Far East Journal of Applied Mathematics
Volume 51, Number 2, 2011, Pages 155-163
Published Online: April 4, 2011
This paper is available online at http://pphmj.com/journals/fjam.htm

AN APPLICATION OF $\left(\frac{G'}{G}\right)$-EXPANSION METHOD TO THE $(2+1)$-DIMENSIONAL BBM EQUATION

YUNJIE YANG

Department of Mathematics
Kunming University
Kunming, Yunnan, 650031, P. R. China
e-mail: dfdyyj@163.com

Abstract

In this paper, a generalized $\left(\frac{G'}{G}\right)$-expansion method is proposed to obtain exact solutions of nonlinear evolution equations. We choose the $(2+1)$-dimensional BBM equation to illustrate the validity and advantages of the proposed method. Solutions in more general forms are obtained. It is shown that the proposed method is direct, effective and can be used for many other nonlinear evolution equations in mathematical physics.

1. Introduction

It is well known that many nonlinear partial differential equations are widely used to describe the complex phenomena. So the study of traveling wave solutions of some nonlinear evolution equations played an important role. In the past several decades, various methods have been developed, such as the inverse scattering transform [1], homotopy perturbation method [2-4], Adomian decomposition method [5], homogeneous balance method [6-8], *F*-expansion method [9, 10], and others.

2010 Mathematics Subject Classification: 35Q51, 35Q25.

Keywords and phrases: $\left(\frac{G'}{G}\right)$-expansion method, $(2+1)$-dimensional BBM equation, exact solutions, traveling wave solutions.

Received December 10, 2010; Revised December 30, 2010

In this paper, we improve the $\left(\frac{G'}{G}\right)$-expansion method [11] to find more types of non-traveling wave and coefficient function solutions. We choose the $(2+1)$-dimensional BBM equation [12] to illustrate the $\left(\frac{G'}{G}\right)$-expansion method, and successfully construct many new and more general non-traveling wave and coefficient function solutions.

2. Description of the $\left(\frac{G'}{G}\right)$-expansion Method

We suppose that the given nonlinear partial differential equation for $u(x, t)$ to be in the form

$$P(u, u_t, u_x, u_{xt}, u_{xx}, \ldots), \tag{2.1}$$

where p is a polynomial in its arguments. The essence of the $\left(\frac{G'}{G}\right)$-expansion method can be presented in the following steps:

Step 1. Seek traveling wave solutions of (2.1) by taking $u(x, t) = u(\xi)$, $\xi = x - Vt$, and transform (2.1) to the ordinary differential equation

$$Q(u, -Vu', u', u'', \ldots) = 0, \tag{2.2}$$

where prime denotes the derivative with respect to ξ.

Step 2. Introduce the solution $u(\xi)$ of (2.2) in the finite series form

$$u(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{2.3}$$

where a_i's are real constants with an $a_n \neq 0$ to be determined. The positive integer will be determined by the homogeneous balance method between the highest order derivatives and highest order nonlinear appearing in (2.2). The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.4}$$

where λ and μ are real constants to be determined.

Step 3. Substituting (2.3) along with (2.4) into (2.2) yields an algebraic equation involving powers of $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, we obtain a system of algebraic equations for a_i, V by using Maple.

Step 4. By substituting the results obtained in the above steps, we can obtain a series of fundamental solutions of (2.1).

3. An Application of the Method

We apply the method to the $(2+1)$-dimensional BBM equation

$$u_t + u_x + u_y + uu_x + uu_y - u_{xxt} - u_{yyt} = 0. \tag{3.1}$$

Let $u(x, y, t) = u(\xi)$, $\xi = \kappa x + \iota y + \omega t$, so (3.1) is carried to

$$\omega u' + \kappa u' + \iota u' + \kappa uu' + \iota uu' - \kappa^2 \omega u''' - \iota^2 \omega u''' = 0. \tag{3.2}$$

Integrating it once, it yields

$$\omega u + \kappa u + \iota u + \frac{1}{2}\kappa u^2 + \frac{1}{2}\iota u^2 - \kappa\omega^2 u'' - \iota^2 \omega u'' = 0. \tag{3.3}$$

Considering the homogeneous balance between u'' and u^2 in (3.3), we get $n = 2$. We assume that (3.1) has the following formal solution:

$$u(\xi) = a_2\left(\frac{G'}{G}\right)^2 + a_1\left(\frac{G'}{G}\right) + a_0, \quad (a_2 \neq 0), \tag{3.4}$$

where $G = G(\xi)$ satisfies (2.4).

By using (2.4), from (3.4), we have

$$u^2(\xi) = a_2^2\left(\frac{G'}{G}\right)^4 + 2a_1a_2\left(\frac{G'}{G}\right)^3 + (2a_0a_2 + a_1^2)\left(\frac{G'}{G}\right)^2 + 2a_0a_1\left(\frac{G'}{G}\right) + a_0^2, \tag{3.5}$$

$$u''(\xi) = 6a_2\left(\frac{G'}{G}\right)^4 + (2a_1 + 10a_2\lambda)\left(\frac{G'}{G}\right)^3 + (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2)\left(\frac{G'}{G}\right)^2$$

$$+ (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2)\left(\frac{G'}{G}\right) + 2a_2\mu^2 + a_1\lambda\mu. \tag{3.6}$$

Substituting (3.4)-(3.6) into (3.3), setting the coefficients of $\left(\frac{G'}{G}\right)^i (i = 0, 1, ..., 4)$ to zero, we obtain the following underdetermined system of algebraic equations for $a_0, a_1, a_2, \kappa, \iota$ and ω:

$$\left(\frac{G'}{G}\right)^4 : \frac{1}{2}\iota a_2^2 + \frac{1}{2}\kappa a_2^2 - 6\kappa^2\omega a_2 - 6\iota^2\omega a_2 = 0,$$

$$\left(\frac{G'}{G}\right)^3 : -\kappa^2\omega(2a_1 + 10a_2\lambda) - \iota^2\omega(2a_1 + 10a_2\lambda) + \iota a_1 a_2 + \kappa a_1 a_2 = 0,$$

$$\left(\frac{G'}{G}\right)^2 : \kappa a_2 - \iota^2\omega(8a_2\mu + 3a_1\lambda + 4a_2\lambda^2) - \kappa^2\omega(8a_2\mu + 3a_1\lambda + 4a_2\lambda^2)$$

$$+ \omega a_2 + \iota a_2 + \frac{1}{2}\iota(2a_0a_2 + a_1^2) + \frac{1}{2}\kappa(2a_0a_2 + a_1^2) = 0,$$

$$\left(\frac{G'}{G}\right)^1 : \iota a_0 a_1 + \kappa a_1 + \kappa a_0 a_1 - \iota^2\omega(6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2)$$

$$+ \omega a_1 + \iota a_1 - \kappa^2\omega(6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2) = 0,$$

$$\left(\frac{G'}{G}\right)^0 : \omega a_0 + \frac{1}{2}\iota a_0^2 + \iota a_0 + \kappa a_0 + \frac{1}{2}\kappa a_0^2 - \iota^2\omega(2a_2\mu^2 + a_1\lambda\mu)$$

$$- \kappa^2\omega(2a_2\mu^2 + a_1\lambda\mu) = 0.$$

Solving the above system with the aid of Maple, we have following two sets of solutions.

Example 1.

$$a_2 = -\frac{12(\iota^2 + \kappa^2)}{\eta},\ a_1 = -\frac{12\lambda(\iota^2 + \kappa^2)}{\eta},\ a_0 = -\frac{2(2\kappa^2\mu + \iota^2\lambda^2 + \kappa^2\lambda^2 + 2\iota^2\mu)}{\eta},$$

$$\omega = -\frac{\iota + \kappa}{\eta}. \tag{3.7}$$

Therefore, substituting (3.7) into (3.4), we obtain that

$$u(\xi) = -\frac{12(\iota^2 + \kappa^2)}{\eta}\left(\frac{G'}{G}\right)^2 - \frac{12\lambda(\iota^2 + \kappa^2)}{\eta} \cdot \left(\frac{G'}{G}\right)$$

$$- \frac{2(2\kappa^2\mu + \iota^2\lambda^2 + \kappa^2\lambda^2 + 2\iota^2\mu)}{\eta}, \tag{3.8}$$

where $\xi = \iota x + \kappa y - \frac{\iota + \kappa}{\eta} t$, $\eta = -4\iota^2\mu - 4\kappa^2\mu + 1 + \kappa^2\lambda^2 + \iota^2\lambda^2$, $\lambda, \kappa, \omega, \iota$ and μ are arbitrary constants.

Substituting (3.8) into (2.4), we get three types of traveling wave solutions of $(2+1)$-dimensional BBM equation.

Case 1. When $\lambda^2 - 4\lambda > 0$, $C_1, C_2 \neq 0$, then

$$u_1(x, y, t) = \frac{3(\iota^2 + k^2)(\lambda^2 - 4\mu)}{-\eta}\left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2$$
$$- \frac{4\kappa^2\mu - \kappa^2\lambda^2 - \iota^2\lambda^2 + 4\iota^2\mu}{\eta},$$

where C_1, C_2 are two arbitrary constants.

When $\lambda^2 - 4\mu > 0$, $C_1 = 0$, $C_2 \neq 0$, then

$$u_2(x, y, t) = \frac{3(\iota^2 + \kappa^2)(\lambda^2 - 4\mu)}{-\eta}\left(\coth \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right)^2$$
$$- \frac{4\kappa^2\mu - \kappa^2\lambda^2 - \iota^2\lambda^2 + 4\iota^2\mu}{\eta}.$$

When $\lambda^2 - 4\mu > 0$, $C_1 \neq 0$, $C_2 = 0$, then

$$u_3(x, y, t) = \frac{3(\iota^2 + \kappa^2)(\lambda^2 - 4\mu)}{-\eta}\left(\tanh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right)^2$$
$$- \frac{4\kappa^2\mu - \kappa^2\lambda^2 - \iota^2\lambda^2 + 4\iota^2\mu}{\eta}.$$

Case 2. When $\lambda^2 - 4\mu < 0$, $C_1, C_2 \neq 0$, then

$$u_4(x, y, t) = \frac{3(\iota^2 + k^2)(4\mu - 4\lambda^2)}{-\eta}\left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right)^2$$
$$- \frac{4\kappa^2\mu - \kappa^2\lambda^2 - \iota^2\lambda^2 + 4\iota^2\mu}{\eta}.$$

When $\lambda^2 - 4\mu < 0,\ C_1 = 0,\ C_2 \neq 0,$ then

$$u_5(x,\ y,\ t) = \frac{3(\iota^2 + k^2)(4\mu - \lambda^2)}{-\eta}\left(\cot\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right)^2$$

$$- \frac{4\kappa^2\mu - \kappa^2\lambda^2 - \iota^2\lambda^2 + 4\iota^2\mu}{\eta}.$$

When $\lambda^2 - 4\mu < 0,\ C_1 \neq 0,\ C_2 = 0,$ then

$$u_6(x,\ y,\ t) = \frac{3(\iota^2 + k^2)(4\mu - \lambda^2)}{-\eta}\left(-\tan\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right)^2$$

$$- \frac{4\kappa^2\mu - \kappa^2\lambda^2 - \iota^2\lambda^2 + 4\iota^2\mu}{\eta}.$$

Case 3. When $\lambda^2 - 4\mu = 0,$ then

$$u_7(x,\ y,\ t) = \frac{12(\iota^2 + k^2)}{-\eta}\left(\frac{C_2}{C_1 + C_2\xi}\right)^2$$

$$- \frac{4\kappa^2\mu - \kappa^2\lambda^2 - \iota^2\lambda^2 + 4\iota^2\mu}{\eta}.$$

Example 2.

$$a_2 = \frac{12(\iota^2 + \kappa^2)}{\rho},\ a_1 = \frac{12\lambda(\iota^2 + \kappa^2)}{\rho},\ a_0 = \frac{12\mu(\iota^2 + \kappa^2)}{\rho},\ \omega = \frac{\iota + \kappa}{\varrho}. \tag{3.9}$$

Therefore, substituting (3.9) to (3.4), we obtain that

$$u(\xi) = \frac{12(\iota^2 + \kappa^2)}{\rho}\left(\frac{G'}{G}\right)^2 + \frac{12\lambda(\iota^2 + \kappa^2)}{\rho}\cdot\left(\frac{G'}{G}\right) + \frac{12\mu(\iota^2 + \kappa^2)}{\rho}, \tag{3.10}$$

where $\xi = \iota x + \kappa y + \frac{\iota + \kappa}{\rho}t,\ \rho = -4\iota^2\mu - 4\kappa^2\mu - 1 + \kappa^2\lambda^2 + \iota^2\lambda^2,\ \lambda,\ \kappa,\ \omega,\ \iota$ and μ are arbitrary constants.

Substituting (3.10) into (2.4), we get three types of traveling wave solutions of $(2 + 1)$-dimensional BBM equation.

Case 1. When $\lambda^2 - 4\lambda > 0,\ C_1,\ C_2 \neq 0,$ then

$$u_1(x,\ y,\ t) = \frac{3(\iota^2 + k^2)(\lambda^2 - 4\mu)}{\rho}\left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2$$

$$+ \frac{(12\mu - 3\lambda^2)(\kappa^2 + \iota^2)}{\rho},$$

where $C_1,\ C_2$ are two arbitrary constants.

When $\lambda^2 - 4\mu > 0,\ C_1 = 0,\ C_2 \neq 0,$ then

$$u_2(x,\ y,\ t) = \frac{3(\iota^2 + \kappa^2)(\lambda^2 - 4\mu)}{\rho}\left(\coth \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right)^2$$

$$+ \frac{(12\mu - 3\lambda^2)(\kappa^2 + \iota^2)}{\rho}.$$

When $\lambda^2 - 4\mu > 0,\ C_1 \neq 0,\ C_2 = 0,$ then

$$u_3(x,\ y,\ t) = \frac{3(\iota^2 + k^2)(4\mu - \lambda^2)}{\rho}\left(\tanh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right)^2$$

$$+ \frac{(12\mu - 3\lambda^2)(\kappa^2 + \iota^2)}{\rho}.$$

Case 2. When $\lambda^2 - 4\mu < 0,\ C_1,\ C_2 \neq 0,$ then

$$u_4(x,\ y,\ t) = \frac{3(\iota^2 + k^2)(4\mu - \lambda^2)}{\rho}\left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right)^2$$

$$+ \frac{(12\mu - 3\lambda^2)(\kappa^2 + \iota^2)}{\rho}.$$

When $\lambda^2 - 4\mu < 0,\ C_1 = 0,\ C_2 \neq 0,$ then

$$u_5(x,\ y,\ t) = \frac{3(\iota^2 + k^2)(4\mu - \lambda^2)}{\rho}\left(\cot \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right)^2$$

$$+ \frac{(12\mu - 3\lambda^2)(\kappa^2 + \iota^2)}{\rho}.$$

When $\lambda^2 - 4\mu < 0,\ C_1 \neq 0,\ C_2 = 0,$ then

$$u_6(x,\ y,\ t) = \frac{3(\iota^2 + k^2)(4\mu - \lambda^2)}{\rho}\left(-\tan\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right)^2$$

$$+\frac{(12\mu - 3\lambda^2)(\kappa^2 + \iota^2)}{\rho}.$$

Case 3. When $\lambda^2 - 4\mu = 0,$ then

$$u_7(x,\ y,\ t) = \frac{12(\iota^2 + \kappa^2)}{\rho}\left(\frac{C_2}{C_1 + C_2\xi}\right)^2$$

$$+\frac{(12\mu - 3\lambda^2)(\kappa^2 + \iota^2)}{\rho}.$$

4. Conclusions

In summary, a generalized-expansion method is proposed to obtain more general exact solutions of nonlinear evolution equations. By using the proposed method, we have successfully obtained exact solutions with parameters of the $(2+1)$-dimensional BBM equation with variable coefficients. These solutions extend or improve the corresponding ones in [13]. So the-expansion method is direct, concise and effective.

References

[1] M. J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.

[2] M. El-Shahed, Int. J. Nonlinear Sci. Numer. Simul. 6 (2005), 163.

[3] J. H. He, Int. J. Nonlinear Sci. Numer. Simul. 6 (2005), 207.

[4] J. H. He, Chaos Solitons Fractals 26 (2005), 695.

[5] T. A. Abassy, M. A. El-Tawil and H. K. Saleh, Int. J. Nonlinear. Sci. Numer. Simul. 5 (2004), 327.

[6] M. L. Wang, Phys. Lett. A 213 (1996), 279.

[7] M. L. Wang, Y. B. Zhou and Z. B. Li, Phys. Lett. A 216 (1996), 67.

[8] E. G. Fan and H. Q. Zhang, Phys. Lett. A 246 (1998), 403.

[9] Y. B. Zhou, M. L. Wang and Y. M. Wang, Phys. Lett. A 308 (2003), 31.

[10] M. A. Abdou, Chaos Solitons Fractals 31 (2007), 95.

[11] M. L. Wang, X. Z. Li and J. L. Zhang, Phys. Lett. A 372 (2008), 417.

[12] B. A. Li, G. W. You and Q. Qin, Periodic wave solutions and solitary wave solutions to BBM equation, Journal of Henan University of Science and Technology (Natural Science) 25(5) (2004), 70-73.

[13] L. Xia, Exact solutions to $(2+1)$-dimensional BBM equation, Journal of Southwest China Normal University (Natural Science) 32(3) (2007), 40-42.