perp)$ is isometric, then $S^\mu : A/J \to B(H^2(\mu) \cap I^\perp)$ is isometric. On the other hand, by the following Proposition 6.3, $S^\mu : A/J \to B(H^2(\mu) \cap I^\perp)$ is not isometric. This is a contradiction. Hence $S^\mu : A/I \to B(H^2(\mu) \cap I^\perp)$ is not isometric. □

Proposition 6.3. *Let* $A = A(\mathbb{T}^2)$ *and let* $d\mu = d\theta_1 d\theta_2 / (2\pi)^2$. *Let* $J = \{ f \in A : f(a, b) = f(c, d) = 0 \}$, *where* $(a, b), (c, d)$ *are distinct points in* $\mathbb{D}^2$. *Then* J *is an ideal of* A *such that* $\dim A/J = 2$. *Let* $S_f^\mu g = P(fg)$ $(g \in H^2(\mu) \cap J^\perp)$. *Then an isomorphism* $S^\mu : A/J \to B(H^2(\mu) \cap J^\perp)$ *is not isometric.*

Proof. Let $k_1(z, w) = 1/(1-\bar{a}z)(1-\bar{b}w)$ and $k_2(z, w) = 1/(1-\bar{c}z)(1-\bar{d}w)$. Let

$$a_{ij} = \langle S_f^{\mu}\psi_j, \psi_i\rangle = \int_{\mathbb{D}} f\psi_j \overline{\psi_i} d\mu \qquad (i, j = 1, 2),$$

where $\{\psi_1, \psi_2\}$ is an orthonormal basis of $H^2(\mu) \cap J^{\perp}$ which is made from $\{k_1, k_2\}$ by the Gram-Schmidt method. By [10, Lemma 3],

$$(a_{ij}) = f(a, b)\begin{pmatrix} 1 & 0 \\ C & 0 \end{pmatrix} + f(c, d)\begin{pmatrix} 0 & 0 \\ -C & 1 \end{pmatrix}, \quad |C| < \sqrt{\frac{1}{\sigma^2} - 1},$$

where

$$\sigma = \sigma((a, b), (c, d)) = \sup\{|f(c, d)| : f(a, b) = 0, \|f\| \leq 1\}$$
$$= \max\left(\left|\frac{a-c}{1-\bar{a}c}\right|, \left|\frac{b-d}{1-\bar{b}d}\right|\right)$$

because

$$|C|^2 = \frac{|\langle k_1, k_2\rangle|^2}{\|k_1\|^2\|k_2\|^2 - |\langle k_1, k_2\rangle|^2}$$

$$= \frac{\dfrac{1}{|1-\bar{a}c|^2|1-\bar{b}d|^2}}{\dfrac{1}{(1-|a|^2)(1-|b|^2)(1-|c|^2)(1-|d|^2)} - \dfrac{1}{|1-\bar{a}c|^2|1-\bar{b}d|^2}},$$

and hence, for

$$x = \left|\frac{a-c}{1-\bar{a}c}\right|^2, \quad y = \left|\frac{b-d}{1-\bar{b}d}\right|^2,$$

$$|C|^2 = \frac{(1-x)(1-y)}{1-(1-x)(1-y)} < \frac{1}{\max(x, y)} - 1 = \frac{1}{\sigma^2} - 1.$$

By [10, Lemma 3], an isomorphism S^{μ} is not isometric. □

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