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HORSESHOES CHAOS IN A DELAYED VAN DER POL-MATHIEU-DUFFING OSCILLATOR WITH FAST HARMONIC EXCITATION

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Abstract

Horseshoes chaos in a delayed van der Pol-Mathieu-Duffing oscillator under a fast harmonic excitation is examined in this paper using the Melnikov method. A delay feedback control scheme is considered to switch the system from chaos to order and vice-versa. Delayed terms are derived from a delayed Duffing potential and a delayed damping. The stability of the delayed system is presented in terms of a theorem. The influence of delay and fast excitation on the threshold for chaos is analyzed.

1. Introduction

Horseshoes chaos appears in nonlinear systems when transverse intersections between stable and unstable manifolds in the Poincaré section occur. The method generally used to study such intersections, due to Melnikov [1], is one of the few

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analytical methods available to determine the existence of chaotic motion in systems subjected to dissipative time dependant perturbation. The main idea of the method is to find a function measuring the distance between the stable and unstable manifolds for a saddle or two saddle of the perturbed system. That is, if the function associated to the Melnikov method, the so-called Melnikov function, vanishes for certain values of the bifurcation parameter, then the stable and unstable manifolds will intersect each other away from the saddle point in the Poincaré section [2]. If the stable and unstable manifolds cross each other once, then they will intersect an infinite number of times, thus forming a type of Smale horseshoes mapping leading to chaos. From Smale-Birhoff theorem [3], the existence of such intersections results in chaotic dynamics. Although the chaos does not manifest itself in the form of permanent chaos, it does in terms of the fractal basin boundaries.

Many contributions have been made to investigate analytically chaos in dynamical systems using Melnikov criteria. One of the early works is due to Holmes [2] for a Duffing oscillator. Wiggins [3] proposed a generalized form of the method and a radon version was proposed by Frey and Simiu [4]. Recently, Zhang et al. used the extended version of the Melnikov criteria to study the multi-pulse global bifurcation and chaos of a cantilever beam [5]. Applications of the method can be found in epidemiology [6], finance [7], biology [8], and in engineering problems [9, 10]. In this later application, having an analytical expression for the prediction of chaos, it is helpful to turn a system from chaos to order or vice-versa. This is done using one of the various control schemes. Amongst them, active control method has received a great attention in last two decades, and it has been applied to various branches of engineering, see [11-13] for details. The most recent control scheme is the one involving delay feedback of the state variables, and is now widely used [14-18]. Ji and Leung [17] considered a Duffing system under parametric excitation and used a delay feedback coupling to control various bifurcations in the system. Sun et al. [18] considered a double well Duffing oscillator with only a positive position and velocity delay feedback coupling, and investigated the effects of time delay on the chaotic behavior of the system. They concluded that a good choice of time delay can affect radically the behavior of the system.

Guided by these previous works, we consider in this paper, the following delayed van der Pol-Mathieu-Duffing (vdPMD) oscillator with a high-frequency excitation (HFE) and investigate the effects of delay and delay gain parameters on the appearance of horseshoes chaos

$$\frac{d^2x}{dt^2} - \varepsilon\frac{dx}{dt}(1-\alpha x^2) + x(1+\gamma\,\mathbf{cos}(\omega\mathbf{t})) + \beta x^3$$

$$= a\Omega^2\mathbf{cos}(\mathbf{x})\,\mathbf{cos}(\mathbf{\Omega t}) + x(t-\tau) + \tilde{\beta}x(t-\tau)^3 + \tilde{\varepsilon}\frac{dx(t-\tau)}{dt}, \quad x = x(t). \qquad (1)$$

Equation (1) is the mathematical model for various phenomena in physics. As in Manoj et al. [19], the meaning of the various terms of the equation for the case of an optically actuated radio frequency MEMS determined as follows:

- The van der Pol term $-\varepsilon\frac{dx}{dt}(1-\alpha x^2)$ models the self oscillation of a disc resonator for a sufficient DC laser power.
- The Mathieu term $x\gamma\,\mathbf{cos}(\omega\mathbf{t})$ models the periodic parametric excitation introduced by modulating the laser using a piezodrive.
- The Φ^4 Duffing potential $\frac{1}{2}x^2 + \frac{1}{4}\beta x^4$ is used to show a soft nonlinear behavior. The parameter β is chosen to be negative in order to be more realistic. In fact, this corresponds to a catastrophic single well potential. That is, for large value of x, the system can escape over the potential barrier and dramatically suffers an unbounded motion. This configuration of the potential possesses three equilibrium points, one stable point $S_0 = \left(x = 0; \frac{dx}{dt} = 0\right)$ and two unstable points $S_1 = \left(x = \sqrt{-1/\beta}; \frac{dx}{dt} = 0\right)$ and $S_2 = \left(x = -\sqrt{-1/\beta}; \frac{dx}{dt} = 0\right)$.
- The fast harmonic excitation is added to modify the nonlinear characteristic spring behavior of the system from softening to hardening and to create entrainment or frequency-locking [20].
- The system is controlled via a delayed Duffing potential $\frac{1}{2}x^2(t-\tau) + \frac{1}{4}\beta x^4(t-\tau)$ and a delayed momentum $\tilde{\varepsilon}\frac{dx(t-\tau)}{dt}$. The gain parameters $\tilde{\beta}$ and $\tilde{\varepsilon}$ have the same sign as β and ε.

- The remaining term is the acceleration of the system.

According to this definition, α is the amplitude of the self oscillation, γ is the magnitude of the parametric force, ω the corresponding frequency, β the Duffing nonlinear term, $a\Omega^2$ the magnitude of the fast harmonic excitation and Ω the corresponding frequency.

The rest of the paper is arranged as follows: Section 2 considers the linear stability of the system with time delay feedback coupling. Lyapunov analysis and Forde and Nelson [22] theorem are used to determine the condition of stability. Section 3 concerns the Melnikov criteria for chaos; focus is made on the effect of time delay and delay gain parameter on the appearance of chaos. Section 4 is for conclusion.

2. Linear Stability

The Lyapunov concept is used to study the stability of equation (1). In this line, the following characteristic equation in λ is obtained from the linear form of equation (1):

$$P_1(\lambda) + e^{-\lambda\tau} P_2(\lambda) = 0, \tag{2}$$

with

$$P_1(\lambda) = \lambda^2 + \lambda\varepsilon(\alpha x_0^2 - 1) + 1 + 3\beta x_0^2 \quad \text{and} \quad P_2(\lambda) = -1 - 3\tilde{\beta} x_0^2 - \lambda\tilde{\varepsilon}. \tag{3}$$

Due to the presence of the exponential term in the characteristic equation (1), it is quite difficult to calculate explicitly its roots. The stability condition can be derived using a theorem due to Forde and Nelson [22] which states that, if one has an idea of the roots of equation (1) for the undelayed case, then it is possible to study the evolution of these roots as the delay grows through positive values. This theorem was recently used by Ghosh et al. [14] while studying the stability of a vdPD equation with position feedback coupling.

For $\tau = 0$, equation (1) becomes

$$\lambda^2 + \lambda[-\tilde{\varepsilon} + \varepsilon(\alpha x_0^2 - 1)] + 3(\beta - \tilde{\beta}) x_0^2 = 0. \tag{4}$$

Routh-Hurwitz criterion dictates that, if $\tilde{\varepsilon} < \varepsilon(1 - \alpha x_0^2)$ and $\beta > \tilde{\beta}$, then all roots of this equation have negative real part. Setting $\lambda = i\mu$, $\mu \in \mathbf{R}$ and using

Lemma 2 in Forde and Nelson [22], one obtains the following algebraic equation in terms of $\nu = \mu^2$ for the stability of the system:

$$\nu^2 + \nu A + B = 0, \tag{5}$$

with

$$A = \varepsilon^2(\alpha x_0^2 - 1)^2 - 2(1 + 3\beta x_0^2) - \tilde{\varepsilon}^2$$

and $B = (1 + 3\beta x_0^2)^2 - (1 + 3\tilde{\beta} x_0^2)^2$. Since the leading coefficient in equation (4) is positive, its has a positive real root in two circumstances: the first case is $B < 0$ and the second case is $B > 0$ with $A < 0$. Thus, we can conclude with the following theorem for the stability of equation (1).

Theorem. *A steady state with characteristic equation given by equation* (1) *is stable in the absence of delay*, *and becomes unstable with increasing delay if and only if*

1. $\tilde{\varepsilon} < \varepsilon(1 - \alpha x_0^2)$ *and* $\beta > \tilde{\beta}$,

2. *either* $(1 + 3\beta x_0^2)^2 < (1 + 3\tilde{\beta} x_0^2)^2$ *or* $(1 + 3\beta x_0^2)^2 > (1 + 3\tilde{\beta} x_0^2)^2$ *and* $\tilde{\varepsilon}^2 > -2(1 + 3\beta x_0^2) + \varepsilon^2(1 - \alpha x_0^2)^2$,

as the delay is increasing from zero to infinity. *For a steady state near one of the two stable equilibrium points of equation* (1),

1. $\tilde{\varepsilon} < \varepsilon$ *and* $\beta > \tilde{\beta}$,

2. $\tilde{\varepsilon}^2 > -2 + \varepsilon^2$. □

3. Delay Parameters and Chaos

3.1. Melnikov's criterion

A key point with unstable and chaotic engineering system is to derive a mathematical condition overlapping the parameters of the system and leading to such phenomenon. The Hamiltonian system from which equation (1) is deduced can be written as

$$\frac{dx}{dt} = y$$

and

$$\frac{dy}{dt} = -x - \beta x^3. \tag{6}$$

The corresponding Hamiltonian function possesses an heteroclinic orbit connecting the two unstable points of the potential. The orbit is given by

$$x_0 = \pm\sqrt{-\frac{1}{\beta}}\mathbf{tanh}\left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\mathbf{t}\right),$$

and

$$y_0 = \pm\sqrt{-\frac{1}{2\beta}}\mathbf{sech^2}\left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\mathbf{t}\right). \tag{7}$$

The Melnikov function [12] is defined by

$$M(t_0) = \int_{-\infty}^{+\infty} \mathbf{F}(\mathbf{x},\ \mathbf{y}) \times \mathbf{G}(\mathbf{x},\ \mathbf{y})\, dt, \tag{8}$$

where

$$\mathbf{F}(\mathbf{x},\ \mathbf{y}) = \begin{pmatrix} y \\ -x - \beta x^3 \end{pmatrix}, \tag{9}$$

and

$$\mathbf{G}(\mathbf{x},\ \mathbf{y}) = \begin{pmatrix} 0 \\ y\varepsilon(1 - \alpha x^2) - x\gamma\, \mathbf{cos}(\omega\eta) + x(t - \tau) + \tilde{\beta} x(t - \tau)^3 \\ + \tilde{\varepsilon} y(t - \tau) + a\Omega^2 \mathbf{cos}(\mathbf{x})\,\mathbf{cos}(\boldsymbol{\Omega}\eta) \end{pmatrix}. \tag{10}$$

The new coefficient is defined as $\eta = t + t_0$ and t_0 is a phase angle. If $M(t_0) = 0$ and $\frac{dM(t_0)}{dt} \neq 0$ for some t_0 and some set of parameters, then a horseshoes exists, and chaos occurs [2]. Carrying out integration of (7), one finds

$$M(t) = \varepsilon(I_1 - \alpha I_2) + \gamma I_3\, \mathbf{sin}(\omega \mathbf{t_0}) + I_4 + \tilde{\varepsilon} I_5 + \tilde{\beta} I_7 + 2a\Omega^2 I_6\, \mathbf{cos}(\boldsymbol{\Omega}\mathbf{t_0}), \tag{11}$$

with

$$\mathbf{I_1} = -\frac{2}{3}\frac{\sqrt{2}}{\beta}, \quad \mathbf{I_2} = \frac{2\sqrt{2}}{15\beta^2}, \quad \mathbf{I_3} = \frac{\omega^2\pi}{2|\beta|}\frac{1}{\mathbf{sinh}\left(\frac{\omega\pi}{\sqrt{2}}\right)},$$

$$\mathbf{I_4} = -\frac{2}{|\beta|}[\delta(1 - \mathbf{cotanh}^2(\delta)) + \mathbf{cotanh}(\delta)],$$

$$\mathbf{I_5} = -\frac{2\sqrt{2}}{\beta}\frac{2\delta\,\mathbf{cotanh}(\delta) - 1}{\mathbf{sinh}^2(\delta)},$$

$$\mathbf{I_7} = \frac{2\,\mathbf{cotanh}\,\delta}{\beta|\beta|}[2(1-\mathbf{cotanh}^2\delta)(-1+3\delta\,\mathbf{cotanh}\,\delta)+\mathbf{cotanh}^2\delta], \; \delta = \frac{\tau}{\sqrt{2}}, \text{ and}$$

$$\mathbf{I_6} = \pm\int_{-1}^{+1}\mathbf{cos(u)\,cos(\Omega\sqrt{2}\,Argtanh(\pm}\sqrt{-\beta}\mathbf{u}))\,du. \tag{12}$$

Using the Melnikov criterion, it is found that chaos appears when one of the following conditions is satisfied [2]:

$$a^2 \geq a_c^2, \text{ with } a_c^2 = \frac{1}{\Omega^4 I_6^2}[(\varepsilon(I_1 - \alpha I_2) + I_4 + \tilde{\varepsilon}I_5 + \tilde{\beta}I_7)^2 - \gamma^2 I_3^2]. \tag{13}$$

Conditions (12) overlap the parameters of the system including time delay such that the system can turn form chaos to order and vice-versa by acting on the parameters.

3.2. Numerical analysis and discussions

Figure 1 shows the shape of the I_3 as a function of the frequency ω of the parametric excitation and I_4, I_5 and I_6 as functions of the time delay τ. It can be observed that, I_3 increases until a maxima for small values of ω and decreases for large values of ω. The maxima is obtained for $\omega = \frac{1.915\sqrt{2}}{\pi}$ as $I_{3\max} = \frac{7}{20\beta}$. The quantities I_4, I_5 and I_6 decrease for small values of the time delay τ and increase for large values of τ. On the other hand, I_4 is negative and I_5 is positive for all τ while I_6 is negative for large value of τ.

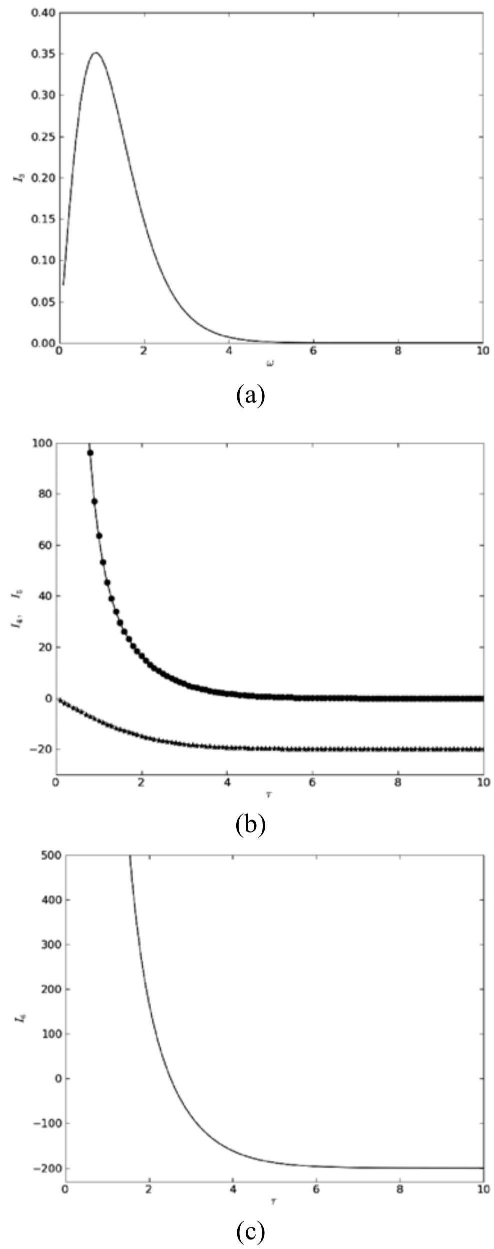
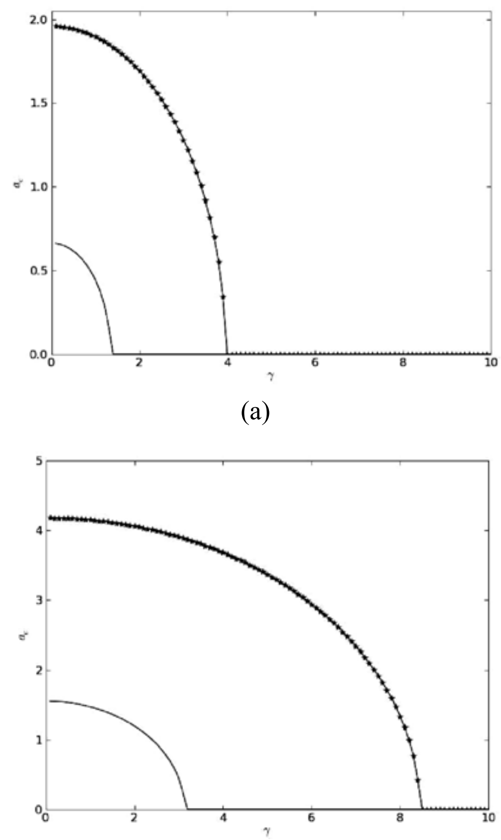


Figure 1. (a) Shape of I_3 as a function of ω. (b) Shape of delay linear terms I_4 and I_5 as functions of time delay τ. (c) Shape of delay nonlinear term I_6 as a function of time delay τ. The figures are plotted for $\beta = 10^1$.



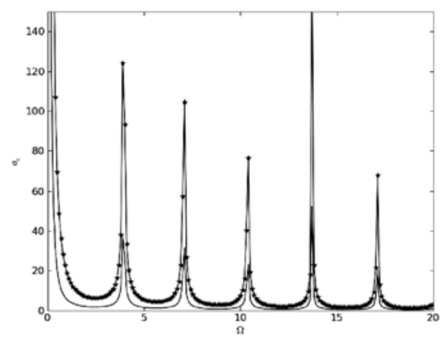
(a)

(b)

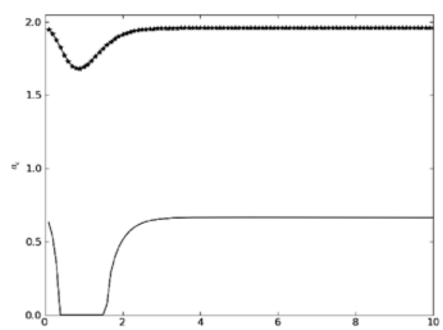
Figure 2. Effects of the amplitude of the parametric excitation on the critical amplitude of the HFE: (a) Case $\alpha = 1.2 > 5|\beta|$. (b) Case $\alpha = 0.2 < 5|\beta|$. The curve with line and cross is plotted for the delayed case while the case without delay is plotted with line.

Figure 2 shows the variation of a_c as a function of γ for $\omega = 1$, $\Omega = 10$, $\beta = 0.11$, $\varepsilon = 0.8$, $\widetilde{\beta} = 0.1$ and $\widetilde{\varepsilon} = 0.82$. The curve in line corresponds to the case without delay while the delayed case is plotted with line and crosses. The case $\alpha = 1.2 > 5|\beta|$ is plotted in Figure 2(a) and the case $\alpha = 0.2 < 5|\beta|$ is plotted in Figure 2(b). The curves show how the presence of time delay increases the critical value of γ from γ_c to γ_{c1}. In each case, the region above the curve corresponds to the region of appearance of fractal basin boundary.

The effects of the frequencies on the critical amplitude are shown in Figure 3 with the values of Figure 2 and $\gamma = 2.$ The influence of Ω on a is an almost periodic function with extremal values (Figure 3(a)). The curve with line and crosses delimits the domain of chaos in the case with delay while a curve with line delimits the region of chaos in the case without delay. This indicates that, as in the previous case, the delay reduces the domain of chaos. The same observation is made considering the frequency of the parametric excitation ω (Figure 3(b)). In this case, the critical amplitude a_c decreases with small values of ω and increases for large values of ω. This is a consequence of the result of Figure 1(a), since I_3 has a negative effect of a_c according to equation (13).



(a)



(b)

Figure 3. Effects of the frequencies on the critical amplitude of the HFE: (a) Effects of the HFE frequency. (b) Effects of the parametric frequency. The curve with line and cross is plotted for the delayed case while the case without delay is plotted with line.

The following proposition resumes the turn off behaviors of the system:

Proposition. *For the system without delay*:

- *If the amplitude of the parametric excitation* γ *verifies*

$$0 < \gamma < \gamma_c, \quad with \quad \gamma_c = \frac{40\sqrt{2}}{21}\varepsilon\left|1 + \frac{\alpha}{5\beta}\right|, \tag{14}$$

  then a nonzero critical amplitude of the HFE may exist. This amplitude a_c *is given as*

$$a_c = \frac{1}{\Omega^2 |I_6|}\sqrt{(\varepsilon(I_1 - \alpha I_2) + I_4 + \tilde{\varepsilon}I_5 + \tilde{\beta}I_7)^2 - \gamma^2 I_3^2} \tag{15}$$

  and for $a > a_c$, *the system has a fractal basin and horseshoe chaos exists.*

- *If the amplitude of the parametric excitation* γ *is defined as*

$$\gamma > \gamma_c > 0, \tag{16}$$

  then a nonzero critical amplitude of the HFE may not exist and the system is always chaotic for all physically meaningful value of a.

For the system with delay:

- *If the amplitude of the parametric excitation* γ *is defined as*

$$0 < \gamma < \gamma_c, \tag{17}$$

  then the nonzero critical amplitude is increased by time delay if and only if the following conditions hold:

1. $(I_1 - \alpha I_2) > 0$ *and some of delay terms* > 0,

2. $(I_1 - \alpha I_2) < 0$ *and some of delay terms* < 0,

  otherwise, time delay reduces the nonzero critical amplitude a_c.

- *If the amplitude of the parametric excitation* γ *is defined as*

$$\gamma > \gamma_c > 0, \tag{18}$$

then a nonzero critical amplitude may exist if

$$\gamma_c < \gamma < \gamma_{c1}, \quad with \quad \gamma_{c1} = \frac{20\beta}{7}\left[I_4 + \tilde{\varepsilon} I_5 + \tilde{\beta} I_7 + \varepsilon\left(1 + \frac{\alpha}{5\beta}\right)\right], \tag{19}$$

otherwise, *the nonzero critical amplitude may not exist*.

Time delay increases the critical amplitude of the Mathieu excitation γ_{c1} *if and only if*

- $(I_1 - \alpha I_2) > 0$ *and some of delay terms* > 0,
- $(I_1 - \alpha I_2) < 0$ *and some of delay terms* < 0.

4. Conclusion

A van der Pol-Mathieu-Duffing equation with linear plus nonlinear delay feedback coupling was considered. Using the Forde and Nelson theorem, the stability of the system was investigated around the equilibrium and the Melnikov method was used to analyze the criterion for the appearance of horseshoes chaos in terms of delay parameters. It was shown that the delay parameters highly influenced the threshold condition for chaos and thus can be used to enhance or suppress chaotic dynamics of the considered oscillator.

References

[1] V. K. Melnikov, On the stability of the center of some periodic perturbations, Trans. of the Moscow Mathematical Society 12 (1963), 1-57.

[2] P. Holmes, A nonlinear oscillator with a strange attractor, Philos. Trans. R. Soc. A 292 (1979), 419-448.

[3] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, Berlin, 1990.

[4] M. Frey and E. Simiu, Noise induced chaos and phase flux, Physica D 63 (1993), 321-340.

[5] W. Zhang, M. H. Yao and J. H. Zhang, Using the extended Melnikov method to study the multi-pulse global bifurcation and chaos of a cantilever beam, J. Sound Vibration 319 (2009), 541-569.

[6] D. Ouatemi and Y. Kone, Melnikov analysis of chaos in a general epidemiological model, Nonlinear Anal.: Real World Appl. 8 (2007), 20-26.

[7] C. L. Chian Abraham, Complex System Approach to Economic Dynamics, Springer, Berlin, Heidelberg, New York, 2007.

[8] Z. Zhang, J. Peng and J. Zhang, Melnikov method to a bacteria-immunity model with bacterial quorum sensing mechanism, Chaos Solitons Fractals 40 (2009), 414-420.

[9] J. S. Lee and K. S. Chang, Applications of chaos and fractals in process systems engineering, J. Process Control 6 (1996), 71-87.

[10] F. Abdel and F. Elhefnawy, Nonlinear electrohydrodynamic instability of two liquid layers, International J. Engineering Science 40 (2002), 319-332.

[11] N. L. Virgin, R. H. Plaut and C. C. Cheng, Prediction of escape from potential well harmonic excitation, International J. Nonlinear Mechanics 27 (1992), 357-365.

[12] C. R. Fuller, S. J. Eliot and P. A. Nelson, Active Control of Vibration, Academic Press, London, 1997.

[13] C. A. Kitio Kwuimy, B. R. Nana Nbendjo and P. Woafo, Optimization of electromechanical control of beam dynamics: Analytical method and finite difference simulation, J. Sound Vibration 298 (2006), 180-195.

[14] D. Ghosh, R. A. Chowdhury and P. Saha, On the various kinds of synchronization in delayed Duffing-van der Pol system, Communication in Nonlinear Science and Numerical Simulation 13 (2008), 790-803.

[15] L. Xinye, J. C. Ji, H. H. Colin and T. Chunxiao, The response of a Duffing-van der Pol oscillator under delayed feedback control, J. Sound Vibration 291 (2006), 644-655.

[16] J. Xu and K. W. Chung, Effects of time delayed position feedback on a van der Pol-Duffing oscillator, Phys. D 180 (2003), 17-39.

[17] J. C. Ji and A. Y. T. Leung, Bifurcation control of a parametrical excited Duffing system, Nonlinear Dyn. 27 (2002), 411-417.

[18] Z. K. Sun, W. Xu, X. L. Yang and T. Fand, Effects of time delays on bifurcation and chaos in a non-autonomous system with multiple time delays, Chaos Solitons Fractals 31 (2007), 39-53.

[19] P. Manoj, R. H. Rand and A. T. Zehnder, Frequency locking in a forced Mathieu-van der Pol-Duffing system, Nonlinear Dyn. 54 (2008), 3-12.

[20] A. Fashi and M. Belhaq, Effect of fast harmonic excitation on frequencing-locking in a van der Pol-Duffing oscillator, Communication in Nonlinear Science and Numerical Simulation 14 (2009), 244-253.

[21] M. Belhaq and A. Fashi, 2:1 and 1:1 frequency-locking in fast excited van der-Mathieu-Duffing oscillator, Nonlinear Dyn. 53 (2008), 139-152.

[22] J. Forde and P. Nelson, Application of Sturm sequences to bifurcation analysis of delay differential equation models, J. Math. Anal. Appl. 300 (2004), 237-284.