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ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF $x'' = -t^{\alpha\lambda-2}x^{1+\alpha}$ WHERE $\alpha > 0$, $\lambda = 0$ OR $\lambda \leq -1$

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Abstract

We consider an initial value problem of the differential equation denoted in the title. We show the domain of the solution and obtain analytical expressions in the neighborhoods of both ends of the domain. For all possible initial values, we do these.

1. Introduction

The second order nonlinear differential equation

$$x'' = -t^{\alpha\lambda-2}x^{1+\alpha} \quad (' = d/dt) \tag{E}$$

has much applicability, because this relates to astrophysics, dynamics, partial differential equations, etc. (cf. [1, 12]). Moreover many authors have treated this in more general form in the papers [2, 4, 5, 6, 7, 8, 9, etc.]. However in these papers they did not obtain all solutions. On the other hand, in the papers [12, 16, 17, 18, 19] we considered (E) under an initial condition and examined asymptotic behavior of all positive solutions. In the case $\alpha < 0$, we completed this research in the papers

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[16, 17, 18]. However, in the case $\alpha > 0$ the case $\lambda < 0$ remains unsettled except the case $0 < \alpha \le -(2\lambda + 1)^2/4\lambda(\lambda + 1)$, $-1 < \lambda < 0$ in [19, 20], while the case $\lambda > 0$ was treated in [12]. So, in this paper we consider the remaining cases:

$$\alpha > 0, \quad \lambda = 0 \quad \text{or} \quad \lambda \le -1.$$

Given an initial condition

$$x(t_0) = A, \qquad x'(t_0) = B \tag{I}$$

$$(0 < t_0 < \infty,\ 0 < A < \infty,\ -\infty < B < \infty),$$

we show the asymptotic behavior of the positive solutions of the initial value problem (E), (I). Since t_0, A, B in (I) are given arbitrarily, we shall discuss all positive solutions.

The case $\lambda = 0$ will be considered in Section 2. In Section 3, we shall get conclusions of the case $\lambda = -1$ and the case $\lambda < -1$ directly from those of Section 2 and [12] respectively.

2. On the Case $\lambda = 0$

Suppose $\lambda = 0$ in (E). Then we get

$$x'' = -t^{-2}x^{1+\alpha} \tag{E_0}$$

where $\alpha > 0$. We consider this in a region $0 < t < \infty$, $0 < x < \infty$. For this, we use a transformation

$$y = x^{\alpha} \ (\text{namely } x = y^{1/\alpha}), \qquad z = ty' \tag{T_0}$$

and have

$$\frac{dz}{dy} = \frac{(\alpha - 1)z^2 + \alpha yz - \alpha^2 y^3}{\alpha yz} \tag{R_0}$$

from (E_0). Using a parameter s, we rewrite this as

$$\frac{dy}{ds} = \alpha yz, \qquad \frac{dz}{ds} = (\alpha - 1)z^2 + \alpha yz - \alpha^2 y^3. \tag{S_0}$$

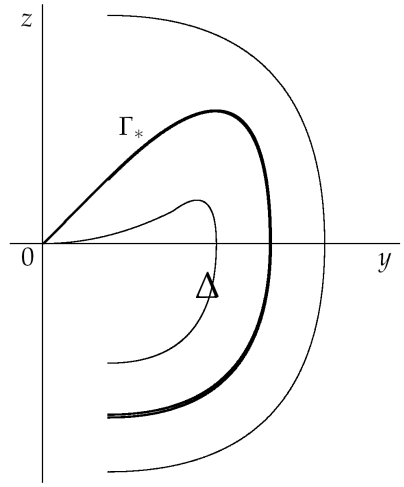

Figure 1. The phase portrait of (S_0).

Notice that the critical point of this is $(0, 0)$ if $\alpha \neq 1$, and every point of the z axis if $\alpha = 1$.

As will be shown, the phase portrait of (S_0) is as in Figure 1. In this figure, the bold curve denotes a unique orbit such that $\lim_{y \to +0} z/y = \alpha$. Let us call this orbit as Γ_*. There are parts of orbits not drawn in the neighborhood of $y = 0$. These are continuable to $y = 0$ and satisfy

$$\lim_{y \to +0} z = \pm\infty,\ c,\ 0$$

if $0 < \alpha < 1$, $\alpha = 1$, $\alpha > 1$, respectively. Here c is a finite constant. Let Δ be the (open) region which Γ_* surrounds.

Now, let $x(t)$ denote a solution of the initial value problem (E_0), (I), and define (y, z) from applying the transformation (T_0) to $x(t)$. Then we get $z = \alpha y t x'/x$. Hence if we suppose $(y, z) = (y_0, z_0)$ as $t = t_0$, then we have

$$y_0 = A^{\alpha}, \qquad z_0 = \alpha t_0 A^{\alpha-1} B.$$

Conversely if (y_0, z_0) is determined, then (t_0, A, B) of (I), and the solution $x(t)$ of (E_0), (I) is determined.

Depending on the place of (y_0, z_0), we state the asymptotic behavior of $x(t)$ as follows:

Theorem 1. *If* $(y_0, z_0) \in \Gamma_*$, *then* $x(t)$ *is defined for* $0 < t < \omega_+$ *where* ω_+ *is a positive constant. In the neighborhood of* $t = 0$, *we get*

$$x(t) = Kt\left(1 + \sum_{n=1}^{\infty} x_n t^{\alpha n}\right), \tag{2.1}$$

and in the neighborhood of $t = \omega_+$,

$$x(t) = L(\omega_+ - t)\left\{1 + \sum_{m+n>0} x_{mn}(\omega_+ - t)^m (\omega_+ - t)^{\alpha n}\right\} \tag{2.2}$$

where $K(> 0)$, x_n, $L(> 0)$, x_{mn} *are constants.*

Of course, ω_+, K, x_n, L, x_{mn} depend on (y_0, z_0), and on (t_0, A, B).

Theorem 2. *Suppose* $(y_0, z_0) \in \Delta$. *Then* $x(t)$ *is defined for* $0 < t < \omega_+$ (ω_+: *a positive constant*). *Moreover we have*

$$x(t) \sim \left(\frac{-1}{\alpha \log t}\right)^{1/\alpha} \quad as \;\; t \to +0, \tag{2.3}$$

and (2.2) *in the neighborhood of* $t = \omega_+$.

Here $f(t) \sim g(t)$ means that $f(t)/g(t)$ tends to 1.

Theorem 3. *If* $(y_0, z_0) \notin \Gamma_* \cup \Delta$, *then* $x(t)$ *is defined for* $\omega_- < t < \omega_+$ *where* $\omega_\pm$ *are positive constants. In the neighborhood of* $t = \omega_-$, *we obtain*

$$x(t) = K(t - \omega_-)\left\{1 + \sum_{m+n>0} x_{mn}(t - \omega_-)^m (t - \omega_-)^{\alpha n}\right\} \tag{2.4}$$

$$(x_{mn} : constants),$$

and in the neighborhood of $t = \omega_+$, (2.2).

Constants appearing in Theorems 2, 3 also depend on (t_0, A, B).

For proving these theorems, we follow the discussion of Section 3 of [15]. First, let us consider (R_0) in the neighborhood of $y = 0$. For this, put $w = y^{-2}z$. Then we get

$$\frac{dw}{dy} = \frac{-(\alpha+1)yw^2 + \alpha w - \alpha^2}{\alpha y^2 w}. \tag{2.5}$$

Now, let γ be a point of accumulation of a solution w of this as $y \to +0$. Then as in Section 3 of [15] we get $\gamma = \alpha, \pm\infty$.

If $\gamma = \alpha$, then for an arbitrary positive number ε there exists a positive number δ such that $0 < y < \delta$ implies $\alpha(1-\varepsilon) < w < \alpha(1+\varepsilon)$. Since $w = y^{-2}z$, $z = ty'$, we get

$$\frac{\alpha(1-\varepsilon)}{t} < y^{-2}y' < \frac{\alpha(1+\varepsilon)}{t}.$$

Hence if $y = y(t)$, $y_0 = y(t_0)$, and $0 < y < y_0 < \delta$, then from the integration of every side from t_0 to t we have

$$\alpha(1-\varepsilon)(\log t - \log t_0) > -\frac{1}{y} + \frac{1}{y_0} > \alpha(1+\varepsilon)(\log t - \log t_0).$$

In fact, from $\gamma = \alpha > 0$ we obtain $w = y^{-2}ty' > 0$ in the neighborhood of $y = 0$, which implies that y is an increasing function of t. Therefore as $y \to +0$, we get

$$\alpha(1+\varepsilon)(\log t - \log t_0) \to -\infty$$

and $t \to +0$. Moreover we have

$$(1-\varepsilon)\left(1 - \frac{\log t_0}{\log t}\right) < -\frac{1}{\alpha y \log t}\left(1 - \frac{y}{y_0}\right) < (1+\varepsilon)\left(1 - \frac{\log t_0}{\log t}\right)$$

and as $y \to +0$ namely $t \to +0$,

$$1 - \varepsilon \leq -\lim_{t \to +0} \frac{1}{\alpha y \log t} \leq 1 + \varepsilon.$$

Since ε is arbitrary, we obtain

$$y \sim -\frac{1}{\alpha \log t} \quad \text{as } t \to +0,$$

and (2.3), for $x = y^{1/\alpha}$ from (T_0).

If $\gamma = \pm\infty$, then we put $w = 1/\theta$, $u = y^{-1}\theta$, and obtain

$$y\frac{du}{dy} = \frac{1}{\alpha}u - u^2 + \alpha y u^3. \tag{2.6}$$

Let δ be a point of accumulation of the solution u of (2.6) as $y \to +0$. Then we have $\delta = 0, 1/\alpha, \pm\infty$ as above.

In the case $\delta = 0$, from following the discussion of Section 5 of [11] we obtain

$$z = C^{-1}y^{-1/\alpha+1}\left\{1 + \sum_{m+n>0} w_{mn}y^m (Cy^{1/\alpha})^n\right\} \tag{2.7}$$

$$(C,\ w_{mn} \text{ : constants})$$

and (2.2), (2.4). Here, as $z > 0$ we have (2.4) and as $z < 0$, (2.2), for $z = ty'$.

In the cases $\delta = 1/\alpha, \pm\infty$, it suffices to use the similar discussion to that of Section 3 of [15]. If $\delta = 1/\alpha$, then we get

$$z = \alpha y\left(1 + \sum_{n=1}^{\infty} w_n y^n\right) \quad (w_n \text{ : constants}) \tag{2.8}$$

uniquely and (2.1). Moreover we conclude that the case $\delta = \pm\infty$ does not occur.

Thus all cases were examined in the case $y \to +0$. Summarizing these, we get the following:

Lemma 1. *There occur only three cases*:

(i) $\gamma = \alpha$, (ii) $\gamma = \pm\infty$, $\delta = 0$, (iii) $\gamma = \pm\infty$, $\delta = 1/\alpha$.

From the case (i), *we have*

$$z \sim \alpha y^2 \quad as \quad y \to +0. \tag{2.9}$$

Moreover from the case (ii), *we obtain* (2.7), *and from the case* (iii), (2.8) *uniquely*.

If we define (y, z) from applying (T_0) to a solution $x(t)$ of (E_0), then z, (y, z) are solutions of (R_0), (S_0) respectively. Let $\omega_- < t < \omega_+$ be the domain of $x(t)$. Then we conclude the following:

Lemma 2. *As* $t \to \omega_{\pm}$, (y, z) *does not tend to a regular point of the phase plane of* (S_0).

This is Lemma 4.1 of [15] whose proof is the same as that of Lemma 2 of [14]. Moreover we state the following:

Lemma 3. *If* (y, z) *denotes an orbit of* (S), *then the following three cases do not occur.*

(i) *z is unbounded as* $y \to c$ $(0 < c < \infty)$

(ii) *z is bounded as* $y \to \infty$

(iii) *z is unbounded as* $y \to \infty$.

Proof. As in the proof of Lemma 4.2 of [15], we conclude that (i), (ii) do not occur. If (iii) takes place, then we put

$$y = 1/\eta, \qquad z = 1/\zeta, \qquad \zeta = \eta^{3/2}w, \qquad \xi = \eta^{1/2}$$

in (R_0) and get

$$\xi \frac{dw}{d\xi} = -\frac{\alpha + 2}{\alpha} w + 2w^2 - 2\alpha w^3.$$

Since $-(\alpha + 2)/\alpha < 0$, from Lemma 2.5 of [13] we have a contradiction $w \equiv 0$. This completes the proof.

This lemma shows that y tends to 0 as s tends to an end of the domain of y. Now, recall that if $\gamma = \pm\infty$, $\delta = 1/\alpha$ we obtain the unique orbit (2.8) of (S_0). We have already named this orbit Γ_*. Hence Γ_* tends to the origin with $\lim_{y\to 0} z/y = \alpha$. Here, notice that the direction of the orbit of (S_0) is judged from $dy/ds = \alpha yz$ of (S_0). If (y, z) denotes Γ_*, then as s increases, y does and Γ_* gets into the region $z < 0$ from Lemma 2. As s increases still more, y decreases and tends to 0 from Lemma 2, again. In this case, we get $\gamma = \pm\infty$, $\delta = 0$, since in the cases $\gamma = \alpha$ and $\gamma = \pm\infty$, $\delta = 1/\alpha$ we have (2.9), (2.8) which implies $z > 0$. In the case $\gamma = \pm\infty$, $\delta = 0$, we obtain (2.7). From this we know the asymptotic behavior of the undrawn parts of the orbits of (S_0).

Recall that Δ denotes the region which Γ_* surrounds. From Lemma 1 orbits in Δ satisfy $\gamma = \alpha$ as s decreases, for the orbits are in $z > 0$ from $dy/ds = \alpha yz$, (2.7) cannot enter Δ as $z > 0$, and Γ_* exists uniquely. On the other hand, as s increases these orbits get into $z < 0$. Hence these satisfy $\gamma = \pm\infty$, $\delta = 0$, and (2.7).

Finally, if the orbits lie outside the closure of Δ, then as s decreases and as s increases, these orbits satisfy $\gamma = \pm\infty$, $\delta = 0$, and (2.7), again. As s decreases, recall that Γ_* exists uniquely.

From the above discussion, we draw the phase portrait of (S_0) as in Figure 1.

Proof of Theorems. If we define (y, z) from a solution $x(t)$ of $(E_0), (I)$ as above, then from Lemma 2, (y, z) draws an orbit of (S_0) passing (y_0, z_0) as t moves all over the domain of $x(t)$. Hence from Lemma 1 it suffices to recall the following: From (2.9) we get (2.3), from (2.7) we have (2.2), (2.4), and from (2.8) we obtain (2.1).

3. On the Case $\lambda \le -1$

First, let us consider

$$x'' = -t^{-\alpha-2}x^{1+\alpha}, \tag{E_-}$$

where $\alpha > 0$. This is obtained from putting $\lambda = -1$ in (E). As in [9], change (x, t) for $(x/t, 1/t)$. Then we get (E_0) from (E_-). Therefore we can apply the discussion of Section 2. Using (T_0), we have (S_0) from (E_0). In the same way, we obtain (S_0) from applying

$$y = \left(\frac{x}{t}\right)^{\alpha} \text{ (namely } x = ty^{1/\alpha}), \quad z = -ty' \tag{T_-}$$

to (E_-). Recall that the phase portrait is drawn in Figure 1.

Now, given an initial condition (I) again, the solution $x(t)$ of $(E_-), (I)$ is transformed into an orbit of (S_0) passing (y_0, z_0) where

$$y_0 = \left(\frac{A}{t_0}\right)^{\alpha}, \quad z_0 = \alpha y_0\left(1 - \frac{t_0 B}{A}\right)$$

from (T_-). In fact, from (T_-) we get $z = \alpha y(1 - tx'/x)$. Owing to Theorems 1, 2, 3, we respectively conclude the following Theorems 4, 5, 6:

Theorem 4. *If* $(y_0, z_0) \in \Gamma_*$, *then* $x(t)$ *is defined for* $\omega_- < t < \infty$ $(0 < \omega_- < \infty)$. *In the neighborhood of* $t = \omega_-$, *we have* (2.4) *and in the neighborhood of* $t = \infty$,

$$x(t) = L\left(1 + \sum_{n=1}^{\infty} x_n t^{-\alpha n}\right) \qquad (L(> 0),\ x_n : \textit{constants}).$$

Theorem 5. *Suppose* $(y_0, z_0) \in \Delta$. *Then* $x(t)$ *is defined for* $\omega_- < t < \infty$ $(0 < \omega_- < \infty)$. *Moreover we have* (2.4) *in the neighborhood of* $t = \omega_-$, *and*

$$x(t) \sim \frac{t}{(\alpha \log t)^{1/\alpha}} \quad \textit{as} \quad t \to \infty.$$

Theorem 6. *If* $(y_0, z_0) \notin \Gamma_* \cup \Delta$, *then* $x(t)$ *is defined for* $\omega_- < t < \omega_+$ $(0 < \omega_- < \omega_+ < \infty)$. *In the neighborhood of* $t = \omega_-$, *we obtain* (2.4), *and in the neighborhood of* $t = \omega_+$, (2.2).

Next, let us consider (E) where $\alpha > 0, \lambda < -1$. Then if we change (x, t) for $(x/t, 1/t)$ as above, we get (E) where $\alpha > 0, \lambda > 0$. For this we can apply discussion of [12].

In both cases, (E) is transformed into

$$\frac{dy}{ds} = \alpha yz, \qquad \frac{dz}{ds} = (\alpha - 1)z^2 + \alpha(2\lambda + 1)yz + \alpha^2\lambda(\lambda + 1)y^2(y - 1) \tag{S}$$

if we put

$$y = -\frac{t^{\alpha\lambda} x^{\alpha}}{\lambda(\lambda + 1)}, \qquad z = ty'. \tag{T}$$

Moreover from the change of variables $(x, t) \to (x/t, 1/t)$, the phase portraits of (S) of the cases $\lambda < -1$ and $\lambda > 0$ are symmetric with respect to the y axis. Therefore from the discussion of [12] we obtain the phase portrait of (S) of the case $\lambda < -1$ as in Figure 2. The bold curve denotes a unique orbit (y, z) such that

$\lim_{y \to -0} z/y = \alpha\lambda$. Furthermore this is represented as

$$z = \alpha\lambda y\left(1 + \frac{\lambda + 1}{\alpha\lambda - 1}y + \cdots\right)$$

in the neighborhood of $(y, z) = (0, 0)$. Here, let us denote this unique orbit as Γ_* and the region which Γ_* surrounds as Δ. Orbits contained in Δ are represented as

$$z = \alpha(\lambda + 1)y\left[1 + \sum_{m+n>0} v_{mn}\eta^m \{\eta^{1/\alpha\lambda}(h \log \eta + C)\}^n\right]$$

in the neighborhood of $(y, z) = (0, 0)$. Here v_{mn}, h, C are constants, $\eta = -y$, and $h = 0$ if $-1/\alpha(\lambda + 1) \notin N$. The other parts of the orbits are represented as

$$z^{-1} = -Cy^{1/\alpha - 1}\left\{1 + \sum_{m+n>0} w_{mn}y^m (Cy^{1/\alpha})^n\right\}$$

$$(C, w_{mn} : \text{constants})$$

in the neighborhood of $y = 0$. Hence on these parts of the orbits, z tends to $\pm\infty, c, 0$ as $y \to -0$, if $0 < \alpha < 1, \alpha = 1, \alpha > 1$, respectively. Here c is a nonzero constant. These parts are not drawn in Figure 2.

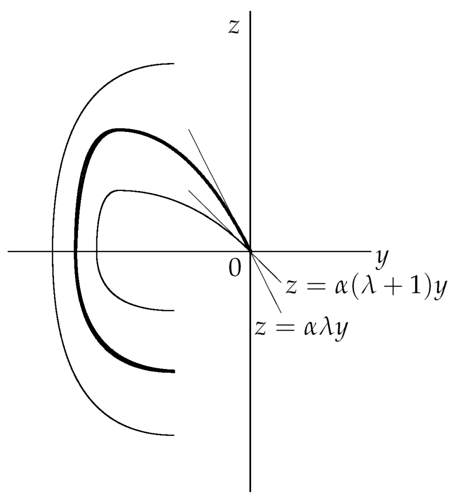

Figure 2. The phase portrait of (S).

Now, recall the initial condition (I) of (E). Since $z = \alpha y(\lambda + tx'/x)$ from (T), if we suppose $(y, z) = (y_0, z_0)$ as $t = t_0$, then we get

$$y_0 = -\frac{t_0^{\alpha\lambda} A^\alpha}{\lambda(\lambda + 1)}, \qquad z_0 = \alpha y_0\left(\lambda + \frac{t_0 B}{A}\right).$$

Therefore if (y_0, z_0) is determined, then (t_0, A, B) of (I), and the solution $x(t)$ of (E), (I) is determined. Restating theorems of [12], we show the asymptotic behavior of $x(t)$ as follows:

Theorem 7. *If* $(y_0, z_0) \in \Delta$, *then* $x(t)$ *is defined for* $\omega_- < t < \infty$ $(0 < \omega_- < \infty)$. *Moreover in the neighborhood of* $t = \infty$, $x(t)$ *is represented as*

$$x(t) = Lt\left\{1 + \sum_{m+n>0} x_{mn} t^{\alpha(\lambda+1)m-n}\right\}$$

if $-1/\alpha(\lambda + 1) \notin \boldsymbol{N}$, *and as*

$$x(t) = Lt\left\{1 + \sum_{k=1}^{\infty} t^{\alpha(\lambda+1)k} p_k(\log t)\right\}$$

if $-1/\alpha(\lambda + 1) \in \boldsymbol{N}$. *Here* $L(> 0)$, x_{mn} *are constants and* p_k *are polynomials with* $\deg p_k \leq [-\alpha(\lambda + 1)k]$.

If $(y_0, z_0) \in \Gamma_*$, *then* $x(t)$ *is defined for* $\omega_- < t < \infty$ $(0 < \omega_- < \infty)$ *and represented as*

$$x(t) = L\left(1 + \sum_{n=1}^{\infty} x_n t^{\alpha\lambda n}\right) \qquad (L(> 0),\ x_n : \textit{constants})$$

in the neighborhood of $t = \infty$. *If* $(y_0, z_0) \notin \Delta \cup \Gamma_*$, *then* $x(t)$ *is defined for* $\omega_- < t < \omega_+$ $(0 < \omega_- < \omega_+ < \infty)$ *and represented as* (2.2) *in the neighborhood of* $t = \omega_+$.

For all (y_0, z_0), $x(t)$ *is represented as* (2.4) *in the neighborhood of* $t = \omega_-$.

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