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# COMAXIMAL GRAPHS AND THEIR COMPLEMENTARY GRAPHS

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**Abstract**

A comaximal graph is introduced in [7] and studied in [4]. In this paper, we investigate the chromatic numbers of comaximal graphs and also of their complementary graphs. Let $R$ be a Noetherian ring and let $\overline{\chi}_0(R)$ be the chromatic number of the complementary graph to a comaximal graph $G_0(R)$. If $\overline{\chi}_0(R)$ is finite, then $R$ is a field or a finite ring. Furthermore, the following assertions hold: (1) If $R$ is a field, then $\overline{\chi}_0(R) = 1$. (2) If $R$ is a finite ring, then $\overline{\chi}_0(R) = \max\{|M_1|, ..., |M_t|\}$, where $M_1, ..., M_t$ are all maximal ideals of $R$ and $|M_i|$ denotes the number of elements of the set $M_i$ for $i = 1, ..., t$. As for comaximal graphs, we give a partial result on chromatic numbers.

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First, we collect the basic notions and results of graph theory for later use. We consider a simple graph $G$. $V(G)$ denotes the set of vertices of $G$ and $E(G)$ denotes the set of edges of $G$. We color the vertices of $G$ so that no two joined vertices have the same color. If we color the vertices, we call it a *coloring* of $G$. The chromatic number $\chi(G)$ of the graph $G$ is the minimum number of colors of colorings of $G$.

Let $C$ be a non-empty subset of $V(G)$. We call $C$ a clique of $G$ if every pair of distinct two elements of $C$ is joined by an edge. The clique number $C(G)$ of $G$ is the maximum number of elements of cliques of $G$.

Our notation is standard and for unexplained terms, our general reference to commutative algebra is [1], [5] and our general reference to graph theory is [2].

**Lemma 1.** *The inequality* $C(G) \leqq \chi(G)$ *holds.*

**Proof.** If $\chi(G)$ is not finite, then it is obvious that $C(G) \leqq \chi(G)$. We may assume that $\chi(G)$ is finite, let $C$ be an arbitrary clique of $G$. Then every vertex of $C$ must be colored with different colors because $C$ is a clique of $G$. Moreover, $G$ needs at least $|C|$ colors because $C$ is a subset of $G$, where $|C|$ denotes the number of elements of $C$. Hence $C(G) \leqq \chi(G)$. □

The symbol $\coprod$ denotes the disjoint union of sets.

**Lemma 2.** *Let* $V_1, V_2, \ldots, V_t$ *be non-empty subsets of* $V(G)$. *Let*

$$V(G) = V_1 \coprod V_2 \coprod \cdots \coprod V_t$$

*be a disjoint union of* $V(G)$ *such that no pair of distinct two elements of* $V_i$ *is joined by an edge for* $i = 1, 2, \ldots, t$. *Then* $\chi(G) \leqq t$.

**Proof.** We color all vertices of $V_i$ by the same color, and we color the vertices of $V_i$ and the vertices of $V_j$ by different colors for $i \neq j$. It is a coloring of $G$. Hence, we need $t$ kinds of colors. Therefore $\chi(G) \leqq t$. □

**Remark 3.** If $\chi(G) = n$ and $c_1, \ldots, c_n$ are colors of minimum coloring of $G$, then we set

$$V_i = \{x \in V(G);\ x \text{ is colored by a color } c_i\}.$$

Then

$$V(G) = V_1 \coprod V_2 \coprod \cdots \coprod V_n$$

is a disjoint union of $V(G)$ such that no pair of distinct two elements of $V_i$ is joined by an edge.

Let $G_1$ and $G_2$ be two simple graphs. We say that $G_1$ is a *subgraph* of $G_2$ if the following conditions hold: (1) $V(G_1) \subset V(G_2)$, (2) $E(G_1) \subset E(G_2)$.

**Lemma 4.** *If* $G_1$ *is a subgraph of* $G_2$, *then* $\chi(G_1) \leqq \chi(G_2)$.

**Proof.** We may assume that $\chi(G_2)$ is finite, say, $\chi(G_2) = n$. Then there exists a coloring of $G_2$ with $n$ colors. Since $G_1$ is a subgraph of $G_2$, we have a coloring of $G_1$ with at most $n$ colors by restricting the coloring of $G_2$ to $G_1$. Hence $\chi(G_1) \leqq \chi(G_2)$. □

Let $G$ be a simple graph. We define the complementary graph $\overline{G}$ to $G$ to be a graph satisfying the following conditions:

(1) $V(\overline{G}) = V(G)$.

(2) Let $x$ and $y$ be distinct two vertices of $\overline{G}$. Then $x$ and $y$ are joined by an edge in $\overline{G}$ if and only if $x$ and $y$ are not joined by an edge in $G$.

Let $R$ be a commutative ring with the identity element. An element $x$ is called a *zero-divisor* of $R$ if there exists a non-zero element $y$ of $R$ such that $xy = 0$. $Z(R)$ denotes the set of zero-divisors of $R$. We consider the simple graph $G(R)$ whose vertices are elements of $R$ and in which distinct two vertices $x$ and $y$ are joined by an edge if $x - y$ is in $Z(R)$. $\chi(R)$ denotes the chromatic number of the graph $G(R)$ and $V(R)$ denotes the set of vertices of $G(R)$.

Let $G_0(R)$ be the comaximal graph of $R$ whose vertices are elements of $R$ and in which distinct two vertices $x$ and $y$ are joined by an edge if $xR + yR = R$. $\chi_0(R)$ denotes the chromatic number of the graph $G_0(R)$.

Let $\overline{G_0}(R)$ be the complementary graph of $G_0(R)$, that is, it is the graph whose vertices are elements of $R$ and in which distinct two vertices $x$ and $y$ are joined by an edge if $xR + yR \neq R$. $\overline{\chi}_0(R)$ denotes the chromatic number of the graph $\overline{G}_0(R)$. First, we consider the graph $\overline{G}_0(R)$ and then we treat $G_0(R)$.

**Lemma 5.** *If R is a field, then* $\overline{\chi}_0(R) = 1$.

**Proof.** Let $x$ and $y$ be distinct two elements of $R$. Since $R$ is a field, every element other than 0 is a unit of $R$. Hence, $x$ or $y$ is a unit of $R$.

Therefore $xR + yR = R$. This means that no pair of distinct two elements of $R$ is joined by an edge. Hence $\overline{\chi}_0(R) = 1$. □

**Proposition 6.** *If* $\overline{\chi}_0(R)$ *is finite, then R is a field or a finite ring.*

**Proof.** Assume that $R$ is not a field. Then there exists an ideal $I$ such that $I \neq (0)$ and $I \neq R$. Furthermore, there exists a non-zero element $a$ of $I$. Let $\operatorname{Ann}_R(a) = \{x \in R;\ ax = 0\}$ be the annihilator ideal of $a$. On the other hand, we know that there is an $R$-module isomorphism $aR \cong R/\operatorname{Ann}_R(a)$. Since $aR$ and $\operatorname{Ann}_R(a)$ are ideals of $R$ such that $aR \neq R$ and $\operatorname{Ann}_R(a) \neq R$, we see that $aR$ and $\operatorname{Ann}_R(a)$ are cliques of $\overline{G}_0(R)$. Then $|aR|$ and $|\operatorname{Ann}_R(a)|$ are finite because $\overline{\chi}_0(R)$ is finite. Hence, $|R|$ is finite. □

To determine $\overline{\chi}_0(R)$ in the case $R$ is a finite ring, we need the following result:

**Lemma 7** [6, Theorem 17(2)]**.** *Let R be a Noetherian ring. If R is finite, then*

$$\chi(R) = \max\{|M_1|, \ldots, |M_t|\},$$

*where* $M_1, \ldots, M_t$ *are all maximal ideals of R.*

Let $A$ be a finite set and $f : A \to A$ be a mapping from $A$ to $A$. If $f$ is injective, then $f$ is surjective. By making use of it we can prove that every element of a finite ring $R$ is a unit or a zero-divisor of $R$.

**Theorem 8.** *Let R be a Noetherian ring. Assume that* $\overline{\chi}_0(R)$ *is finite. Then the following assertions hold*:

(1) *If R is a field, then* $\overline{\chi}_0(R) = 1$.

(2) *If R is a finite ring, then*

$$\overline{\chi}_0(R) = \max\{|M_1|, \ldots, |M_t|\},$$

*where* $M_1, \ldots, M_t$ *are all maximal ideals of R.*

**Proof.** The assertion (1) is proved in Lemma 5. We shall show the assertion (2). Assume that $R$ is a finite ring. Let $M$ be an arbitrary maximal ideal of $M$. Then $M$ is a clique of $\overline{G}_0(R)$ because $M \neq R$. Hence $|M| \leqq \overline{\chi}_0(R)$. This implies that $\overline{\chi}_0(R) \geqq \max\{|M_1|, ..., |M_t|\}$.

We will prove that $\overline{G}_0(R)$ is a subgraph of $G(R)$. Let $x$ and $y$ be arbitrary distinct two elements of $R$ such that $xR + yR \neq R$. Then we have $(x - y)R \subset xR + yR \neq R$. Hence $(x - y)R \neq R$, that is, $x - y$ is not a unit of $R$. Since $R$ is a finite ring, $x - y$ is a zero-divisor of $R$. This shows that $\overline{G}_0(R)$ is a subgraph of $G(R)$.

By Lemmas 4 and 7, we get

$$\overline{\chi}_0(R) \leqq \chi(R) = \max\{|M_1|, ..., |M_t|\}.$$

Therefore $\overline{\chi}_0(R) = \max\{|M_1|, ..., |M_t|\}$. □

We use a notation in [4]. Let $\Gamma_2(R)$ be a subgraph of $G_0(R)$ whose vertices are non-units of $R$. $\chi_0(\Gamma_2(R))$ denotes the chromatic number of $\Gamma_2(R)$.

The following have been proved in [7]:

(1) $\chi_0(R)$ is finite if and only if $R$ is a finite ring.

(2) If $\chi_0(R)$ is finite, then $\chi_0(R) = C(G_0(R)) = n + l$, where $n$ is the number of maximal ideals of $R$ and $l$ is the number of units of $R$.

We consider $\chi_0(\Gamma_2(R))$.

**Proposition 9.** *Let $R$ be a semi-local ring and let $M_1, M_2, ..., M_n$ be all maximal ideals of $R$. Then $\chi_0(\Gamma_2(R)) = n$.*

**Proof.** Since $M_1, M_2, ..., M_n$ are maximal ideals of $R$, we see that

$$M_i \not\subset M_1 \cup \cdots \cup \overset{\vee}{M_i} \cup \cdots \cup M_n$$

for every $i$ with $1 \leqq i \leqq n$, where $\overset{\vee}{M_i}$ means the deletion of $M_i$ (cf. [3, Theorem 81]). Then there exists an element $x_i$ of $M_i$ such that

$$x_i \notin M_1 \cup \cdots \cup \overset{\vee}{M_i} \cup \cdots \cup M_n.$$

This implies that $x_iR + x_jR = R$ for $i \neq j$. Hence, $\{x_1, x_2, ..., x_n\}$ is a clique of $\Gamma_2(R)$ and $n \leqq \chi_0(\Gamma_2(R))$.

We show that $\chi_0(\Gamma_2(R)) \leqq n$. Let $R^*$ be the set of units of $R$. Then we have

$$R - R^* = M_1 \coprod (M_2 - M_1) \coprod (M_3 - (M_1 \cup M_2)) \coprod \cdots \coprod (M_n - (M_1 \cup \cdots \cup M_{n-1})).$$

Note that $M_2 - M_1, M_3 - (M_1 \cup M_2), ..., M_n - (M_1 \cup \cdots \cup M_{n-1})$ are not empty. Set

$$V_1 = M_1, \ V_2 = M_2 - M_1, \ V_3 = M_3 - (M_1 \cup M_2), ...,$$

$$V_n = M_n - (M_1 \cup \cdots \cup M_{n-1}).$$

Then $R - R^* = V_1 \coprod V_2 \coprod \cdots \coprod V_n$ and no pair of distinct two elements of $V_i$ is joined by an edge for $i = 1, 2, ..., n$. Hence by Lemma 2, $\chi_0(\Gamma_2(R)) \leqq n$. Therefore $\chi_0(\Gamma_2(R)) = n$. □

We give a conjecture on the converse of Proposition 9.

**Conjecture.** If $\chi_0(\Gamma_2(R))$ is finite, then $R$ is a semi-local ring.

A partial result of this conjecture is given in Proposition 11.

We call $Q$ a *maximal prime divisor* of (0) if $Q$ is in $\mathrm{Ass}_R(R)$ and $Q$ is a maximal element in $\mathrm{Ass}_R(R)$ with respect to inclusion. We know that $Z(R) = \bigcup Q$, where the union is taken over all maximal prime divisors $Q$'s of (0) in $\mathrm{Ass}_R(R)$ ([3, Theorem 80 and its proof]).

**Lemma 10.** *Let R be a Noetherian ring. Then the following assertions hold*:

(1) *If every element of R is a unit of R or a zero-divisor of R, then R is a semi-local ring.*

(2) *Let* $M_1, ..., M_t$ *be some of maximal ideals of R. If every element of* $R - M_1 \cup \cdots \cup M_t$ *is a unit of R or a zero-divisor of R, then R is a semi-local ring.*

**Proof.** Let $Q_1, ..., Q_r$ be all maximal prime divisors of (0). Then by the assumption we have

$$R - R^* = Z(R) = Q_1 \cup \cdots \cup Q_r.$$

Let $N_i$ be a maximal ideal of $R$ such that $Q_i \subset N_i$ for $i = 1, 2, ..., r$. Note that $N_i \subset R - R^*$ for $i = 1, 2, ..., r$. Hence $R - R^* = N_1 \cup \cdots \cup N_r$.

Let $M$ be an arbitrary maximal ideal of $R$. Then $M \subset R - R^* = N_1 \cup \cdots \cup N_r$. Therefore $M \subset N_j$ for some $j$ by [3, Theorem 81]. Furthermore, $M = N_j$ because $M$ is a maximal ideal of $R$. Hence, $N_1, ..., N_r$ are all maximal ideals of $R$. This shows that $R$ is a semi-local ring.

(2) By the assumption we get

$$R - R^* = M_1 \cup \cdots \cup M_t \cup Z(R).$$

Let $N_1, ..., N_r$ be the same as in the proof of the assertion (1). Then

$$R - R^* = M_1 \cup \cdots \cup M_t \cup N_1 \cup \cdots \cup N_r.$$

Thus $M_1, ..., M_t, N_1, ..., N_r$ are all maximal ideals of $R$. Hence, $R$ is a semi-local ring. □

**Proposition 11.** *Let $R$ be a Noetherian ring with* Krull $\dim R = 1$. *Then the following conditions are equivalent*:

(i) $\chi_0(\Gamma_2(R))$ *is finite*.

(ii) *$R$ is a semi-local ring*.

**Proof.** (i) $\Rightarrow$ (ii). Suppose the contrary. Then there are infinitely many maximal ideals of $R$. If every element of $R$ is a unit of $R$ or a zero-divisor of $R$, then $R$ is a semi-local ring by Lemma 10. So we may assume that we can take an element $a_1$ which is a non-zero divisor of $R$ and not a unit of $R$. Let $M$ be a maximal ideal of $R$ such that $a_1 \in M$. We shall show that $M \in \operatorname{Ass}_R(R/a_1R)$. Since $a_1 \in M$, there exists a minimal prime divisor $P$ of $a_1R$ such that $P \subset M$. If $\operatorname{ht} P = 0$, then $P$ is a minimal prime divisor of (0). This implies that $P \subset Z(R)$ and $a_1$ is a zero-divisor of $R$. This is a contradiction. Hence by Principal Ideal Theorem of Krull we know that $\mathrm{ht} P = 1$. On the other hand, $\mathrm{ht} M \leqq 1$ because Krull $\dim R = 1$. This means that $P = M$ and $M \in \mathrm{Ass}_R(R/a_1R)$. Since $|\mathrm{Ass}_R(R/a_1R)|$ is finite, the number of a set $\{M;\, M$ is a maximal ideal of $R$ and $a_1 \in M\}$ is finite. Let $M_{11}, \ldots, M_{1e_1}$ be all maximal ideals of $R$ which contain $a_1$.

If every element of $R - M_{11} \cup \cdots \cup M_{1e_1}$ is a unit of $R$ or a zero-divisor of $R$, then $R$ is a semi-local ring by Lemma 10. Hence, we may assume that we can take an element $a_2$ of $R$ such that $a_2$ is not a unit of $R$, $a_2$ is a non-zero divisor of $R$ and $a_2 \notin M_{11}, \ldots, M_{1e_1}$. Let $M_{21}, \ldots, M_{2e_2}$ be all maximal ideals of $R$ which contain $a_2$. Then $a_1R + a_2R = R$ because $M_{11}, \ldots, M_{1e_1}, M_{21}, \ldots, M_{2e_2}$ are distinct maximal ideals of $R$.

If every element of $R - M_{11} \cup \cdots \cup M_{1e_1} \cup M_{21} \cup \cdots \cup M_{2e_2}$ is a unit of $R$ or a zero-divisor of $R$, then $R$ is a semi-local ring by Lemma 10. Hence, we may assume that we can take an element $a_3$ of $R$ such that $a_3$ is not a unit of $R$, $a_3$ is a non-zero divisor of $R$ and $a_3 \notin M_{11}, \ldots, M_{1e_1} \cup M_{21} \cup \cdots \cup M_{2e_2}$. Then we get $a_1R + a_3R = R$ and $a_2R + a_3R = R$. Continuing this process, we obtain a set $\{a_1, a_2, a_3, \ldots\}$ and it is a clique of $\Gamma_2(R)$. Hence $|\chi_0(\Gamma_2(R))| = \infty$ by Lemma 1. This is a contradiction.

(ii) $\Rightarrow$ (i). It is clear from Proposition 9. □

Let $R$ be a Noetherian ring with Krull $\dim R = 1$. Let $a$ be an element of $R$. If $a$ is a non-zero divisor of $R$, then there are finitely many maximal ideals of $R$ which contain $a$ by the argument of the proof of Proposition 11. In the case $a$ is a zero-divisor of $R$, it is not so as the following example shows:

**Example 12.** Let $\mathbf{C}$ be the field of complex numbers and let $\mathbf{C}[X]$ be a polynomial ring over $\mathbf{C}$ in an indeterminate $X$. Let $x$ be the residue class of $X$ in $\mathbf{C}[X]/(X^2)$ and set $R = \mathbf{C}[X]/(X^2)[Y]$, where $Y$ denotes an indeterminate. Then $R$ is Noetherian ring and Krull $\dim R = 1$. Furthermore, there are infinitely many maximal ideals $(x, Y - \alpha)(\alpha \in \mathbf{C})$ which contain $x$.

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