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A NOTE ON COMPLETELY DISTRIBUTIVE LATTICES

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Abstract

In this short note, we prove that the following three conditions are equivalent for a complete lattice L: (1) L is a completely distributive lattice, (2) $\lhd$ is the smallest approximating auxiliary on L, and (3) L is a Heyting algebra and there is a smallest approximating auxiliary relation on L.

1. Introduction and Preliminaries

The theory of continuous lattices arose independently in a variety of mathematical contexts. Due to their strong connections to computer science, general topology and topological algebra, continuous lattices have been extensively studied by people coming from various areas [1, 2]. There are several different equivalent ways to define continuous lattices, the most straightforward one is formulated by using the way below relation. For x and y in a complete lattice L, we say that x is way below y, in symbols $x \ll y$, iff for each directed subset $D \subseteq L$, $y \leq \vee D$ implies $x \leq d$ for some $d \in D$. A complete lattice L is continuous if $x = \vee\{y \in L : y \ll x\}$ for every $x \in L$. In order to take a closer look at the way

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below relation and detect how it fits into a more general framework, Smyth [4] introduced the concept of auxiliary relation and located the way below relation within $Aux(L)$. Taking arbitrary subsets instead of directed ones leads to the strong way below relation $\lhd$, where $x \lhd y$ if for any nonempty set S, $y \leq \vee S$ implies $x \leq s$ for some $s \in S$. Raney [3] proved that a complete lattice L is a completely distributive lattice iff $x = \vee\{y \in L : y \lhd x\}$ for all $x \in L$. A natural question is how to locate the strong way below relation $\lhd$ within $Aux(L)$. In this note, we will discuss this question.

The following definitions and propositions are cited from [1, 2].

Definition 1.1. We say that a binary relation $\prec$ on a poset L is an *auxiliary relation* or an *auxiliary order*, if it satisfies the following conditions for all u, x, y, z:

(i) $x \prec y$ implies $x \leq y$;

(ii) $u \leq x \prec y \leq z$ implies $u \prec z$;

(iii) if a smallest element 0 exists, then $0 \prec x$.

The set of all auxiliary relations on L is denoted by $Aux(L)$. As $Aux(L)$ is closed under arbitrary intersections in $2^{L \times L}$, $(Aux(L), \subseteq)$ is a complete lattice.

For a poset L, let Low L denote the set of all lower sets in L.

Proposition 1.2. *Let L be a poset and let M be the set of all monotone functions* $s : L \to Low\, L$ *satisfying* $s(x) \subseteq \downarrow x$ *for all* $x \in L$ *- considered as a poset relative to the ordering* $s \leq t$ *iff* $s(x) \subseteq t(x)$ *for all* $x \in L$. *Then the mapping*

$$\phi : Aux(L) \to M,\ \phi(\prec) = s_{\prec} = (x \mapsto \{y : y \prec x\})$$

is an order isomorphism between $Aux(L)$ *and M, whose inverse associates to each function* $s \in M$ *the relation* $\prec_s$ *given by*

$$\forall s \in M,\quad x \prec_s y \Leftrightarrow x \in s(y).$$

Definition 1.3. An auxiliary relation $\prec$ on a complete lattice L is called *approximating* iff we have $x = \vee s_{\prec}(x) = \vee\{u \in L : u \prec x\}$ for all $x \in L$. The set of all such relations is denoted by $App(L)$.

Proposition 1.4. *Let L be a complete lattice. Then the following conditions are equivalent*:

(1) *L is a continuous lattice*;

(2) *$\ll$ is the smallest approximating auxiliary relation on L*;

(3) *L is meet-continuous and there is a smallest approximating auxiliary relation on L.*

2. Main Results

From the definition of strong way below relation $\lhd$,

$$s_{\lhd}(x) = \bigcap\{T \in Low\ L : x \leq \vee T\}$$

for all $x \in L$.

Lemma 2.1. *Let L be a complete lattice. For every $T \in Low\ L$, we define the function $m_T : T \to Low\, L$ by*

$$m_T(x) = \begin{cases} \downarrow x \cap T, & \text{if } x \leq \vee T; \\ \downarrow x, & \text{otherwise.} \end{cases}$$

Then $m_T \in M$ for all $T \in Low\ L$ and $\lhd = \bigcap\{\prec_{m_T} : T \in Low\ L\}$.

Proof. It is obvious that $m_T \in M$ for all $T \in Low\, L$. From Proposition 1.2, we need only to prove that $\phi(\bigcap\{\prec_{m_T} : T \in Low\ L\}) = \phi(\lhd) = s_{\lhd}$. Since ϕ is an isomorphism, $\phi(\bigcap\{\prec_{m_T} : T \in Low\ L\}) = \wedge_{T \in Low\, L}\phi(\prec_{m_T}) = \wedge_{T \in Low\, L} s_{\prec_{m_T}}$. From the definitions of $\prec_T$ and $s_{\prec}(x)$, $s_{\prec_{m_T}}(x) = m_T(x)$. Now, we prove that $(\wedge_{T \in Low\, L}\, s_{\prec_{m_T}})(x) = s_{\lhd}(x)$ for all $x \in L$. In fact,

$$\begin{aligned} (\bigwedge\nolimits_{T \in Low\, L}\, s_{\prec_{m_T}})(x) &= \bigwedge\{s_{\prec_{m_T}}(x) : T \in Low\ L\} \\ &= \bigcap\{m_T(x) : T \in Low\ L\} \\ &= (\bigcap\{\downarrow x \cap T : x \leq \bigvee T\}) \cap (\bigcap\{\downarrow x : x \nleq \bigvee T\}) \\ &= \bigcap\{T \in Low\, L : x \leq \bigvee T\} \\ &= s_{\lhd}(x). \end{aligned}$$

Lemma 2.2. *In a complete Heyting algebra L, all relations* $\prec_{m_T}$ *belong to the functions* m_T *for* $T \in Low\ L$ *are approximating*.

Proof. Note that for every $x \in L$,

$$s_{\prec_{m_T}}(x) = m_T(x) = \begin{cases} \downarrow x \cap T, & \text{if } x \leq \vee T; \\ \downarrow x, & \text{otherwise.} \end{cases}$$

If $x \leq \vee T$, then $\vee s_{\prec_{m_T}}(x) = \vee m_T(x) = \vee \downarrow x \cap T = \vee(x \wedge T) = x \wedge (\vee T) = x$ since L is Heyting algebra; if $x \not\leq \vee T$, then $\vee s_{\prec_{m_T}}(x) = \vee \downarrow x = x$. Therefore, $\prec_{m_T}$ are approximating for all $T \in Low\ L$.

Lemma 2.3. *In a complete lattice L, the strong below relation* $\lhd$ *is contained in all approximating auxiliary relations, and is equal to their intersection, if L is a Heyting algebra*.

Proof. Suppose that $y \lhd x$, and $\prec$ is an approximating relation. Then $x = \vee s_{\prec}(x) = \vee\{u \in L : u \prec x\}$. It follows from the definition of $\lhd$ that $y \leq u$ for some $u \prec x$, thus $y \prec x$ since $\prec$ is an auxiliary relation. Therefore, $\lhd \subseteq \prec$ and $\lhd \subseteq \bigcap\{\prec : \prec \in App(L)\}$. If L is a Heyting algebra, then $\prec_{m_T} \in App(L)$ by Lemma 2.2, and $\bigcap\{\prec : \prec \in App(L)\} \subseteq \bigcap\{\prec_{m_T} : T \in Low\, L\}$. By Lemma 2.1, $\lhd = \bigcap\{\prec_{m_T} : T \in Low\, L\}$.

Thus $\bigcap\{\prec : \prec \in App(L)\} \subseteq \lhd$. Therefore, $\lhd = \bigcap\{\prec : \prec \in App(L)\}$.

Now we have the main results which are analogous to Proposition 1.4.

Proposition 2.4. *Let L be a complete lattice and consider the following conditions*:

(1) *L is a completely distributive lattice*;

(2) $\lhd$ *is the smallest approximating auxiliary relation on L*;

(3) *there is a smallest approximating auxiliary relation on L*.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *Moreover, if L is a Heyting algebra, then these three conditions are equivalent*.

Proof. (1) $\Leftrightarrow$ (2) Since L is a completely distributive lattice iff $\forall x \in L$, $x = \vee s_{\lhd}(x) = \vee\{y \in L : y \lhd x\}$, $\lhd$ is an approximating. Thus the equivalence of (1) and (2) follows from the first part in Lemma 2.3.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) If L is a Heyting algebra, then $\lhd$ is the intersection of all approximating auxiliary relations by Lemma 2.3. Thus, if there is a smallest approximating auxiliary relation, then this has to be $\lhd$, and we see that (3) implies (1).

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