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A MONTAGE FOR THE BINOMIAL DISTRIBUTION STIELTJES TRANSFORM

K. O. BOWMAN and L. R. SHENTON

Computational Sciences and Engineering Division
Oak Ridge National Laboratory
P. O. Box 2008, 4500N, MS-6191
Oak Ridge, TN 37831-6191
U. S. A.
e-mail: bowmanko@ornl.gov

Department of Statistics
University of Georgia
Athens, Georgia 30602
U. S. A.

Abstract

We provide Stieltjes transform for the binomial function, the negative binomial and a recent model for the binomial. Stieltjes transform given by $\int_0^\infty \frac{f(x)}{z+x} dx$, continued fractions are S-form and under certain circumstances, when $z > 0$ upper and lower bounds exist.

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1. Introduction

Anscombe [2] introduce the probability function

$$b(x;\ \alpha,\ \theta) = \left(\frac{1}{1+\frac{\theta}{\alpha}}\right)^{x}\left(\frac{1}{1+\frac{\alpha}{\theta}}\right)^{-\theta}\frac{\Gamma(\theta+x)}{\Gamma(\theta)x!} \quad (x = 0, 1, \ldots, \alpha > 0, \theta > 0) \qquad (1)$$

with p.g.f.

$$\left(1+\frac{\alpha}{\theta}-\frac{\alpha}{\theta}t\right)^{-\theta}.$$

There are two other binomial models:

(i) $(pt+q)^{n} \quad (0 < p < 1,\ p+q = 1,\ n = 1, 2, \ldots)$

(ii) $(1+p-pt)^{-k} \quad (p > 0,\ k > 0).$

Here we study the Stieltjes transformations and some symbolic form due to Aitken and Gonin [1].

2. Stieltjes Transform $\int_0^{\infty}\frac{f(x)}{x+z}dx$

The simple case relates to the p.g.f. $(pt+q)^{n}$. Defining

$$b(x,\ n,\ p) = \binom{n}{x}p^{x}(1-p)^{n-x},$$

there is the transform

$$\sum_{x=0}^{n}\frac{b(x,\ n,\ p)}{z+t}$$

with Stieltjes fraction development

$$F(x) = \frac{1}{1+}\frac{np}{1+}\frac{q}{z+}\frac{(n-1)p}{1+}\frac{2q}{z+}\frac{(n-2)p}{1+}\frac{3q}{z+}\cdots\frac{nq}{z+}.$$

Similarly, for a negative binomial model with

$$b(k, t;\ p) = \frac{\Gamma(k+t)}{t!\Gamma(k)} p^t (1+p)^{-k-t}, \quad (k > 0,\ p > 0;\ t = 0, 1, \ldots)$$

and Stieltjes integral

$$\sum_{t=0} \frac{B(k, t;\ p)}{z+t} = \frac{1}{z+}\,\frac{kp}{1+}\,\frac{q}{z+}\,\frac{(k+1)\,p}{1+}\,\frac{2q}{z+}\,\frac{(k+2)\,p}{1+}\cdots.$$

Lastly for the Anscombe [2] model,

$$b(\theta,\ t;\ \alpha) = \left(\frac{1}{1+\dfrac{\theta}{\alpha}}\right)^{\alpha}\left(\frac{1}{1+\dfrac{\alpha}{\theta}}\right)^{-\theta}\frac{\Gamma(\theta+x)}{x!\,\Gamma(\theta)}$$

$$\sum_{x=0}^{\infty} \frac{b(\theta,\ t;\ \alpha)}{z+x} = \frac{1}{z+}\,\frac{p_1}{1+}\,\frac{q_1}{z+}\,\frac{p_2}{1+}\,\frac{q_2}{z+}\cdots,$$

where

$$q_s = s\left(1+\frac{\alpha}{\theta}\right),$$

$$p_s = (\theta + s - 1)\frac{\alpha}{\theta},\ s = 1,\ 2,\ \ldots.$$

The continued fractions so far mentioned are in Stieltjes form and referred to as S-fractions: they have the property that z alternates in the denominators, and for $z > 0$ the odd convergent are decreasing upper bound, the even convergent increasing lower bound. For example, for the basic binomial model (p.g.f $(pt+q)^n$) the continued fraction is, less than $1/z$ but greater than $1/(z+np)$, z is real and positive.

We have not mentioned convergent questions - for that questions see Wall [6, p. 120]. In the meantime for $z > 0$ bounds are $\frac{1}{z}$ and $\frac{1}{z+\alpha}$.

3. Symbolic Binomials

We refer to Aitken and Gonin [1]. They show a formula for the orthogonal set related to the function $\binom{n}{x} p^x q^{n-x}$. In fact,

$$G(x) = (1 + p\Delta_x)^{-(n-r+1)} x^{(r)}, \quad (x = 1, 2, \ldots)$$

where $x^{(r)} = r(r-1)\cdots(x-r+1)$, and Δ is the forward difference operator Δ with

$$\Delta_x f(x) = f(x+1) - f(x).$$

Actually Δ was mentioned by Euler. Newton introduced the main concept of the binomial around 1665, about 345 years ago (Boyer [4]).

Aitken and Gonin show that

$$\Delta G_r(x) = rG_{r-1}(x;\ p,\ n-1),$$

and

$$x^{(r)} = G_r(x) + rp(n-r+1)G_{r-1}(x;\ p,\ n-1)/1!$$

$$+ r_{(2)} p^2 (n-r+1)^{(2)} G_{r-2}(x;\ p,\ n_2)/2! + \cdots.$$

There is an alternative form due to Shenton [5] namely

$$x^{(r)} = G_r(x) + rp(n-r+1)G_{r-1}(x)$$

$$+ r_{(2)} p^2 (n-r+2)^{(2)} G_{r-2}(x) + r_{(3)} p^3 (n-r+3)^{(3)} G_{r-3}(x) \cdots.$$

There is the orthogonality statement

$$\sum_{x=0}^{n} G_r(x) G_s(x) \binom{n}{x} p^x q^{n-x} = \delta_{r,s} n^{(r)} p^r q^r r!, \quad (r, s = 0, 1, \ldots)$$

where δ is the δ operator.

For the traditional negative binomial

$$\sum_{x=0}^{\infty} G_r(x) G_s(x) (1+p)^{-k} \left(\frac{p}{p+1}\right)^x \frac{\Gamma(k+x)}{x!\Gamma(k)} = \delta_{r,s}(k+r-1)^{(r)} p^r (1+p)^r r!.$$

Lastly, for the new negative binomial

$$\sum_{x=0}^{\infty} G_r(x) G_s(x) \left(\frac{1}{1 + \frac{\theta}{\alpha}} \right)^x \left(\frac{1}{1 + \frac{\alpha}{\theta}} \right)^{-\theta} \frac{\Gamma(\theta + x)}{x! \Gamma(\theta)}$$

$$= \delta_{\alpha, s} (\theta + r - 1)^{(r)} \left(\frac{\alpha}{\theta} \right)^r \left(1 + \frac{\alpha}{\theta} \right)^r r!.$$

4. Conclusion

We have given a comprehensive account of the Stieltjes transform related to the binomial. Note that the binomial in some form or other is due to Newton, about 345 years ago.

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