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CONSTRUCTION OF BAYER'S THEOREM IN THE SAMPLE SPACE $S_{N,r}^{p}$ $(S_{N,r}^{c})$ OF *r*-PERMUTATIONS/*r*-COMBINATIONS

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Abstract

Let $X = \{x_1, x_2, ..., x_N\}$ be a collection and $S_{N,r}^{*}$ be the set of all *r*-combinations of distinct elements of *X*. Let *Y* be any nonempty subset of *X* with cardinality *k* and let $S_{N,r,k}^{*}$ be the set of all $S_{N,r}^{*}$ that contains *Y*. We extend the Bayer's Theorem to the class of sets

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$S^* = \{S^*_{N,r};\ r = 1, 2, ..., N\}$ which is more general and give an explicit formula for practical purposes.

1. Introduction and Preliminaries

Let $X \neq \emptyset;\ n(X) = N$, i.e., $X = \{x_1, x_2, ..., x_N\}$ and let $S_{N,r}$ and $S^*_{N,r}$ be the sets of all r-permutations and r-combinations of distinct elements of X. Let $Y \subset X;\ n(Y) = k$, i.e., $Y = \{y_1, y_2, ..., y_k\}$ and let $S^*_{N,r,k}$ and $S_{N,r,k}$ be the sets of all elements of $S^*_{N,r}$ and $S_{N,r}$ that contain Y (completely if $r \geq k$ and partly if $r < k$). In contrast, let $S^*_{N,r,k-1}$ and $S_{N,r,k-1}$ be the sets of all elements of $S^*_{N,r}$ and $S_{N,r}$ that do not contain Y. Then we define $S^* = \{S^*_{N,r};\ r = 1, 2, ..., N\}$ and $S = \{S_{N,r};\ r = 1, 2, ..., N\}$.

Observe that

$$S_{N,0} = \{\emptyset\} = S^*_{N,0},$$

$$S_{N,1} = X^* = S^*_{N,1}; \quad X^* = \{\{x\};\ x \in X\},$$

$$S_{N,1,k} = Y^* = S^*_{N,1,k}; \quad Y^* = \{\{y\};\ y \in Y\}.$$

Let U be the random variable corresponding to an event of $S^*_{N,r,k}\ (S_{N,r,k})$ chosen randomly. Then the probability of picking $u \in S^*_{N,r,k}$ is given by

$$p_u = \frac{n(S_{N,r,k})}{n(S_{N,r})} = \frac{n(S^*_{N,r,k})}{n(S^*_{N,r})}.$$

Similarly, let V be the random variable corresponding to an event of $S^*_{N,r,k-1}\ (S_{N,r,k-1})$ chosen randomly. Then the probability of picking $v \in S^*_{N,r,k-1}$ is given by

$$q_u = \frac{n(S_{N,r,k-1})}{n(S_{N,r})} = \frac{n(S^*_{N,r,k})}{n(S^*_{N,r})}.$$

Observe that

$$n(S^*_{N,r,k}) + n(S^*_{N,r,k-1}) = n(S^*_{N,r}).$$

Hence, it is easy to see that $p_u + q_v = 1$.

Theorem 1.1. *Let* $A_1, A_2, \ldots, A_N$ *be a finite set of events of a sample S. Then*

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{1\le i\le N} P(A_i) - \sum_{1\le i\le j\le N} P(A_i \cap A_j) + \cdots + (-1)^{N+1} P(A_1 \cap A_2 \cap \cdots \cap A_N). \quad (1)$$

Proof. It suffices to prove that the following equation holds (where $n(A)$ denotes the cardinality of A):

$$n\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{1\le i\le N} n(A_i) - \sum_{1\le i\le j\le N} n(A_i \cap A_j) + \cdots + (-1)^{N+1} n(A_1 \cap A_2 \cap \cdots \cap A_N). \quad (2)$$

So from (2) above, we observe that $a \in A_i$ $(i = 1, 2, \ldots, N)$ will be counted

$\binom{N}{1}$ times in $\sum_{1\le i\le N} n(A_i)$,

$\binom{N}{2}$ times in $\sum_{1\le i\le j\le N} n(A_i \cap A_j)$, ..., $\binom{N}{N}$ times in $n(A_1 \cap A_2 \cap \cdots \cap A_N)$.

Thus, the number of times '*a*' is counted on the right hand side $n_a(\text{RHS})$ is given by

$$n_a(\text{RHS}) = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} = 1$$

which is the desired result. Now, the result follows by the definition of relative frequency probability.

Definition 1.2. Let $A_1, A_2, \ldots, A_N$ be a finite set of events of a sample *S* such that $P(A_i) \ge 0$, $\forall i = 1, 2, \ldots, N$. Then the events are said to be

(i) *dependent* if

$$P\left(\bigcap_{i=1}^{N} A_i\right) = \prod_{k=1}^{N} P\left(A_k / \bigcap_{i=1}^{k-1} A_i\right),$$

(ii) *independent* if

$$P\left(\bigcap_{i=1}^{N} A_i\right) = \prod_{i=1}^{N} p(A_i),$$

(iii) *exhaustive* if

$$P\left(\bigcup_{i=1}^{N} A_i\right) = 1,$$

(iv) *mutually exclusive* if

$$P(A_i \cap A_j) = 0, \quad \forall i \neq j, \quad i, \ j = 1, 2, \dots, N.$$

Lemma 1.3. *Let* $A_1, A_2, \dots, A_N$ *be a finite collection of mutually exclusive and exhaustive events with* $P(A_i) > 0, \quad \forall i = 1, 2, \dots, N.$ *Then for any other event B, such that* $P(A_i) > 0,$

$$P(A_k / B) = \frac{P(A_k) P(B/A_k)}{\sum_{i=1}^{N} P(A_i) P(B/A_i)}, \quad k = 1, 2, \dots, N.$$

Lemma 1.4. *Let* $X = \{x_1, x_2, \dots, x_N\}$. *Then the number of r-permutations and r-combinations of N distinct elements of X with the inclusion of a fixed k-number of elements of* $Y \subset X$ *is*

$$n(S^*_{N,r,k}) = \begin{cases} \binom{N-k}{r-k} & \text{if } r \geq k, \\ \binom{k}{r} & \text{if } r < k, \end{cases}$$

where

$$n(S_{N,r,k}) = r!\, n(S^*_{N,r,k}).$$

Lemma 1.5. *Let* $X = \{x_1, x_2, \dots, x_N\}$. *Then the number of r-permutations and r-combinations of N distinct elements of X with the non-inclusion of a fixed k-number of elements of* $Y \subset X$ *is*

$$n(S^*_{N,r,k-1}) = \begin{cases} \sum_{j=0}^{k-1} \binom{N-k}{r-j}\binom{k}{j} & if\ r \geq k\ and\ r+k \leq N, \\ \binom{N}{r} & if\ r < k, \\ \sum_{j=r+k-N}^{k-1} \binom{N-k}{r-j}\binom{k}{j} & if\ r \geq k\ and\ r+k > N, \end{cases}$$

where

$$n(S_{N,r,k-1}) = r!\,n(S^*_{N,r,k-1}).$$

2. Main Results

At this juncture, we are ready to give an explicit formula on the extension of Bayer's Theorem on the set (sample) S^* containing all r-combinations of distinct elements of X.

Theorem 2.1. *Let* $X = \{x_1, x_2, ..., x_N\}$ *and* Y *be a subset of* X *such that* $n(Y) = k$. *Then for any* $B \subset S^*$ *partitioned by* $S^*_{N,r}$ $(r = 1, 2, ..., N)$,

$$P(S^*_{N,r}/B) = \frac{P(S^*_{N,r})P(B/S^*_{N,r})}{\sum_{i=1}^{N} P(S^*_{N,t})P(B/S^*_{N,t})}, \quad r = 1, 2, ..., N. \tag{3}$$

Proof. By the definition, $S^* = \{S^*_{N,r};\ r = 1, 2, ..., N\}$.

It is easy to see that $S^*_{N,r_i} \cap S^*_{N,r_j} = \varnothing$, $\forall i \neq j$, hence, S^* is a collection of pair-wise disjoint sets so that we can write

$$S^* = \bigcup_{r=i}^{N} S^*_{N,r}.$$

Hence, the collections $S^*_{N,r}$ $(r = 1, 2, ..., N)$ are of mutually exclusive and exhaustive events. Elements of the set $B \subset S^*$ are generated by $Y \subset X$ in $S^*_{N,r}$, that is, B contains element(s) of $S^*_{N,r}$ that has the k-inclusion property, completely or partly depending if $r \geq k$ or $r < k$.

Observe that

$$B = (S^*_{N,1} \cap B) \overset{\circ}{\bigcup} (S^*_{N,2} \cap B) \overset{\circ}{\bigcup} \cdots \overset{\circ}{\bigcup} (S^*_{N,N} \cap B)$$

$$\Rightarrow n(B) = n(S^*_{N,1} \cap B) + n(S^*_{N,2} \cap B) + \cdots + n(S^*_{N,N} \cap B)$$

$$\Rightarrow P(B) = P(S^*_{N,1} \cap B) + P(S^*_{N,2} \cap B) + \cdots + P(S^*_{N,N} \cap B).$$

But

$$P(S^*_{N,r}/B) = \frac{P(S^*_{N,r})P(B/S^*_{N,r})}{P(B)} = \frac{P(S^*_{N,r})P(B/S^*_{N,r})}{\sum_{t=1}^{N} P(S^*_{N,t})P(B/S^*_{N,t})}, \quad r = 1, 2, \ldots, N.$$

An immediate consequence of Theorem 2.1 is the following corollary:

Corollary 2.2. *Let B be a subset of S^* that has been partitioned by $S^*_{N,r}$ $(r = 1, 2, \ldots, N)$ and whose elements satisfy the inclusion condition for any subset Y of X with cardinality k. Then*

$$P(B) = \sum_{r=1}^{N} P(S^*_{N,r})P(B/S^*_{N,r}).$$

In the next two theorems, we shall give an explicit formula for computing the results obtained in Theorem 2.1 and Corollary 2.2 for practical purposes.

Theorem 2.3. *Let B be any subset of S^* and $P : S^* \to [0, 1]$ be such that equation* (3) *holds in S^*. Then*

$$P(S^*_{N,r}/B) = \begin{cases} \dfrac{\dbinom{N-k}{r-k}}{\displaystyle\sum_{t=1}^{k-1}\binom{k}{t} + \sum_{t=k}^{N}\binom{N-k}{t-k}} & \text{if } r \geq k,\ r = k, k+1, \ldots, N, \\[4ex] \dfrac{\dbinom{k}{r}}{\displaystyle\sum_{t=1}^{k-1}\binom{k}{t} + \sum_{t=k}^{N}\binom{N-k}{t-k}} & \text{if } r < k,\ r = 1, 2, \ldots, k-1. \end{cases}$$

Proof. For $r \geq k$, observe that

$$P(S^*_{N,r}) = \frac{n(S^*_{N,r})}{n(S^*)} = \frac{\binom{N}{r}}{2^N - 1}$$

and

$$P(B/S^*_{N,r}) = \frac{n(B)}{n(S^*_{N,r})} = \frac{\binom{N-k}{r-k}}{\binom{N}{r}}.$$

Hence, it follows that

$$P(S^*_{N,r}/B) = \frac{\dfrac{\binom{N}{r}}{2^N - 1} \times \dfrac{\binom{N-k}{r-k}}{\binom{N}{r}}}{\sum_{t=1}^{k-1} P(S^*_{N,t})P(B/S^*_{N,t}) + \sum_{t=k}^{N} P(S^*_{N,t})P(B/S^*_{N,t})}$$

$$= \frac{\dfrac{\binom{N}{r}}{2^N - 1} \times \dfrac{\binom{N-k}{r-k}}{\binom{N}{r}}}{\sum_{t=1}^{k-1} \dfrac{\binom{N}{t}}{2^N - 1} \times \dfrac{\binom{k}{t}}{\binom{N}{t}} + \sum_{t=k}^{N} \dfrac{\binom{N}{t}}{2^N - 1} \times \dfrac{\binom{N-k}{t-k}}{\binom{N}{t}}}$$

$$= \frac{\binom{N-k}{r-k}}{\sum_{t=1}^{k-1} \binom{k}{t} + \sum_{t=k}^{N} \binom{N-k}{t-k}} \quad \text{if} \quad r \geq k, \quad r = k,\ k+1,\ \ldots,\ N.$$

If we assume that $r < k$, then by a similar argument the result follows.

Corollary 2.4. *Let* $B \subset S^*$, $Y \subset X$ *and* $P : S^* \to [0, 1]$ *be such that* $n(Y) = k$. *Then*

(i) $P(B) = \dfrac{1}{2^N - 1}\left\{\sum_{t=1}^{k-1}\binom{k}{r} + \sum_{t=k}^{N}\binom{N-k}{r-k}\right\}, \quad r = 1, 2, \ldots, N,$

(ii) $n(B) = \sum_{t=1}^{k-1}\binom{k}{r} + \sum_{t=k}^{N}\binom{N-k}{r-k}; \quad r = 1, 2, \ldots, N.$

Proof. By Corollary 2.2,

$$P(B) = \sum_{r=1}^{N} P(S^*_{N,r})P(B/S^*_{N,r}) = \sum_{r=1}^{k-1} P(S)P(B) + \sum_{r=k}^{N} P(S)P(B)$$

$$= \sum_{r=1}^{k-1}\binom{k}{r} + \sum_{r=k}^{N}\binom{N-k}{r-k}$$

$$= \sum_{r=1}^{k-1}\frac{\binom{N}{r}}{2^N - 1} \times \frac{\binom{k}{r}}{\binom{N}{r}} + \sum_{r=k}^{N}\frac{\binom{N}{r}}{2^N - 1} \times \frac{\binom{N-k}{r-k}}{\binom{N}{r}}.$$

Thus, we have (i).

Further, since $P(B) = \dfrac{n(B)}{n(S^*)}$, it follows (ii).

Remark 2.5. It is important to note the following, which is a consequence of Corollary 2.4:

$$P(B) = \begin{cases} \dfrac{1}{2^N - 1}\displaystyle\sum_{r=k}^{N}\binom{N-k}{r-k} & \text{if } r \ge k, \quad r = k, k+1, \ldots, N, \\ \dfrac{1}{2^N - 1}\displaystyle\sum_{r=1}^{k-1}\binom{k}{r} & \text{if } r < k, \quad r = 1, 2, \ldots, k-1, \end{cases}$$

and consequently, we have

$$n(B) = \begin{cases} \displaystyle\sum_{r=k}^{N}\binom{N-k}{r-k} & \text{if } r \ge k, \quad r = k, k+1, \ldots, N, \\ \displaystyle\sum_{r=1}^{k-1}\binom{k}{r} & \text{if } r < k, \quad r = 1, 2, \ldots, k-1. \end{cases}$$

Corollary 2.6. *Let* $B \subset S^*$, $Y \subset X$ *and* $P : S^* \to [0, 1]$ *be such that* $n(Y) = k$ *and* $n(X) = N$. *Then*

$$\sum_{r=1}^{N} P(S^*_{N,r}/B) = 1.$$

For a practical purpose, we provide a simple illustration of how some of the results obtained in this paper can be applied to real life problems.

Example 2.7. Four different boxes contain element of the set $\{a, b, c, d\}$, a man was asked to choose an element from the 1st box one at time, from the 2nd box two at a time, from the 3rd box three at a time and from the 4th box four at a time. What is the probability of (a) choosing an element of the set $\{a, b\}$ from (i) the 1st box (ii) the 2nd box, (b) Picking the 4th box given that the elements of $\{a, b\}$ has been chosen.

Consider the diagram in appendix *i*.

Observe that the elements selected from 1st, 2nd, 3rd and 4th boxes are contained in the sets of the form $S^*_{4,1}$, $S^*_{4,2}$, $S^*_{4,3}$ and $S^*_{4,4}$, respectively. Hence from the diagram, we have

(a) (i) $P(B/S_{4,1}) = 1/2$, (ii) $P(B/S_{4,2}) = 1/6$,

(b) $P(S_{4,4}/B) = 1/6$.

Now, applying the result obtained

(a) (i) $P(B/S_{4,1}) = \dfrac{\binom{k}{r}}{\binom{N}{r}} = \dfrac{\binom{2}{1}}{\binom{4}{1}} = 1/2,$

(ii) $P(B/S_{4,2}) = \dfrac{\binom{N-k}{r-k}}{\binom{N}{r}} = \dfrac{\binom{4-2}{2-2}}{\binom{4}{2}} = 1/6.$

$$\text{(b)}\quad P(S_{4,4}/B) = \frac{\binom{N-k}{r-k}}{\sum_{t=1}^{k-1}\binom{k}{t} + \sum_{t=k}^{N}\binom{N-k}{t-k}} \quad \text{since} \quad r \geq k$$

$$= \frac{\binom{4-2}{4-2}}{\sum_{t=1}^{1}\binom{2}{t} + \sum_{t=2}^{4}\binom{4-2}{t-2}} \quad \text{since} \quad r \geq k$$

$$= 1/6.$$

Conclusion

We observed that the result obtained by Thomas Bayer's has been given a suitable explicit formula in S^* which is a larger class of sets. The concepts have been used to modify most of the well-known discrete probability functions of which this work is a follow up.

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