Far East Journal of Applied Mathematics
Volume 52, Number 2, 2011, Pages 131-146
Published Online: May 22, 2011
This paper is available online at http://pphmj.com/journals/fjam.htm

ON HOCHSTADT-LIEBERMAN THEOREM FOR STURM-LIOUVILLE OPERATORS

CHUNG-TSUN SHIEH*, S. A. BUTERIN and MIKHAIL IGNATIEV

*Department of Mathematics
Tamkang University
Danshui Dist.
New Taipei City 25137
Taiwan (R.O.C.)
e-mail: ctshieh@math.tku.edu.tw

Department of Mathematics and Mechanics
Saratov State University
Astrakhan-skaya 83
Saratov 410012, Russia
e-mail: buterinsa@info.sgu.ru
ignatievmu@info.sgu.ru

Abstract

The inverse spectral problem of the Sturm-Liouville operator $L_q = -\frac{d^2}{dx^2} + q(x)$ is considered, where $q(x)$ is an integrable function on $(0, 1)$. Some analogies of the Hochstadt-Lieberman Theorem for Sturm-Liouville operators are proved.

2010 Mathematics Subject Classification: Primary 34A55; Secondary 34B24, 34L05.

Keywords and phrases: inverse spectral problem, Sturm-Liouville equations, Hochstadt-Lieberman theorem.

This research is partially supported by Grants 07-01-00003 and 07-01-92000-NSCa of Russian Foundation for Basic Research and 96-2923-M-032-001-MY3 of Taiwan National Science Council.

Received February 25, 2011

1. Introduction

The inverse spectral problem for Sturm-Liouville equations is to determine the potential function $q(x) \in L_1(0, 1)$ from the spectral data of the boundary value problem

$$y'' + (\lambda - q(x))\, y = 0,\ 0 < x < 1, \tag{1.1}$$

$$y'(0) - hy(0) = 0, \tag{1.2}$$

$$y'(1) + Hy(1) = 0. \tag{1.3}$$

It often arises from vibration of a string, quantum mechanics, geophysics and other branches of sciences. We can study the inverse spectral problem of (1.1)-(1.3) by several approaches: the transformation operator theory, the Weyl-Titchmarsh $\mathfrak{M}$-function, the boundary control method, the method of spectral mappings or some other methods; varieties of (1.1)-(1.3) were studied by several mathematicians, e.g., inverse spectral problems of Sturm-Liouville equations on graphs, of Sturm-Liouville equations with interior discontinuities, of Sturm-Liouville equations with an eigenparameter on boundary conditions and inverse spectral problem for vectorial or matrix-valued Sturm-Liouville equations (see [1, 3, 4, 5-13, 18-22, 26-38]).

In 1978, Hochstadt and Lieberman (see [23] for details) proved a uniqueness theorem for a Sturm-Liouville equation with mixed-data. The statement is as following:

Theorem 1.1. *Consider the equation* (1.1) *subject to boundary conditions* (1.2) *and* (1.3), *where* $q \in L_1(0, 1)$. *Then the spectrum* $\sigma(h, H, q)$ *and* $q|_{(1/2, 1)}$ *determine* $q(x)$ *uniquely*.

The purpose of the paper is to prove some analogies of this theorem for Sturm-Liouville equations. In the second section, some preliminaries are reviewed. In Section 3, we prove analogies of the Hochstadt-Lieberman Theorem for Sturm-Liouville equations with interior discontinuities, with eigenparameter on boundary conditions and the mixed problems.

2. Preliminaries

The main idea we use in this paper is the Weyl-Titchmarsh $\mathfrak{M}$-function. Roughly speaking, the $\mathfrak{M}$-function for a Sturm-Liouville equation (1.1) is a meromorphic function of the form

$$\mathfrak{M}(\lambda) = \frac{\varphi(1, \lambda)}{\varphi'(1, \lambda)}, \tag{2.1}$$

where $\varphi(x, \lambda)$ is a solution of (1.1) and satisfies the initial condition

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h, \tag{2.2}$$

where h is scalar. It is well known that the $\mathfrak{M}$ -function determined $q(x)$ uniquely, or equivalently two spectra determine $q(x)$ uniquely.

In order to prove our main results, we need more specifically to focus on the properties of $\varphi(1, \lambda)$. It is not difficult to derive the asymptotic behavior of $\varphi(x, \lambda)$ (see Chapter 1 of [36]).

Lemma 2.1. $q(x) \in L_1(0, 1)$. *Let* $\lambda = \rho^2$ *and* $\tau = \operatorname{Im} \rho$. *Then, for* $|\lambda| \to \infty$,

$$\varphi(x, \lambda) = \cos(\rho x) + H(x)\frac{\sin \rho x}{\rho} + o\left(\frac{1}{\rho}\exp|\tau|x\right) \tag{2.3}$$

and

$$\varphi'(x, \lambda) = -\rho \sin(\rho x) + H(x)\cos(\rho x) + o(\exp|\tau|x), \tag{2.4}$$

uniformly for $x \in [0, 1]$, *where* $H(x) = h + \frac{1}{2}\int_0^x q(t)\,dt$.

The following theorem is also necessary for our analysis.

Theorem 2.2 ([15, Proposition B.6])**.** *Let* $f(z)$ *be an entire function that satisfies*

(1) $\sup_{|z|=R_k} |f(z)| \le C_1 \exp(C_2 R_k^{\rho})$ *for some* $0 < \rho < 1$, $C_1, C_2 > 0$, *and some sequence* $R_k \to \infty$ *as* $k \to \infty$.

(2) $\lim_{|x|\to\infty} |f(ix)| = 0$.

Then $f \equiv 0$.

We can apply Theorem 2.2 to establish some Hochstadt-Lieberman type theorems (see [15] for details). We shall use this method to obtain some analogies.

Let $\varphi_h(x, \lambda)$ and $\psi_H(x, \lambda)$ be solutions of (1.1) with

$$\varphi_h(0, \lambda) = \psi_H(1, \lambda) = 1, \quad \varphi'_h(0, \lambda) = h, \quad \psi'_H(1, \lambda) = -H. \tag{2.5}$$

Then the characteristic function for Problems (1.1)-(1.3) is

$$\Delta(h, H, q)(\lambda) = \begin{vmatrix} \psi_H(x, \lambda) & \varphi_h(x, \lambda) \\ \psi'_H(x, \lambda) & \varphi'_h(x, \lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \psi_H(1/2, \lambda) & \varphi_h(1/2, \lambda) \\ \psi'_H(1/2, \lambda) & \varphi'_h(1/2, \lambda) \end{vmatrix} = C \prod_{n \geq 0} \left(1 - \frac{\lambda}{\lambda_n}\right), \tag{2.6}$$

where $\{\lambda_n\}_{n\geq 0}$ is the spectrum of Problems (1.1)-(1.3) and C is a constant depending only on the spectrum $\{\lambda_n\}_{n\geq 0}$. We should remind the readers that all the zeros of that characteristic function $\Delta(h, H, q)(\lambda)$ is geometrically simple. By Lemma 2.1,

$$\Delta(h, H, q)(\lambda) = -\rho \sin \rho + \omega \cos \rho + O\left(\frac{1}{\rho} \exp(|\tau|)\right), \tag{2.7}$$

where $\omega = h + H + \frac{1}{2}\int_0^1 q(t)dt$. Also, note if we let $G_\delta = \{\rho \in \mathbb{C} \,|\, |\rho - k\pi| > \delta,$ $k \in \mathbb{Z}\}$, where $0 < \delta < \pi$, then $|\sin \rho| \geq C_\delta \exp(|\tau|)$ for $\rho \in G_\delta$, hence

$$|\Delta(h, H, q)(\lambda)| \geq C_0 |\rho \exp(|\tau|)| \quad \text{for } \rho \in C_\delta \text{ and } |\rho| \gg 1. \tag{2.8}$$

If we let

$$\begin{cases} y_1(x) = y(x), \\ y_2(x) = y(1-x), \\ q_1(x) = q(x), \\ q_2(x) = q(1-x) \end{cases} \tag{2.9}$$

for $0 \leq x \leq 1/2$, then (1.1)-(1.3) can be transformed to

$$y''_i + (\lambda - q_i(x))y_i = 0, \quad 0 < x < 1/2, \quad i = 1, 2, \tag{2.10}$$

$$y'_1(0) - hy_1(0) = 0, \tag{2.11}$$

$$y'_2(0) - Hy_2(0) = 0, \tag{2.12}$$

$$y_1(1/2) = y_2(1/2), \tag{2.13}$$

$$y_1'(1/2) + y_2'(1/2) = 0, \tag{2.14}$$

or equivalently,

$$\begin{cases} \vec{y}'' + (\lambda I_2 - Q(x))\vec{y} = 0, \ 0 < x < 1/2, \\ \vec{y}'(0) - \mathcal{H}\vec{y}(0) = 0, \\ \mathcal{A}\vec{y}'(1/2) + \mathcal{B}\vec{y}(1/2) = 0, \end{cases} \tag{2.15}$$

where

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \tag{2.16}$$

$$Q(x) = \begin{bmatrix} q_1(x) & 0 \\ 0 & q_2(x) \end{bmatrix}, \tag{2.17}$$

$$\mathcal{H} = \begin{bmatrix} h & 0 \\ 0 & H \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}. \tag{2.18}$$

Problem (2.15) is equivalent to Problems (1.1)-(1.3).

For a matrix-valued Sturm-Liouville equation, we have

Lemma 2.3. *Assuming that* $\mathcal{Q}(x)$ *is an integrable* 2×2 *matrix-valued function. Let* $Y(x, \lambda)$ *denote the solution of matrix-valued equation of*

$$Y'' + (\lambda I_2 - \mathcal{Q}(x))Y = 0, \quad 0 < x < 1/2 \tag{2.19}$$

with $Y(0) = I_2$, $Y'(0) = \mathcal{K}$, *where* I_2 *is the* 2×2 *identity matrix and* $\mathcal{K}$ *is a complex-valued* 2×2 *matrix. Then*

$$Y(x, \lambda) = \cos\sqrt{\lambda}xI_2 + \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\mathcal{K} + O\left(\frac{\exp|\tau|x}{\sqrt{\lambda}}\right) \quad \text{for} \quad |\lambda| \gg 1,$$

where

$$\tau = \operatorname{Im}(\sqrt{\lambda}).$$

Note if $\mathcal{Q}(x) = Q(x) = diag(q_1(x), q_2(x))$ (see (2.17)) and $\mathcal{K} = \mathcal{H}$ (see (2.18)), then

$$Y(x, \lambda) = \begin{bmatrix} y_{1,h}(x, \lambda) & 0 \\ 0 & y_{2,H}(x, \lambda) \end{bmatrix},$$

where $y_{i,h}$ denotes the solution of

$$y'' + (\lambda - q_i(x))y = 0$$

on $0 < x < 1/2$ and satisfies the initial conditions

$$y(0) = 1, \quad y'(0) = h.$$

The characteristic function of (2.15) is

$$\begin{aligned} \Delta(H, \mathcal{Q})(\lambda) &= \det(\mathcal{A}Y'(1/2, \lambda) + \mathcal{B}Y(1/2, \lambda)) \\ &= \begin{vmatrix} y_{1,h}(1/2, \lambda) & -y_{2,H}(1/2, \lambda) \\ y'_{1,h}(1/2, \lambda) & y'_{2,H}(1/2, \lambda) \end{vmatrix} \\ &= \varphi'_h(1/2, \lambda)\psi_H(1/2, \lambda) - \varphi_h(1/2, \lambda)\psi'_H(1/2, \lambda) \\ &= \Delta(h, H, q)(\lambda). \end{aligned} \tag{2.20}$$

All the eigenvalues of (2.15) consist of the zeros of $\Delta(H, \mathcal{Q})(\lambda)$ and are algebraically simple. This viewpoint will enable us to avoid some complicated computation while studying inverse spectral problems of Sturm-Liouville equations with interior discontinuities, inverse spectral problems of Sturm-Liouville equations with an eigenparameter in boundary conditions or the mixed problems of the last two types of problems.

3. Inverse Spectral Problems of Sturm-Liouville Equations with Interior Discontinuities or Eigenparameter on Boundary Conditions

In this section, we prove some Hochstadt-Lieberman type theorems. The first case we want to treat is the Sturm-Liouville equation with interior discontinuities. This problem arises from several physical models, for example, the oscillation of the Earth (see [2, 24]). The Hochstadt-Lieberman Theorem for Sturm-Liouville equations with interior discontinuities have been studied by some mathematicians (see [22, 35, 33] and references therein). Here we provide an alternative proof for the following theorem:

Theorem 3.1 ([33, Theorem 2])**.** *$q(x) \in L_1(0, 1)$, $(h_1, h_2, a_1, a_2) \in \mathbb{R}^4$ and $a_1 \neq 0$. Let $\sigma(a_1, a_2, h, H, q)$ denote the spectrum of*

$$\begin{cases} y'' + (\lambda - q(x))y = 0, \ 0 < x < 1, \\ y'(0) - h_1 y(0) = 0, \\ y'(1) + h_2 y(1) = 0, \\ y\left(\frac{1}{2}^{+}\right) = a_1 y\left(\frac{1}{2}^{-}\right), \\ y'\left(\frac{1}{2}^{+}\right) = a_1^{-1} y'\left(\frac{1}{2}^{-}\right) + a_2 y\left(\frac{1}{2}^{-}\right). \end{cases} \tag{3.1}$$

Then $\sigma(a_1, a_2, h_1, h_2, q)$ and $q(x)|_{\left(\frac{1}{2}, 1\right)}$ determine $q(x)$ uniquely.

Proof. Using the transformation in (2.9), Problem (3.1) can be transformed to the problem

$$\vec{y}'' + (\lambda I_2 - Q(x))\vec{y} = 0, \quad 0 < x < 1/2, \tag{3.2}$$

with boundary conditions

$$\begin{cases} \vec{y}'(0) - \mathcal{H}\vec{y}(0) = 0, \\ \mathcal{A}_1 \vec{y}'(1/2) + \mathcal{B}_1 \vec{y}(1/2) = 0, \end{cases} \tag{3.3}$$

where $Q(x) = \begin{bmatrix} q_1(x) & 0 \\ 0 & q_2(x) \end{bmatrix}$, $\mathcal{H} = diag(h_1, h_2)$, $\mathcal{A}_1 = \begin{bmatrix} 0 & 0 \\ a_1^{-1} & 1 \end{bmatrix}$, $\mathcal{B}_1 = \begin{bmatrix} a_1 & -1 \\ 0 & a_2 \end{bmatrix}$ and

$$\begin{cases} q_1(x) = q(x), \\ q_2(x) = q(1 - x) \end{cases} \quad \text{for} \quad 0 < x < 1/2.$$

Let $\varphi_i(x, \lambda)$ denote the solution of

$$y'' + (\lambda - q_i(x))y = 0, \quad 0 < x < 1/2,$$

with $\varphi_i(0, \lambda) = 1$ and $\varphi_i'(0, \lambda) = h_i$ for $i = 1, 2,$ and

$$Y(x, \lambda) = \begin{bmatrix} \varphi_1(x, \lambda) & 0 \\ 0 & \varphi_2(x, \lambda) \end{bmatrix}. \tag{3.4}$$

Then the characteristic function $\Delta(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, Q)(\lambda)$ for (3.2) is

$$\begin{aligned} &\Delta(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, Q)(\lambda) \\ &= \det\{\mathcal{A}_1 Y'(1/2, \lambda) + \mathcal{B}_1 Y(1/2, \lambda)\} \\ &= \begin{vmatrix} a_1\varphi_1(1/2, \lambda) & -\varphi_2(1/2, \lambda) \\ a_1^{-1}\varphi_1'(1/2, \lambda) & a_2\varphi_2(1/2, \lambda) + \varphi_2'(1/2, \lambda) \end{vmatrix} \\ &= \varphi_1(1/2, \lambda)[a_1a_2\varphi_2(1/2, \lambda) + a_1\varphi_2'(1/2, \lambda)] + a_1^{-1}\varphi_1'(1/2, \lambda)\varphi_2(1/2, \lambda) \\ &= O(\rho\exp(|\tau|)), \quad \text{for} \quad |\lambda| \gg 1. \end{aligned} \tag{3.5}$$

Suppose that there are two potential functions $q(x)$ and $\tilde{q}(x)$ which satisfy

$$\sigma(a_1, a_2, h_1, h_2, q) = \sigma(a_1, a_2, h_1, h_2, \tilde{q}) \quad \text{and} \quad q(x)|_{\left(\frac{1}{2}, 1\right)} = \tilde{q}(x)|_{\left(\frac{1}{2}, 1\right)}.$$

Then we have the corresponding potential matrix $Q(x) = diag(q_1(x), q_2(x))$ and $\tilde{Q}(x) = diag(\tilde{q}_1(x), \tilde{q}_2(x)) = diag(\tilde{q}_1(x), q_2(x))$ so that

$$\sigma(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, Q) = \sigma(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, \tilde{Q}).$$

The readers should note that all eigenvalues of (3.1) are algebraically simple. Denote

$$Y(x, \lambda, Q) = \begin{bmatrix} \varphi_1(x, \lambda) & 0 \\ 0 & \varphi_2(x, \lambda) \end{bmatrix}$$

and

$$Y(x, \lambda, \tilde{Q}) = \begin{bmatrix} \tilde{\varphi}_1(x, \lambda) & 0 \\ 0 & \tilde{\varphi}_2(x, \lambda) \end{bmatrix} = \begin{bmatrix} \tilde{\varphi}_1(x, \lambda) & 0 \\ 0 & \varphi_2(x, \lambda) \end{bmatrix}$$

the fundamental matrices of (3.2) corresponding to potential matrix $Q(x)$ and $\tilde{Q}(x)$, respectively. Since $\sigma(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, Q) = \sigma(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, \tilde{Q})$, we have $\Delta(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, Q) = \Delta(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, \tilde{Q})$, and this leads to

$$\begin{bmatrix} \varphi_1(1/2, \lambda) & \varphi_1'(1/2, \lambda) \\ \tilde{\varphi}_1(1/2, \lambda) & \tilde{\varphi}_1'(1/2, \lambda) \end{bmatrix} \begin{bmatrix} a_1 a_2 \varphi_2(1/2, \lambda) + a_1 \varphi_2'(1/2, \lambda) \\ a_1^{-1} \varphi_2(1/2, \lambda) \end{bmatrix}$$

$$= \begin{bmatrix} \Delta(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, Q)(\lambda) \\ \Delta(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, Q)(\lambda) \end{bmatrix}. \tag{3.6}$$

Denote

$$F_1(\lambda) = \frac{\varphi_1(1/2, \lambda)\tilde{\varphi}_1'(1/2, \lambda) - \tilde{\varphi}_1(1/2, \lambda)\varphi_1'(1/2, \lambda)}{\Delta(\mathcal{H}, \mathcal{A}_1, \mathcal{B}_1, Q)(\lambda)}.$$

Since $\begin{bmatrix} a_1 a_2 \varphi_2(1/2, \lambda) + a_1 \varphi_2'(1/2, \lambda) \\ a_1^{-1} \varphi_2(1/2, \lambda) \end{bmatrix}$ never vanishes,

$$\varphi_1(1/2, \lambda)\tilde{\varphi}_1'(1/2, \lambda) - \tilde{\varphi}_1(1/2, \lambda)\varphi_1'(1/2, \lambda) = 0$$

for $\lambda \in \sigma(a_1, a_2, h_1, h_2, q)$. Hence, $F_1(\lambda)$ is an entire function and satisfies all assumptions in Theorem 2.2. Applying Theorem 2.2, we have $F_1(\lambda) = 0$. This leads to

$$\frac{\varphi_1(1/2, \lambda)}{\varphi'(1/2, \lambda)} = \frac{\tilde{\varphi}_1(1/2, \lambda)}{\tilde{\varphi}_1'(1/2, \lambda)}.$$

Hence, $q_1(x) = \tilde{q}_2(x)$. This completes the proof. □

The readers shall see in Theorem 3.5 that the number and positions of discontinuities are not important.

Next, we are going to study the Sturm-Liouville equations with eigenparameter in boundary conditions. Binding et al. have done a lot work on this topic (see [5-10] and the references therein for details). Here we shall prove the Hochstadt-Lieberman Theorem for this type of Sturm-Liouville problems. From now on, we denote

$$f(\lambda) = \frac{a\lambda + b}{c\lambda + d}, \; (a, b, c, d) \in \mathbb{R}^4, \; c \neq 0, \quad ad - bc > 0.$$

Theorem 3.2. *Assuming that* $q(x) \in L_1(0, 1)$ *and* $h \in \mathbb{R}$. *The spectrum* $\sigma(h, f(\lambda), q)$ *of*

$$\begin{cases} y'' + (\lambda - q(x))y = 0, \ 0 < x < 1, \\ y'(0) - hy(0) = 0, \\ y'(1) + f(\lambda)y(1) = 0 \end{cases} \tag{3.7}$$

and $q(x)|_{\left(\frac{1}{2}, 1\right)}$ *determine* $q(x)$ *uniquely.*

Proof. Using the transformation in (2.9), (3.7) can be transformed to the problem

$$\begin{cases} \vec{y}'' + (\lambda I_2 - Q(x))\vec{y} = 0, \ 0 < x < 1/2, \\ \vec{y}'(0) - \mathcal{H}_\lambda \vec{y}(0) = 0, \\ \mathcal{A}\vec{y}'(1/2) + \mathcal{B}\vec{y}(1/2) = 0, \end{cases} \tag{3.8}$$

where $\mathcal{H}_\lambda = diag(h, f(\lambda))$, $\mathcal{A} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and $Q(x)$ is as that in the proof of Theorem 3.1. Let $\varphi_1(x, \lambda)$ be as that in Theorem 3.1 and $\varphi_2(x, \lambda; f)$ be the solution of

$$y'' + (\lambda - q_2(x))y = 0, \quad 0 < x < 1/2,$$

with $\varphi_2(0, \lambda; f) = 1$ and $\varphi_2'(0, \lambda; f) = f(\lambda)$. Then the characteristic function of (3.8) is

$$\Delta(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q)(\lambda) = \det(\mathcal{A}Y'(1/2, \lambda; f) + \mathcal{B}Y(1/2, \lambda; f)), \tag{3.9}$$

where $Y(x, \lambda; f) = diag(\varphi_1(x; \lambda), \varphi_2(x, \lambda; f))$. Note that $\Delta(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q)(\lambda)$ is a meromorphic function with a simple pole $\lambda_* = \frac{-c}{d}$. Hence, the function $\Delta_1(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q)(\lambda)$ defined by

$$\begin{aligned} &\Delta_1(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q)(\lambda) \\ &= (c\lambda + d)\Delta(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q)(\lambda) \\ &= \varphi_1(1/2, \lambda)(c\lambda + d)\varphi_2'(1/2, \lambda; f) + \varphi_1'(1/2, \lambda)(c\lambda + d)\varphi_2(1/2, \lambda; f) \\ &= O(\rho^2 \exp|\tau|) \end{aligned}$$

is analytic, where $\rho = \sqrt{\lambda}$, $\tau = \mathrm{Im}(\rho)$. The zeros of $\Delta_1(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q)(\lambda)$ are also algebraically simple (see [6, Theorem 2.1]). The remaining of the proof is the same as that of Theorem 3.1. Suppose there exists a $\tilde{q}(x) \in L_1(0, 1)$ so that $q(x) = \tilde{q}(x)$ for $x \in (1/2, 1)$ and $\sigma(h, f(\lambda), q) = \sigma(h, f(\lambda), \tilde{q})$. Then we shall obtain a corresponding $\tilde{Q}(x) = \begin{bmatrix} \tilde{q}_1(x) & 0 \\ 0 & q_2(x) \end{bmatrix}$ such that

$$\sigma(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q) = \sigma(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, \tilde{Q}).$$

This implies

$$\Delta_1(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q)(\lambda) = \Delta_1(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, \tilde{Q})(\lambda).$$

Hence, we have

$$\begin{bmatrix} \varphi_1(1/2, \lambda) & \varphi_1'(1/2, \lambda) \\ \tilde{\varphi}_1(1/2, \lambda) & \tilde{\varphi}_1'(1/2, \lambda) \end{bmatrix} \begin{bmatrix} (c\lambda + d)\varphi_2'(1/2, \lambda) \\ (c\lambda + d)\varphi_2(1/2, \lambda) \end{bmatrix} = \begin{bmatrix} \Delta_1(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q)(\lambda) \\ \Delta_1(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q)(\lambda) \end{bmatrix}. \tag{3.10}$$

Denote $F_2(\lambda) = \dfrac{\varphi_1(1/2, \lambda)\tilde{\varphi}_1'(1/2, \lambda) - \tilde{\varphi}_1(1/2, \lambda)\varphi_1'(1/2, \lambda)}{\Delta_1(\mathcal{H}_\lambda, \mathcal{A}, \mathcal{B}, Q)}$, then $F_2(\lambda)$ is an entire function and satisfies all the assumptions of Theorem 2.2. Applying Theorem 2.2 again, we have $F_2(\lambda) = 0$. This leads to

$$\frac{\varphi_1(1/2, \lambda)}{\varphi_1'(1/2, \lambda)} = \frac{\tilde{\varphi}_1(1/2, \lambda)}{\tilde{\varphi}_1'(1/2, \lambda)}.$$

Hence, $q_1(x) = \tilde{q}_1(x)$. This leads to the assertion. □

Combining the last two theorems, we have

Theorem 3.3. *Assuming* $q(x) \in L_1(0, 1)$, $(h_1, a_1, a_2) \in \mathbb{R}^3$ *and* $a_1 \neq 0$. *The spectrum* $\sigma(h_1, h_2, a_1, a_2, a_2, q)$ *of*

$$\begin{cases} y'' + (\lambda - q(x))y = 0, \ 0 < x < 1, \\ y'(0) - hy(0) = 0, \\ y'(1) + f(\lambda)y(1) = 0, \\ y\left(\dfrac{1}{2}^+\right) = a_1 y\left(\dfrac{1}{2}^-\right), \\ y'\left(\dfrac{1}{2}^+\right) = a_1^{-1} y'\left(\dfrac{1}{2}^-\right) + a_2 y\left(\dfrac{1}{2}^-\right) \end{cases} \tag{3.11}$$

and $q(x)|_{\left(\frac{1}{2}, 1\right)}$ *determine* $q(x)$ *uniquely.*

Proof. The proof is similar to the proof of Theorem 3.1 and Theorem 3.2. We omit the details. □

The readers might ask: "Does the Hochstadt-Lieberman Theorem hold for Sturm-Liouville Problems with arbitrarily finite number of interior discontinuities?" The answer is positive. Note that our arguments depend only on the fact: "the Weyl-Titchmarsh $\mathfrak{M}$ -function can determine the potential uniquely" and this is also true for Sturm-Liouville Problems with arbitrary number of interior discontinuities (see Sec. 4.4 in [14]). Hence, we can conclude that

Theorem 3.4. *Assuming that* $q(x) \in L_1(0, 1)$, $0 < x_i < x_{i+1} < 1$, $a_1^i, a_2^i \in \mathbb{R}$ *and* $a_1^i \neq 0$ *for* $i = 1, 2, 3, \ldots, k$. *The spectrum* $\sigma(a_1^i, a_2^i, h_1, h_2, q; k)$ *of*

$$\begin{cases} y'' + (\lambda - q(x))y = 0, \ 0 < x < 1, \\ y'(0) - h_1 y(0) = 0, \\ y'(1) + h_2 y(1) = 0, \\ y(x_i^+) = a_1^i y(x_i^-), \\ y'(x_i^+) = (a_1^i)^{-1} y'(x_i^-) + a_2^i y(x_i^-), \ i = 1, 2, 3, \ldots, k \end{cases} \tag{3.12}$$

and $q(x)|_{\left(\frac{1}{2}, 1\right)}$ *determine* $q(x)$ *uniquely.*

Proof. If $x = 1/2$ is not a point of interior discontinuity, then we can label it as some x_j and the corresponding coefficients can be $a_1^j = 1$ and $a_2^j = 0$. Hence, we can always assume $x_l = 1/2$ for some integer $1 < l < k$. Suppose that $x_i \in (0, 1/2)$ for $i = 1, 2, 3, \ldots, l - 1$ and $x_j \in (1/2, 1)$ for $j = l + 1, l + 2, \ldots, k$. Let $\varphi_1(x; \lambda)$ and $\varphi_2(x; \lambda)$ be solutions of

$$y'' + (\lambda - q(x))y = 0, \quad 0 < x < 1,$$

where $\varphi_1(x; \lambda)$ satisfies the initial condition $\varphi_1(0; \lambda) = 1$, $\varphi_1'(0; \lambda) = h_1$ and the discontinuity condition at x_i for $i = 1, 2, \ldots, l$; $\varphi_2(x; \lambda)$ satisfies the initial condition $\varphi_2(1; \lambda) = 1$, $\varphi_1'(1; \lambda) = -h_2$ and the discontinuity condition at x_j for $j = l + 1$, $l + 2, \ldots, n$. The asymptotic behaviors of $\varphi_i(x, \lambda)$ and $\varphi_i'(x, \lambda)$ can be obtained (see Sec. 4.4 in [14]). Applying the same arguments as that in the proofs of Theorem 3.1 and Theorem 3.2, we can uniquely determine the Weyl-Titchmarsh $\mathfrak{M}$-function $\frac{\varphi_1'(1/2, \lambda)}{\varphi_1(1/2, \lambda)}$ for (3.12) on $(0, 1/2)$. Since Weyl-Titchmarsh $\mathfrak{M}$-function for Sturm-Liouville equation with interior discontinuities can uniquely determine potential function (see Sec. 4.4 of [14]), we can conclude the assertion. $\square$

Moreover, we have

Theorem 3.5. *Assuming that* $q(x) \in L_1(0, 1)$, $0 < x_i < x_{i+1} < 1$, $a_1^i, a_2^i \in \mathbb{R}$ *and* $a_1^i \neq 0$ *for* $i = 1, 2, 3, \ldots, k$. *The spectrum* $\sigma(a_1^i, a_2^i, h_1, f(\lambda), q; k)$ *of*

$$\begin{cases} y'' + (\lambda - q(x))\, y = 0,\ 0 < x < 1, \\ y'(0) - h_1 y(0) = 0, \\ y'(1) + f(\lambda)(1) = 0, \\ y(x_i^+) = a_1^i y(x_i^-), \\ y'(x_i^+) = (a_1^i)^{-1} y'(x_i^-) + a_2^i y(x_i^-),\ i = 1, 2, 3, \ldots, k \end{cases} \tag{3.13}$$

and $q(x)|_{\left(\frac{1}{2}, 1\right)}$ *determine* $q(x)$ *uniquely*.

The readers may notice that some of our results are similar to the results in [35] and [22], but with the approach in this paper, we can simplify the proofs for some natural generalizations of Hochstadt-Lieberman theorem which is what we wanted to emphasize.

References

[1] E. Andersson, On the *M*-function and Borg-Marchenko theorems for vector-valued Sturm-Liouville equations, J. Math. Phys. 44(12) (2003), 6077-6100.

[2] R. S. Anderssen, The effect of discontinuities in density and shear velocity on the asymptotic overtone structure of torsional eigenfrequencies of the earth, Geophysical Journal International 50(2) (1977), 303-309.

[3] Avdonin Sergei, Lenhart Suzanne and Protopopescu Vladimir, Solving the dynamical inverse problem for the Schrödinger equation by the boundary control method, Inverse Problems 18(2) (2002), 349-361.

[4] M. I. Belishevm, Boundary spectral inverse problem on a class of graphs (trees) by the BC method, Inverse Problems 20(3) (2004), 647-672.

[5] P. A. Binding, P. J. Browne and K. Seddighi, Sturm-Liouville problems with eigenparameter dependent boundary conditions, Proc. Edinburgh Math. Soc. 37 (1993), 57-72.

[6] P. A. Binding, P. J. Browne and B. A. Waston, Inverse spectral problems for Sturm-Liouville equations with eigenparameter dependent boundary conditions, J. London Math. Soc. 62(1) (2000), 161-182.

[7] P. A. Binding, P. J. Browne and B. A. Waston, Transformations between Sturm-Liouville problems with eigenvalue dependent and independent boundary conditions, Bull. London Math. Soc. 33 (2001), 749-757.

[8] P. A. Binding, R. Hryniv, H. Langer and B. Najman, Elliptic eigenvalue problems with eigenparameter dependent boundary conditions, J. Differential Equations 174 (2001), 30-54.

[9] P. A. Binding, P. J. Browne and B. A. Waston, Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, II, J. Comput. Appl. Math. 148(1) (2002), 147-168.

[10] P. J. Browne and B. D. Sleeman, A uniqueness theorem for inverse eigenparameter dependent Sturm-Liouville problems, Inverse Problems 13 (1997), 1453-1462.

[11] Carlson Robert, An inverse problem for the matrix Schrödinger equation, J. Math. Anal. Appl. 267 (2002), 564-575.

[12] Chern Hua-Huai and Shen Chao-Liang, On the *n*-dimensional Ambarzumyan's theorem, Inverse Problems 13(1) (1997), 15-18.

[13] Clarka Steve, Gesztesy Fritz, Holdenc Helge and M. Levitan Boris, Borg-type theorems for matrix-valued Schrödinger operators, J. Differential Equations 167(1) (2000), 181-210.

[14] G. Freiling and V. A. Yurko, Inverse Sturm-Liouville Problems and their Applications, NOVA Science Publishers, Inc., New York, 2001.

[15] Gesztesy Fritz and Simon Barry, Inverse spectral analysis with partial information on the potential, II. The case of discrete spectrum, Trans. Amer. Math. Soc. 352(6) (2000), 2765-2787.

[16] Gesztesy Fritz, Kiselev Alexander and A. Konstantin, Uniqueness results for matrix-valued Schrödinger, Jacobi, and Dirac-type operators, Math. Nachr. 239-240 (2002), 103-145.

[17] Jodeit Max and B. M. Levitan, Isospectral vector-valued Sturm-Liouville problems, Lett. Math. Phys. 43 (1998), 117-122.

[18] Jodeit Max and B. M. Levitan, The isospectrality problem for the classical Sturm-Liouville equation, Adv. Differential Equations 2(2) (1997), 297-318.

[19] O. Martinyuk and V. Pivovarchik, On the Hochstadt-Lieberman theorem, Inverse Problems 26(3) (2010), 035011.

[20] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1984.

[21] Yaroslav Kurylev, Matti Lassas and Ricardo Weder, Multidimensional Borg-Levision theorem, Inverse Problems 21(5) (2005), 1685-1696.

[22] O. Hald, Discontinuous inverse eigenvalue problems, Comm. Pure. Appl. Math. XXXVII (1984) 539-577.

[23] H. Hochstadt and B. Lieberman, An inverse Sturm-Liouville problem with mixed data, SIAM J. Appl. Math. 34 (1978), 676-680.

[24] E. R. Lapwood and T. Usami, Free Oscillations of the Earth, Cambridge University Pres, 1981.

[25] B. M. Levitan and M. G. Gasymov, Determination of a differential equation by two spectra, Russian Math. Surveys 19(2) (1964), 3-63.

[26] B. Levitan, Inverse Sturm-Liouville Problems, VNU Science Press, Utrecht, 1987.

[27] V. A. Marchenko, Sturm-Liouville Operators and Applications, Birkhäuser Verlag, 1986.

[28] K. Mochizuki and I. Trooshin, Inverse problem for interior spectral data of the Sturm-Liouville operator, J. Inverse Ill-Posed Probl. 9(4) (2001), 425-433.

[29] Shen Chao-Liang, Some eigenvalue problems for the vectorial Hill's equation, Inverse Problems 16(3) (2000), 749-783.

[30] Chao-Liang Shen, Some inverse spectral problems for vectorial Sturm-Liouville equations, Inverse Problems 17(5) (2001), 1253-1294.

[31] V. N. Pivovarchik, Inverse problem for the Sturm-Liouville equation on a star-shaped graph, Math. Nachr. 280(13-14) (2007), 1595-1619.

[32] Remling Christian, Inverse spectral theory for one-dimensional Schrödinger operators: the *A* function, Math. Z. 245(3) (2003), 597-617.

[33] C. T. Shieh and V. A. Yurko, Inverse nodal and inverse spectral problems for discontinuous boundary value problems, J. Math. Anal. Appl. 347(1) (2008), 266-272.

[34] Simon Barry, A new approach to inverse spectral theory, I. Fundamental formalism, Ann. of Math. 150(3) (1999), 1029-1057.

[35] Willis Catherine, Inverse Sturm-Liouville problems with two discontinuities, Inverse Problems 1(3) (1985), 263-289.

[36] V. Yurko, Method of Spectral Mappings in the Inverse Problem Theory, VSP, 2002.

[37] V. Yurko, Inverse spectral problems for Sturm-Liouville operators on graphs, Inverse Problems 21 (2005), 1075-1086.

[38] V. Yurko, Inverse problems for the matrix Sturm-Liouville equation on a finite interval, Inverse Problems 22 (2006), 1139-1149.