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DETERMINING THE SIZE OF EXPERIMENTS FOR ANOVA MODELS

DIETER RASCH

Center of Experimental Design
University of Life Sciences
Vienna, Austria

Abstract

In this paper, the theoretical background for determining the minimal size of an experiment that should be analyzed by analysis of variance with at least one fixed factor about which a null hypothesis has to be tested, is given. Fixed and mixed models in cross, nested and mixed classifications are described. Corresponding R-programs are demonstrated by examples.

1. Introduction

The process of gaining knowledge in the empirical sciences can be considered as follows:

(i) Formulation of the problem,

(ii) Fixing the precision requirements,

(iii) Selecting the statistical model for planning and analysis,

(iv) Determining the (optimal) design of the experiment or survey,

(v) Performing the experiment or the survey,

(vi) Statistical analysis of the observed results,

(vii) Interpretation of the results.

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The first four steps belong to the pre-experimental statistics whereas the two last belong to the post-experimental statistics.

The *statistical planning of an experiment* includes the construction of an optimum statistical experimental design and the determination of the *minimum sample size*, which is necessary to achieve, predetermined precision requirements, in the context of a chosen statistical model for the analysis of the results.

In this paper, we consider only the problem of the determination of the *minimum sample size* of an experiment for the best estimators, the in expectation shortest confidence intervals and the uniformly best unbiased tests which in linear models are sometimes different from determining the sample size.

Point estimation

Choose the size of the experiment (random sample) so that the variance of the best estimator is below a given bound B. As an example, we consider the estimation of the expectation. At first, we have to choose the best unbiased estimator, which is the mean of the sample. Its variance is $\frac{\sigma^2}{n}$ and from $\frac{\sigma^2}{n} \leq B$ it follows $n \geq \frac{\sigma^2}{B}$ or the integer solution $n = \left\lceil \frac{\sigma^2}{B} \right\rceil$, where $\lceil x \rceil$ is the smallest integer $\geq x$.

Interval estimation

If the precision requirement states that the expected half-width of a confidence interval must be less than or equal to δ, then, for a given α, n has to be determined so that with the upper bound $\boldsymbol{u}$ and the lower bound $\boldsymbol{l}$ of the $(1 - \alpha)$-confidence interval

$$\frac{1}{2} E[\boldsymbol{u}(\alpha) - \boldsymbol{l}(\alpha)] \leq \delta.$$

Of course at first we have to find the $(1 - \alpha)$-confidence interval with the smallest expected length.

As an example, we consider the confidence estimation of the expectation of a normal distribution (variance unknown). The shortest (in expectation) two-sided $(1 - \alpha)$-confidence interval is given by

$$\left[\bar{\boldsymbol{y}} - t\left(n - 1; 1 - \frac{\alpha}{2}\right) \frac{\boldsymbol{s}}{\sqrt{n}}; \ \bar{\boldsymbol{y}} + t\left(n - 1; 1 - \frac{\alpha}{2}\right) \frac{\boldsymbol{s}}{\sqrt{n}} \right],$$

its half expected length is

$$E(\boldsymbol{H}) = t\left(n-1; 1-\frac{\alpha}{2}\right)\frac{E(\boldsymbol{s})}{\sqrt{n}} = \frac{t\left(n-1; 1-\frac{\alpha}{2}\right)}{\sqrt{n}}\frac{\Gamma\left(\frac{n}{2}\right)\cdot\sqrt{2}}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{n-1}}\sigma.$$

$E(\boldsymbol{H}) \leq \delta$ leads to the equation for n:

$$n = \left\lceil t^2\left(n-1; 1-\frac{\alpha}{2}\right)\frac{2\cdot\Gamma^2\left(\frac{n}{2}\right)\cdot}{\Gamma^2\left(\frac{n-1}{2}\right)(n-1)}\frac{\sigma^2}{\delta^2}\right\rceil,$$

where $\lceil x \rceil$ is the smallest integer larger or equal to x.

Hypothesis testing

The problem of determining the size of an experiment we explain for the one-sample problem for testing an expectation, and then the size of the experiment is again the sample size.

A random sample $\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_n^1$ of size n will be drawn from a normally distributed population with mean μ and variance σ^2, with the purpose of testing the null hypothesis:

$$H_0 : \mu = \mu_0 \ (\mu_0 \text{ is a given constant})$$

against the alternative hypothesis:

$$H_A : \mu \neq \mu_0 \text{ (two-sided alternative).}$$

The test statistic of a uniformly most powerful unbiased test is

$$\boldsymbol{t} = \frac{\bar{\boldsymbol{y}} - \mu_0}{\boldsymbol{s}}\sqrt{n}$$

which is non-central t-distributed with $n-1$ *d.f.* and non-centrality parameter

$$\lambda = \frac{\mu - \mu_0}{\sigma}\sqrt{n}.$$

Under the null hypothesis, the distribution is central t.

If the Type I error probability is α, then H_0 will be rejected if:

$$|t| > t(n-1;\, 1-\alpha/2).$$

Our precision requirement is given by α and the risk of the second kind β if $|\mu - \mu_0| = \delta$.

From this, we have the requirement

$$t(n-1;\, 1-\alpha/2) = t(n-1;\, \lambda;\, \beta), \tag{1}$$

where $t(n-1;\, \lambda;\, \beta)$ is the β-quantile of the non-central *t*-distribution with $n-1$ *d.f.* and non-centrality parameter λ.

Using the approximation $t(n-1;\, \lambda;\, \beta) = t(n-1,\, \beta) + \lambda$ leads to the approximate formula

$$n \approx \left\lceil \left[\left\{ t\left(n-1; 1-\frac{\alpha}{2}\right) + t(n-1; 1-\beta) \right\} \frac{\sigma}{\delta} \right]^2 \right\rceil.$$

Equation (1) is crucial for determining minimal sizes when testing location parameters, a generalization will be considered in the main part of the paper.

2. Tests in the Analysis of Variance about the Effects of a Fixed Factor

For all models in the analysis of variance (ANOVA), the linear model equation has the form

$$\boldsymbol{y} = E(\boldsymbol{y}) + \boldsymbol{e}.$$

In this equation, the random variable $\boldsymbol{y}$ models the observed character. The observation $\boldsymbol{y}$ is the sum of the expectation (mean) $E(\boldsymbol{y})$ of $\boldsymbol{y}$ and an error term $\boldsymbol{e}$, containing observational errors with $E(\boldsymbol{e}) = 0$, $\operatorname{var}(\boldsymbol{e}) = \sigma^2$. The variability in $E(\boldsymbol{y})$ between experimental units depends linearly on model parameters. The models for the analysis of variance differ in the number and the nature of these parameters.

The observations in an analysis of variance are allocated to at least two classes, which are determined by the levels of the factors.

Each of the models of the analysis of variance contains the general mean μ, i.e., we write $E(\boldsymbol{y})$ in the form

$$E(\boldsymbol{y}) = \mu + EC(\boldsymbol{y}),$$

where $EC(\boldsymbol{y})$ is the mean deviation from μ within the corresponding class. In the case of p factors, the analysis of variance is called *p-way*.

It follows that the total set of the $\boldsymbol{y}$ does not constitute a random sample because not all the $\boldsymbol{y}$ have the same expectation. Furthermore, in models with random factors, the $\boldsymbol{y}$ within the same class are not independent.

For all the models, we assume that the variance σ^2 of the error terms in the equations is the same in all sub-classes and that all the random variables in the right hand side (r.h.s.) of the model equations are mutually independent and have expectation zero.

We assume that $\boldsymbol{y}$ has a normal distribution and that we have equal sub-class numbers n. We then can test the following null hypothesis in all the models with a fixed factor A having effects a_i $(i = 1, ..., a)$.

H_0: "The factor A has no effect on the dependent variable $\boldsymbol{y}$". In other words: "All the a_i are equal". If it is assumed that the sum of the a_i is zero, then this is the same as "All the a_i are equal to zero".

The alternative hypothesis is

H_A: "At least two of the a_i are different".

The test statistic for this test is a variate $\boldsymbol{F}$ that (if the null hypothesis is true) follows a (central) F-distribution with f_1 and f_2 degrees of freedom. The $(1 - \alpha)$-quantile of the distribution of $F(f_1;\ f_2)$ is denoted by $F(f_1;\ f_2;\ 1 - \alpha)$.

This test statistic is generally calculated by following the next 8 steps - here "generally" means that these steps should not only be used for all situations and models in this paper but also for any other ANOVA situation.

1. Define the null hypothesis.

2. Choose the appropriate model (I, II, or mixed).

3. Find the $E(\boldsymbol{MS})$ column in the ANOVA-table (if there are several such columns, then find the one that corresponds to your model).

4. In the same table, find the row for the factor that appears in your null hypothesis.

5. Change the $E(\boldsymbol{MS})$ in this row to what it would be if the null hypothesis were true.

6. Search in the same table (in the same column) for the row, which now has the same $E(\boldsymbol{MS})$ as you found in the 5th step.

7. The F-value is now the value of the $\boldsymbol{MS}$ of the row you found in the fourth step divided by the value of the $\boldsymbol{MS}$ of the row you found in the 6th step.

8. Note: in ANOVA with higher classifications, the 6th step may not be successful, in which case we can use the so-called Satterthwaite approximation.

The minimum size of the experiment should be determined so that the precision requirements are fulfilled. The size of the experiment depends on the degrees of freedom (*d.f.*) of the nominator f_1 and the denominator f_2 of the F-statistic. f_2 depends not always on the sub-class number n. If we sample factor levels of random factors, then the size of those samples also determines the size of the experiment.

The minimal size is determined in dependence on a lower bound δ of the difference between the maximum and the minimum of the effects to be tested for equality by an F-test, further on, the risks α and β of the test and on a presumable value of the common residual variance.

The problem of the determination of the size of an experiment for the analysis of variance has been investigated by, among others, Tang [15], Lehmer [9], Fox [5], Tiku [16, 17], Gupta [4], Bratcher et al. [3], Kastenbaum et al. [7, 8], Bowman [1], Bowman and Kastenbaum [2], Rasch et al. [10], Herrendörfer et al. [6] and Rasch [11].

The solution λ of the following equation plays a crucial role:

$$F(f_1,\ f_2,\ 0,\ 1-\alpha) = F(f_1,\ f_2,\ \lambda,\ \beta), \tag{2}$$

where $F(f_1,\ f_2,\ 0,\ 1-\alpha)$ is the $(1-\alpha)$-quantile of the (central) F-distribution with degrees of freedom f_1 and f_2 and non-centrality parameter 0 and where $F(f_1,\ f_2,\ \lambda,\ \beta)$ is the β-quantile of the F-distribution with degrees of freedom f_1 and f_2 and non-centrality parameter λ.

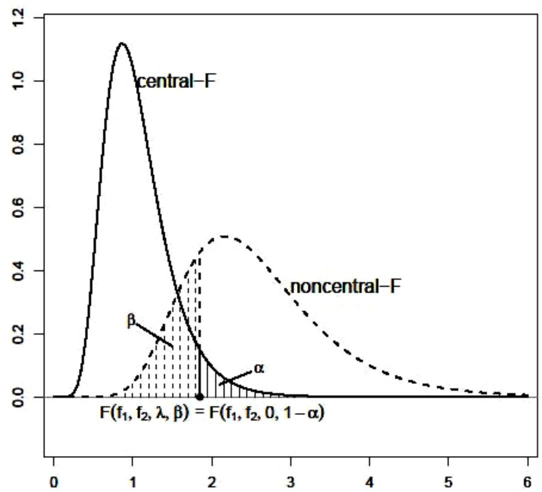

Figure 1. Demonstrating the relation of equation (2).

Figure 1 shows the relation of equation (2), it is a generalization of equation (1).

But contrary to the *t*-test, we have at least 3 expectations or effects in ANOVA models.

For the null hypothesis about a fixed factor *A* having effects a_i $(i = 1, \ldots, a)$ the sample size depends not only on the difference between the extreme effects but also on the position of the other effects.

Remark. The non-centrality parameter λ is proportional to $\sum_{i=1}^{a} a_i^2$ (when we assume that $\sum a_i = 0$) and from Figure 1 we see that as larger λ as smaller the minimal size needed.

The most favorable case leads to the maximum λ and the smallest minimum sample size $n_{\min}$; $b_{\min}, \ldots,$ the *minimin* size and the least favorable case leads to the biggest minimum sample size, the *maximin* size $n_{\max}$; $b_{\max}, \ldots$.

Lemma 1. *Without loss of generality* (*w.l.o.g.*), *we assume*: $\sum_{i=1}^{a} a_i = 0$, $a_1 \leq a_2 \leq \cdots \leq a_a$, $a_{\min} = -\frac{\delta}{2}$ *and* $a_{\max} = \frac{\delta}{2}$ *and further w.l.o.g.* $\delta = \sigma$. *We consider a Model* I *of ANOVA* (*all the factors are fixed*) *with a cross classification and equal sub-class numbers*.

(a) *Under the conditions above, the minimin size* $n_{\min}$ (*the most favorable case*) *occurs if we split the* a_i *into two groups of size* a_I *and* a_{II}, *respectively, with* $a = a_I + a_{II}$ *and* $|a_I - a_{II}| \le 1$ *and the* a_I *elements of the first group equal* $-\frac{a_{II}}{a}\delta$ *and the* a_{II} *remaining equal* $\frac{a_I}{a}\delta$. *Thus, there are two solutions for odd a and for even a half of the effects are equal to* $-\frac{\delta}{2}$ *and half of them are equal to* $\frac{\delta}{2}$.

Then $\sum_{i=1}^{a} a_i^2 = \frac{\delta^2}{a}(a_I \cdot a_{II})$ *and is a maximum.*

(b) *Under the conditions above, the maximin size* $n_{\max}$ (*the least favorable case*) *occurs if* $a_1 = -\frac{\delta}{2}$, *and* $a_a = \frac{\delta}{2}$ *and all the other effects are zero. Then* $\sum_{i=1}^{a} a_i^2 = \frac{\delta^2}{2}$ *and a minimum.*

(c) *In the singular case* (*two-sample problem*), $a = 2$ *both sizes are identical.*

Proof. It is easy to see that the condition $\sum_{i=1}^{a} a_i = 0,\ a_1 \le a_2 \le \cdots \le a_a$ is fulfilled as well in case (a) as also in case (b).

(a) In this case with even *a*, the statement is evident. In general, we know that with $a = a_I + a_{II}$ the product $a_I a_{II}$ is maximum if a_I and a_{II} are as equal as possible. That makes

$$\sum_{i=1}^{a} a_i^2 = a_I \frac{a_{II}^2}{a^2} \cdot \delta^2 + a_{II} \frac{a_I^2}{a^2} \cdot \delta^2 = \frac{a}{a^2} a_I \cdot a_{II} \delta^2 = \frac{a_I \cdot a_{II}}{a} \delta^2$$

a maximum if a_I and a_{II} differ at most by 1. In Table 1, some values of $\frac{1}{\delta^2}\sum_{i=1}^{a} a_i^2$ in the most favorable case are given.

Table 1

a	2	3	4	5	6	7	8
$\frac{1}{\delta^2}\sum_{i=1}^{a} a_i^2$	0.5	0.667	1	1.2	1.5	1.714	2

In Table 1, values of $\frac{1}{\delta^2}\sum_{i=1}^{a} a_i^2$ for the most favorable case in dependence on a.

For even a, the result of case (b) follows from the theory of D-optimal designs in regression. For odd a, we obtain the result by equating the partial derivatives with respect to the effects and κ of $w = x - \kappa z$ to zero. Hereby are

$$x = \sum_{i=1}^{a} a_i^2 = a_1^2 + \cdots + a_{a-1}^2 + (\delta + a_1)^2$$

and

$$z = \sum_{i=1}^{a} a_i = a_1 + \cdots + a_{a-1} + (\delta + a_1).$$

This completes the proof because w is a convex function.

We already called the minimin size by $n_{\min}$ and the maximin size by $n_{\max}$. The experimenter now has to choose the number of observations n per factor level (class) between the lower bound n_l and the upper bound n_u:

$$n_{\min} \leq n \leq n_{\max}.$$

All that remains to be done is to calculate the bounds $n_{\min}$ and $n_{\max}$ for different classifications and models.

3. The One-way Analysis of Variance

The model equation of the one-way analysis of variance with a fixed factor A is written in the form

$$\boldsymbol{y}_{ij} = E(\boldsymbol{y}_{ij}) + \boldsymbol{e}_{ij} = \mu + a_i + \boldsymbol{e}_{ij} \quad (i = 1, \ldots, a;\ j = 1, \ldots, n).$$

The α_i are the main effects of the *factor levels* A_i; they are real numbers, i.e., not random. The model is completed by the following constraints (sometimes called *side conditions*): the $\boldsymbol{e}_{ij}$ are mutually independent with $E(\boldsymbol{e}_{ij}) = 0$ and $\operatorname{var}(\boldsymbol{e}_{ij}) = \sigma^2$ and that the sum of the a_i is zero. The pair of hypotheses:

H_0: "All the a_i are equal to zero".

H_A: "At least two of the a_i are different".

is tested with help of the test statistics

$$\boldsymbol{F} = \frac{\boldsymbol{MS}}{\boldsymbol{MS}_{res}}$$

with the mean squares from the corresponding ANOVA-table with $a-1$ and $a(n-1)$ degrees of freedom.

$\boldsymbol{F}$ is with these *d.f.* *F* distributed with the non-centrality parameter

$$\lambda = \frac{n}{a-1}\sum_{i=1}^{a} a_i^2$$

which in the most and least favorable case increases with *n* and decreases with *a*. This means that under otherwise equal conditions, the necessary sample sizes and thus, the size of the experiment increase with increasing number of factor levels.

Examples and R-programs

To calculate the minimum sample sizes $n_{\min}$ and $n_{\max}$, we use the R-package OPDOE.

We plan to perform an experiment with four levels of a fixed factor *A* and measure the yield of a crop in *dt* per ha. The four levels are four varieties of a cereal crop.

The number *n* of plots per variety has to be determined to satisfy the following conditions: Type I error probability $\alpha = 0.05$, and Type II error probability $\beta \le 0.1$ if $a_{\max} - a_{\min} \ge 2\sigma$.

>size_n.one_way_*model*_1(0.05, 0.1, 2, 4, "maximin")

[1] 9

>size_n.one_way_model_1(0.05, 0.1, 2, 4, "minimin")

[1] 5

This means that $n_{\min} = 5$ and $n_{\max} = 9$.

4. The Two-way Analysis of Variance

The model equation of the two-way analysis of variance with the factor *A* with *a* levels A_i and the factor *B* with the *b* levels B_j and equal number of observations in the sub-classes is

$$\boldsymbol{y}_{ijk} = \mu + a_i + b_j + (ab)_{ij} + \boldsymbol{e}_{ijk} \quad (i = 1, ..., a;\ j = 1, ..., b;\ k = 1, ..., n)$$

if both factors are fixed (Model I) and

$$\boldsymbol{y}_{ijk} = \mu + a_i + \boldsymbol{b}_j + (\boldsymbol{ab})_{ij} + \boldsymbol{e}_{ijk} \quad (i = 1, ..., a;\ j = 1, ..., b;\ k = 1, ..., n)$$

if factor *A* is fixed and factor *B* is random (mixed model).

In model I, we assume in addition to the assumptions for the one-way ANOVA that the sums of the $(ab)_{ij}$ (separately over each index) all equal zero.

As in the one-way classification, we like to test the pair of hypotheses:

H_0: “All the a_i are equal to zero”.

H_A: “At least two of the a_i are different”.

The test statistics is for Model I

$$F_A = \frac{\boldsymbol{MS}_A}{\boldsymbol{MS}_R}$$

which is *F*-distributed with $a - 1$ and $a(n-1)$ d.f. and non-centrality parameter

$$\lambda = \frac{bn}{a-1} \sum_{i=1}^{a} a_i^2.$$

For the mixed model, the test statistics is

$$\boldsymbol{F}_A = \frac{\boldsymbol{MS}_A}{\boldsymbol{MS}_{A \times B}}$$

which is *F*-distributed with $a - 1$ and $(a-1)(b-1)$ d.f. and non-centrality parameter $\lambda = \frac{bn}{a-1} \sum_{i=1}^{a} a_i^2$.

This means that under otherwise equal conditions, the necessary sample sizes and thus, the size of the experiment increases with an increasing number of levels of the factor *A* but decreases with the number of levels of the factor *B*.

For determining the minimin and the maximin size, we can use Lemma 1 in both the cases.

In model I, we also can test a hypothesis about the interaction effects.

H_0: "All $(ab)_{ij}$ are zero".

The alternative hypothesis is:

H_A: "At least two $(ab)_{ij}$ differs from zero, respectively".

The test statistic is

$$\boldsymbol{F}_{AB} = \boldsymbol{MS}_{A\times B} / \boldsymbol{MS}_R$$

and is under H_0 $F((a-1)(b-1);\ ab(n-1))$ distributed. The non-centrality parameter is

$$\lambda = \frac{n}{(a-1)(b-1)} \cdot \sum_{i,\,j} (ab)_{ij}^2.$$

Before we give some examples, we have to show the least and the most favorable situation for the interaction effects.

Lemma 2. *We consider a model* I *of the balanced two-way ANOVA or an analogue balanced multi-way ANOVA with two fixed factors A and B under the condition that the sums of the interaction effects* $(ab)_{ij}$ *of the two factors A and B* (*separately over each index*) *equal zero*.

Further, let $\max\lfloor (ab)_{ij} \rfloor - \min\lfloor (ab)_{ij} \rfloor = \delta = \sigma$ with σ^2 as the error variance of the model.

(a) Then the minimum of $\sum_{i=1}^{a} \sum_{j=1}^{b} (ab)_{ij}^2$ is w.l.o.g. obtained for

$$(ab)_{11} = \frac{(a-1)(b-1)}{a(b-1)} \delta = \frac{a-1}{a} \delta;$$

$$(ab)_{i1} = -\frac{(b-1)}{a(b-1)}\delta = -\frac{1}{a}\delta; \quad i = 2, \ldots, a;$$

$$(ab)_{1j} = -\frac{(a-1)}{a(b-1)}\delta; \quad j = 2, \ldots, b;$$

$$(ab)_{ij} = \frac{1}{a(b-1)}\delta; \quad i = 2, \ldots, a;\; j = 2, \ldots, b.$$

(b) If as well a as also b are even the maximum of $\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^2$ is given if half of the interaction effects equal $-\frac{1}{2}\delta$ and the remaining equal $\frac{1}{2}\delta$. Then

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^2 = \frac{ab}{4}\delta^2.$$

Proof. (a) We assume w.l.o.g. that $a \le b$. It is easy to see that $\sum_{i=1}^{a}(ab)_{ij} = \sum_{j=1}^{b}(ab)_{ij} = 0$ and $\max\lfloor(ab)_{ij}\rfloor - \min\lfloor(ab)_{ij}\rfloor = (ab)_{11} - (ab)_{i1} = \delta$ and all side conditions are fulfilled. We now consider

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^2 = \delta^2\left\{\left[\frac{(a-1)(b-1)}{a(b-1)}\right]^2 + (a-1)\left[\frac{(b-1)}{a(b-1)}\right]^2\right.$$

$$\left. + (b-1)\left[\frac{(a-1)}{a(b-1)}\right]^2 + (a-1)(b-1)\left[\frac{1}{a(b-1)}\right]^2\right\}$$

$$= \frac{ab(a-1)(b-1)}{a^2(b-1)^2}\delta^2 = \frac{b(a-1)}{a(b-1)}\delta^2 \le \delta^2.$$

The equality sign occurs if and only if $a = b$. This expression depends besides δ only on a and b and is invariant against permutations of rows and/or columns which all are also solutions. The solution $(ab)_{11} = (ab)_{ab} = \frac{\delta}{2}$; $(ab)_{a1} = (ab)_{1b} = -\frac{\delta}{2}$ and all other effects equal to zero fulfill the side conditions as well but lead to a larger value of $\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^{2}$ if $a < b$. This completes the proof. Thus, the least favorable case is known. We give in Table 2, the value of $\frac{1}{\delta^2}\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^{2}$ for some values of *a* and *b*.

Table 2. Values of $\frac{1}{\delta^2}\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^{2}$ for the least favorable case and some values of *a* and *b*

	b			
a	2	3	4	5
2		0.75	0.667	0.625
3		1	0.888	0.833
4			1	0.938

(b) Under the side condition above, no larger value of $\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^{2}$ is possible.

For the most favorable case with at least one of the values *a* and *b* odd, we only have a conjecture.

Conjecture

We consider a model I of the balanced two-way ANOVA or an analogue balanced multi-way ANOVA with two fixed factors *A* and *B* under the condition that the sums of the interaction effects $(ab)_{ij}$ of the two factors *A* and *B* (separately over each index) equal zero.

Further, let $\max\lfloor(ab)_{ij}\rfloor - \min\lfloor(ab)_{ij}\rfloor = \delta = \sigma$ with σ^2 as the error variance of the model.

Under the conditions above, the maximum of $\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^{2}$ is

(i) for *a* even, *b* odd,

(ii) for *a* odd, *b* even,

(iii) *a* and *b* both odd

obtained as the value occurring if the odd number is reduced to the next smaller even number.

The maximum of $\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^{2}$ is for

(i) by $\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^{2} = \frac{a(b-1)}{4}$,

(ii) by $\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^{2} = \frac{b(a-1)}{4}$,

(iii) by $\sum_{i=1}^{a}\sum_{j=1}^{b}(ab)_{ij}^{2} = \frac{(a-1)(b-1)}{4}$.

Some arguments supporting this conjecture are shown in Rasch et al. [12].

Examples and R-programs

To calculate the minimum sample sizes $n_{\min}$ and $n_{\max}$, we use the R-package OPDOE.

Assume that six wheat varieties should be compared concerning their yield. For that the varieties will be cultivated at several farms. Four farms have been selected, for model I, the farms are randomly sampled from all farms in a country. The number n of plots per variety in model I and the number of farms sampled in model II have to be determined to satisfy the following conditions: Type I error probability $\alpha = 0.05$, and Type II error probability $\beta \leq 0.1$ if $a_{\max} - a_{\min} \geq 2\sigma$.

The OPDOE programs have the following structure:

>size_x.two_way_cross.model_r_E(α,β,δ,a,b,q) with x = n or x = b (in this case, the b in the bracket is replaced by n); r = 1 or mixed, E = a or axb (when testing interaction effects) and q = "maximin" or "minimin".

Model I: Testing the main effects:

>size_n.two_way_cross.model_1_a(0.05,0.1,2,6,4,"maximin")

[1] 9

>size_n.two_way_cross.model_1_a(0.05,0.1,2,6,4,"minimin")

[1] 4

Model I: Testing the interaction effects:

>size_n.two_way_cross.model_1_axb(0.05,0.1,2,6,4,“maximin”)

[1] 48

>size_n.two_way_cross.model_1_axb(0.05,0.1,2,6,4,“minimin”)

[1] 5

Mixed model: Testing the main effects:

If we put $n = 1$, then we obtain

>size_b.two_way_cross.mixed_model_a(0.05,0.1,2,6,1,“maximin”)

[1] 35

>size_n.two_way_cross.model_1_a(0.05,0.1,2,6,4,1,“minimin”)

[1] 13

If we put $n = 2$, then we obtain

>size_b.two_way_cross.mixed_model_a(0.05,0.1,2,6,2,“maximin”)

[1] 18

>size_n.two_way_cross.model_1_a(0.05,0.1,2,6,4,2,“minimin”)

[1] 7

It is the product *bn* what is (up to integer rounding) constant.

For the nested classification, an analogue procedure is applied. This and the different models and classification together with the R-programs can be found in [12, Chapter 3 of Rasch et al.].

5. The Three-way Analysis of Variance

In the three-way analysis of variance, we have four classifications (cross, nested and two mixed) and several models with at least one fixed factor. For most of these cases, sample size formula using the two lemmas above and the conjecture could be derived. In two cases a special approach was needed because no exact *F*-test exists and an approximate *F*-test using the Satterthwaite approximation (Satterthwaite [14]) leads to problems with determining sample sizes, see Rasch et al. [13].

Writing $F \succ G$ or $G \prec F$ if factor G is nested within factor F and GxF if both factors are cross classified and printing factor symbols in bold if a factor is random, then we have the following cases:

Classification	Model equation
$A \times B \times C$	$\boldsymbol{y}_{ijkv} = \mu + a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk} + (abc)_{ijk} + \boldsymbol{e}_{ijkv}$
$A \times B \times \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + a_i + b_j + \boldsymbol{c}_k + (ab)_{ij} + (\boldsymbol{ac})_{ik} + (\boldsymbol{bc})_{jk} + (\boldsymbol{abc})_{ijk} + \boldsymbol{e}_{ijkv}$
$A \times \boldsymbol{B} \times \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + a_i + \boldsymbol{b}_j + \boldsymbol{c}_k + (\boldsymbol{ab})_{ij} + (\boldsymbol{ac})_{ik} + (\boldsymbol{bc})_{jk} + (\boldsymbol{abc})_{ijk} + \boldsymbol{e}_{ijkv}$
$\boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + \boldsymbol{b}_j + \boldsymbol{c}_k + (\boldsymbol{ab})_{ij} + (\boldsymbol{ac})_{ik} + (\boldsymbol{bc})_{jk} + (\boldsymbol{abc})_{ijk} + \boldsymbol{e}_{ijkv}$
$A \succ B \succ C$	$\boldsymbol{y}_{ijkv} = \mu + a_i + b_{j(i)} + c_{k(ij)} + \boldsymbol{e}_{ijkv}$
$\boldsymbol{A} \succ B \succ C$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + b_{j(i)} + c_{k(ij)} + \boldsymbol{e}_{ijkv}$
$A \succ \boldsymbol{B} \succ C$	$\boldsymbol{y}_{ijkv} = \mu + a_i + \boldsymbol{b}_{j(i)} + c_{k(ij)} + \boldsymbol{e}_{ijkv}$
$A \succ B \succ \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + a_i + b_{j(i)} + \boldsymbol{c}_{k(ij)} + \boldsymbol{e}_{ijkv}$
$\boldsymbol{A} \succ \boldsymbol{B} \succ C$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + \boldsymbol{b}_{j(i)} + c_{k(ij)} + \boldsymbol{e}_{ijkv}$
$\boldsymbol{A} \succ B \succ \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + b_{j(i)} + \boldsymbol{c}_{k(ij)} + \boldsymbol{e}_{ijkv}$
$A \succ \boldsymbol{B} \succ \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + a_i + \boldsymbol{b}_{j(i)} + \boldsymbol{c}_{k(ij)} + \boldsymbol{e}_{ijkv}$
$\boldsymbol{A} \succ \boldsymbol{B} \succ \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + \boldsymbol{b}_{j(i)} + \boldsymbol{c}_{k(ij)} + \boldsymbol{e}_{ijkv}$
$(A \times B) \succ C$	$\boldsymbol{y}_{ijkv} = \mu + a_i + b_j + (ab)_{ij} + c_{k(ij)} + \boldsymbol{e}_{ijkv}$
$(A \times \boldsymbol{B}) \succ C$	$\boldsymbol{y}_{ijkv} = \mu + a_i + \boldsymbol{b}_j + (\boldsymbol{ab})_{ij} + c_{k(ij)} + \boldsymbol{e}_{ijkv}$
$(\boldsymbol{A} \times \boldsymbol{B}) \succ C$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + \boldsymbol{b}_j + (\boldsymbol{ab})_{ij} + c_{k(ij)} + \boldsymbol{e}_{ijkv}$
$(A \times B) \succ \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + a_i + b_j + (ab)_{ij} + \boldsymbol{c}_{k(ij)} + \boldsymbol{e}_{ijkv}$
$(A \times \boldsymbol{B}) \succ \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + a_i + \boldsymbol{b}_j + (\boldsymbol{ab})_{ij} + \boldsymbol{c}_{k(ij)} + \boldsymbol{e}_{ijkv}$
$(A \times \boldsymbol{B}) \succ \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + \boldsymbol{b}_j + (\boldsymbol{ab})_{ij} + \boldsymbol{c}_{k(ij)} + \boldsymbol{e}_{ijkv}$
$(A \succ B) \times C$	$\boldsymbol{y}_{ijkv} = \mu + a_i + b_{j(i)} + c_k + (ac)_{ik} + (bc)_{j(i)k} + \boldsymbol{e}_{ijkv}$
$(\boldsymbol{A} \succ B) \times C$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + b_{j(i)} + c_k + (\boldsymbol{ac})_{ik} + (bc)_{j(i)k} + \boldsymbol{e}_{ijkv}$

$(A \succ \boldsymbol{B}) \times C$	$\boldsymbol{y}_{ijkv} = \mu + a_i + \boldsymbol{b}_{j(i)} + c_k + (ac)_{ik} + (\boldsymbol{bc})_{j(i)k} + \boldsymbol{e}_{ijkv}$
$(\boldsymbol{A} \succ \boldsymbol{B}) \times C$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + \boldsymbol{b}_{j(i)} + c_k + (\boldsymbol{ac})_{ik} + (\boldsymbol{bc})_{j(i)k} + \boldsymbol{e}_{ijkv}$
$(A \succ B) \times \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + a_i + b_{j(i)} + \boldsymbol{c}_k + (\boldsymbol{ac})_{ik} + (\boldsymbol{bc})_{j(i)k} + \boldsymbol{e}_{ijkv}$
$(\boldsymbol{A} \succ B) \times \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + b_{j(i)} + \boldsymbol{c}_k + (\boldsymbol{ac})_{ik} + (\boldsymbol{bc})_{j(i)k} + \boldsymbol{e}_{ijkv}$
$(A \succ \boldsymbol{B}) \times \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + a_i + \boldsymbol{b}_{j(i)} + \boldsymbol{c}_k + (\boldsymbol{ac})_{ik} + (\boldsymbol{bc})_{j(i)k} + \boldsymbol{e}_{ijkv}$
$(\boldsymbol{A} \succ \boldsymbol{B}) \times \boldsymbol{C}$	$\boldsymbol{y}_{ijkv} = \mu + \boldsymbol{a}_i + \boldsymbol{b}_{j(i)} + \boldsymbol{c}_k + (\boldsymbol{ac})_{ik} + (\boldsymbol{bc})_{j(i)k} + \boldsymbol{e}_{ijkv}$

We only show how to determine the size of the experiment for the case $(A \succ B) \times \boldsymbol{C}$.

We test the null hypothesis that the factor A has no effect on the observed random variable.

Let us consider the case of $a = 6$ levels of A and $b = 5$ levels of B with $\alpha = 0.05$ and $\beta = 0.1$ and $\delta = 0.5\sigma$. In OPDOE, we use the R-program

```
> size_c.three_way_mixed_cxbina.model_5_a(0.05, 0.1, 0.5, 6, 5,
+ 2, “maximin”)
[1]   15
> size_c.three_way_mixed_cxbina.model_5_a(0.05, 0.1, 0.5, 6, 5,
+ 2, “minimin”)
[1]   6.
```

As a result, we found the minimin size 6 and the maximin size 15 for the number of levels of the random factor C.

We also can derive the test statistic for testing

$$H_0 : b_{j(i)} = 0,\ \forall j,\ i;\ H_A \text{ at least one } b_{j(i)} \neq 0.$$

$\boldsymbol{F} = \dfrac{\boldsymbol{MS}_{BinA}}{\boldsymbol{MS}_{BxCinA}}$ is under H_0 $F[a(b-1);\ a(b-1)(c-1)]$-distributed with $a(b-1)$ and $a(b-1)(c-1)$ degrees of freedom.

Let us again consider the case of $a = 6$ levels of A and $b = 5$ levels of B with $\alpha = 0.05$ and $\beta = 0.1$ and $\delta = 0.5\sigma$. In OPDOE, we use the R-program

```
> size_c.three_way_mixed_cxbina.model_5_b(0.05, 0.1, 0.5, 6, 5,
+ 2, “maximin”)
[1]  113
> size_c.three_way_mixed_cxbina.model_5_b(0.05, 0.1, 0.5, 6, 5,
+ 2, “minimin”)
[1]  9.
```

As a result, we found the minimin size 9 and the maximin size 113 for the number of levels of the random factor C.

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