Far East Journal of Mathematical Sciences (FJMS)
Volume 50, Number 1, 2011, Pages 79-86
Published Online: March 28, 2011
This paper is available online at http://pphmj.com/journals/fjms.htm

SOME PROPERTIES FOR THE COMPATIBLE MAPPINGS IN INTUITIONISTIC FUZZY METRIC SPACE

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Abstract

In this paper, we investigate that the concepts of compatible mappings, compatible mappings of type(α) and weak compatible mappings of type(α) are equivalent under some conditions on intuitionistic fuzzy metric spaces.

1. Introduction and Preliminaries

Grabiec [2] obtained the Banach contraction principle in setting of fuzzy metric spaces, and Jungck et al. [3] introduced the concept of compatible maps of type(α) in metric space. Also, Mishra [4] and Cho [1] introduced the concept of compatible maps of type(α) on Menger spaces and fuzzy metric spaces. Furthermore, Park et al. [5, 6] defined the intuitionistic fuzzy metric space and introduced the some properties. Recently, Pathak et al. [7] proved properties for the compatible maps in Menger spaces.

2010 Mathematics Subject Classification: 46S40, 47H10, 54H25.

Keywords and phrases: compatible map, compatible of type (α), weak compatible of type (α).

Received October 2, 2010

In this paper, we investigate that the concepts of compatible mappings, compatible mappings of $\text{type}(\alpha)$ and weak compatible mappings of $\text{type}(\alpha)$ are equivalent under some conditions on intuitionistic fuzzy metric space.

Let us recall (see [8]) that a continuous t-norm is a operation $* : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (a) $*$ is commutative and associative, (b) $*$ is continuous, (c) $a * 1 = a$ for all $a \in [0, 1]$, (d) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ $(a, b, c, d \in [0, 1])$. Also, a continuous t-conorm is a operation $\Diamond : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (a) $\Diamond$ is commutative and associative, (b) $\Diamond$ is continuous, (c) $a \Diamond 0 = a$ for all $a \in [0, 1]$, (d) $a \Diamond b \ge c \Diamond d$ whenever $a \le c$ and $b \le d$ $(a, b, c, d \in [0, 1])$.

Definition 1.1 [5]**.** The 5-tuple $(X, M, N, *, \Diamond)$ is said to be an *intuitionistic fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm, and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions; for all $x, y, z \in X$, such that

(a) $M(x, y, t) > 0$,

(b) $M(x, y, t) = 1 \Leftrightarrow x = y$,

(c) $M(x, y, t) = M(y, x, t)$,

(d) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(e) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous,

(f) $N(x, y, t) > 0$,

(g) $N(x, y, t) = 0 \Leftrightarrow x = y$,

(h) $N(x, y, t) = N(y, x, t)$,

(i) $N(x, y, t) \Diamond N(y, z, s) \ge N(x, z, t + s)$,

(j) $N(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

Note that (M, N) is called an *intuitionistic fuzzy metric* on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

In this paper, X is considered to be the intuitionistic fuzzy metric space with the following condition:

$$\lim_{t \to \infty} M(x, y, t) = 1, \quad \lim_{t \to \infty} N(x, y, t) = 0 \tag{1.1}$$

for all $x, y \in X$ and $t > 0$.

Definition 1.2 [6]**.** Let A, B be mappings from intuitionistic fuzzy metric space X into itself. Then the mappings are said to be *compatible* if

$$\lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1, \quad \lim_{n \to \infty} N(ABx_n, BAx_n, t) = 0$$

for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x$ for some $x \in X$.

Definition 1.3 [6]**.** Let A, B be mappings from intuitionistic fuzzy metric space X into itself. Then the mappings are said to be *compatible* of $\text{type}(\alpha)$ if

$$\lim_{n \to \infty} M(ABx_n, BBx_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} M(BAx_n, AAx_n, t) = 1,$$

$$\lim_{n \to \infty} N(ABx_n, BBx_n, t) = 0 \quad \text{and} \quad \lim_{n \to \infty} N(BAx_n, AAx_n, t) = 0$$

for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x$ for some $x \in X$.

Definition 1.4. Let A, B be mappings from intuitionistic fuzzy metric space X into itself. Then the mappings are said to be *weak compatible* of $\text{type}(\alpha)$ if

$$\lim_{n \to \infty} M(ABx_n, BBx_n, t) \geq \lim_{n \to \infty} M(BAx_n, BBx_n, t),$$

$$\lim_{n \to \infty} N(ABx_n, BBx_n, t) \leq \lim_{n \to \infty} N(BAx_n, BBx_n, t)$$

and

$$\lim_{n \to \infty} M(BAx_n, AAx_n, t) \geq \lim_{n \to \infty} M(ABx_n, AAx_n, t),$$

$$\lim_{n \to \infty} N(BAx_n, AAx_n, t) \leq \lim_{n \to \infty} N(ABx_n, AAx_n, t)$$

for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x$ for some $x \in X$.

2. Some Properties of Compatible Mappings

Proposition 2.1 [6]**.** *Let X be an intuitionistic fuzzy metric space and A, B be continuous mappings from X into itself. Then A and B are compatible mappings iff they are compatible mappings of* $type(\alpha)$.

Proposition 2.2. *Let X be an intuitionistic fuzzy metric space and A, B be continuous mappings from X into itself. Then A, B are weak compatible mappings of* $type(\alpha)$ *if they are compatible mappings of* $type(\alpha)$.

Proof. Suppose that A, B are compatible mappings of $\text{type}(\alpha)$. Then, we have, for all $t > 0$,

$$\begin{aligned}
1 &= \lim_{n\to\infty} M(ABx_n, BBx_n, t) \\
&\geq \lim_{n\to\infty} M\left(ABx_n, BAx_n, \frac{t}{2}\right) * \lim_{n\to\infty} M\left(BAx_n, BBx_n, \frac{t}{2}\right) \\
&= \lim_{n\to\infty} M\left(BAx_n, BBx_n, \frac{t}{2}\right) \\
&\geq \lim_{n\to\infty} M(BAx_n, BBx_n, t),
\end{aligned}$$

$$\begin{aligned}
0 &= \lim_{n\to\infty} N(ABx_n, BBx_n, t) \\
&\leq \lim_{n\to\infty} N\left(ABx_n, BAx_n, \frac{t}{2}\right) \lozenge \lim_{n\to\infty} N\left(BAx_n, BBx_n, \frac{t}{2}\right) \\
&= \lim_{n\to\infty} N\left(BAx_n, BBx_n, \frac{t}{2}\right) \\
&\leq \lim_{n\to\infty} N(BAx_n, BBx_n, t).
\end{aligned}$$

Also, we establish with same methods for another condition of definition. Hence, A, B are weak compatible of $\text{type}(\alpha)$. □

Proposition 2.3. *Let X be an intuitionistic fuzzy metric space and A, B be continuous mappings from X into itself. If A, B are weak compatible mappings of* $type(\alpha)$, *then they are compatible mappings of* $type(\alpha)$.

Proof. Let $\{x_n\}$ be a sequence in intuitionistic fuzzy metric space such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$. Since A and B are continuous, we have, for all $t \in (0, 1)$,

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) \geq \lim_{n\to\infty} M(BAx_n, BBx_n, t)$$

$$= M(Bx, Bx, t) = 1,$$

$$\lim_{n\to\infty} N(ABx_n, BBx_n, t) \leq \lim_{n\to\infty} N(BAx_n, BBx_n, t)$$

$$= N(Bx, Bx, t) = 0.$$

Also,

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) \geq \lim_{n\to\infty} M(ABx_n, AAx_n, t)$$

$$= M(Ax, Ax, t) = 1,$$

$$\lim_{n\to\infty} N(BAx_n, AAx_n, t) \leq \lim_{n\to\infty} N(ABx_n, AAx_n, t)$$

$$= N(Ax, Ax, t) = 0.$$

Therefore, A and B are compatible mappings of type(α). □

Proposition 2.4. *Let X be an intuitionistic fuzzy metric space and A, B be weak compatible mappings of type*(α) *from X into itself. If one of A, B is continuous, then they are compatible mappings.*

Proof. Let $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$. Suppose that A and B are weak compatible mappings of type(α) and A is continuous without loss of generality. Then $\lim_{n\to\infty} ABx_n = Ax = \lim_{n\to\infty} AAx_n$ and so, for $t, \lambda > 0$, there exists an integer $U(t, \lambda)$ such that

$$M\left(ABx_n, Ax, \frac{t}{2}\right) > 1 - \lambda, \quad N\left(ABx_n, Ax, \frac{t}{2}\right) < \lambda,$$

$$M\left(AAx_n, Ax, \frac{t}{2}\right) > 1 - \lambda, \quad N\left(AAx_n, Ax, \frac{t}{2}\right) < \lambda$$

for all $n \geq U(t, \lambda)$. Further, since A, B are weak compatible mappings of type(α), we have

$$\lim_{n\to\infty} M\left(BAx_n, AAx_n, \frac{t}{2}\right) \geq \lim_{n\to\infty} M\left(ABx_n, AAx_n, \frac{t}{2}\right) = 1,$$

$$\lim_{n\to\infty} N\left(BAx_n, AAx_n, \frac{t}{2}\right) \leq \lim_{n\to\infty} N\left(ABx_n, AAx_n, \frac{t}{2}\right) = 0.$$

By (c), (d), (h) and (i) of Definition 1.1, it follows that

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1, \quad \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0.$$

This completes the proof. □

Proposition 2.5. *Let X be an intuitionistic fuzzy metric space and $A, B: X \to X$ be continuous mappings. Then A and B are compatible mappings if and only if they are weak compatible mappings of type(α).*

Proof. This proof is following from Propositions 2.1, 2.2, 2.3 and 2.4. □

Proposition 2.6. *Let X be an intuitionistic fuzzy metric space and A, B be mappings from X into itself. If A, B are weak compatible mappings of type(α) and $Ax = Bx$ for some $x \in X$, then $AAx = ABx = BAx = BBx$.*

Proof. Suppose that $\{x_n\} \subset X$ defined by $x_n = x$, $n = 1, 2, \ldots$ for some $x \in X$ and $Ax = Bx$. Then we have $\lim_{n\to\infty} Ax_n = Ax$, $\lim_{n\to\infty} Bx_n = Ax$. Since A, B are weak compatible mappings of type(α) for every $t > 0$,

$$\begin{aligned} M(ABx, BBx, t) &= \lim_{n\to\infty} M(ABx_n, BBx_n, t) \\ &\geq \lim_{n\to\infty} M(BAx_n, BBx_n, t) \\ &= M(BAx, BBx, t) = 1, \end{aligned}$$

$$\begin{aligned} N(ABx, BBx, t) &= \lim_{n\to\infty} N(ABx_n, BBx_n, t) \\ &\leq \lim_{n\to\infty} N(BAx_n, BBx_n, t) \\ &= N(BAx, BBx, t) = 0. \end{aligned}$$

Hence, we have $ABx = BBx$. Therefore, we have $ABx = AAx = BBx = BAx$ since $Ax = Bx$. This completes the proof. □

Proposition 2.7. *Let X be an intuitionistic fuzzy metric space and A, B be mappings from X into itself. Also, let A, B be weak compatible mappings of type*(α) *and* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ *for some* $x \in X$. *Then*

(a) $\lim_{n\to\infty} BAx_n = Ax$ *if A is continuous at* $x \in X$,

(b) $\lim_{n\to\infty} ABx_n = Bx$ *if B is continuous at* $x \in X$,

(c) $ABx = BAx$ *and* $Ax = Bx$ *if A and B are continuous at* $x \in X$.

Proof. (a) Suppose that A is continuous at $x \in X$. Since $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$, we have $\lim_{n\to\infty} AAx_n = Ax$, or equivalently, for any $t, \lambda > 0$, there exists an integer $U(t, \lambda)$ such that $M\left(AAx_n, Ax, \frac{t}{2}\right) > 1 - \lambda$, $N\left(AAx_n, Ax, \frac{t}{2}\right) < \lambda$ for all $n \geq U(t, \lambda)$. Since A, B are weak compatible mappings of type(α), for every $t > 0$,

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) \geq \lim_{n\to\infty} M\left(ABx_n, AAx_n, \frac{t}{2}\right),$$

$$\lim_{n\to\infty} N(BAx_n, AAx_n, t) \leq \lim_{n\to\infty} N\left(ABx_n, AAx_n, \frac{t}{2}\right)$$

and we have

$$M(BAx_n, Ax, t) \geq M\left(BAx_n, AAx_n, \frac{t}{2}\right) * M\left(AAx_n, Ax, \frac{t}{2}\right) > 1 - \lambda,$$

$$N(BAx_n, Ax, t) \leq N\left(BAx_n, AAx_n, \frac{t}{2}\right) \lozenge N\left(AAx_n, Ax, \frac{t}{2}\right) < \lambda$$

for all $n \geq U(t, \lambda)$. Now, we have

$$\begin{aligned}\lim_{n\to\infty} M(BAx_n, Ax, t) &\geq \lim_{n\to\infty} M\left(BAx_n, AAx_n, \frac{t}{2}\right) * \lim_{n\to\infty} M\left(AAx_n, Ax, \frac{t}{2}\right)\\ &\geq \lim_{n\to\infty} M\left(ABx_n, AAx_n, \frac{t}{2}\right) * \lim_{n\to\infty} M\left(AAx_n, Ax, \frac{t}{2}\right)\\ &= M\left(Ax, Ax, \frac{t}{2}\right) * M\left(Ax, Ax, \frac{t}{2}\right) = 1,\end{aligned}$$

$$\lim_{n\to\infty} N(BAx_n,\ Ax,\ t) \le \lim_{n\to\infty} N\left(BAx_n,\ AAx_n,\ \frac{t}{2}\right) \lozenge \lim_{n\to\infty} N\left(AAx_n,\ Ax,\ \frac{t}{2}\right)$$

$$\le \lim_{n\to\infty} N\left(ABx_n,\ AAx_n,\ \frac{t}{2}\right) \lozenge \lim_{n\to\infty} N\left(AAx_n,\ Ax,\ \frac{t}{2}\right)$$

$$= N\left(Ax,\ Ax,\ \frac{t}{2}\right) \lozenge N\left(Ax,\ Ax,\ \frac{t}{2}\right) = 0.$$

Therefore, $\lim_{n\to\infty} BAx_n = Ax$.

(b) This proof is following on the similar lines as argued in (a).

(c) Suppose that $A, B : X \to X$ are continuous at $x \in X$. Since $\lim_{n\to\infty} Bx_n = x$ and A is continuous at $x \in X$, $\lim_{n\to\infty} BAx_n = Ax$ from (a). On the other hand, since $\lim_{n\to\infty} Ax_n = x$ and B is continuous at $x \in X$, $\lim_{n\to\infty} BAx_n = Bx$. Thus, we have $Ax = Bx$ from the uniqueness of the limit and so, by Proposition 2.6, $ABx = BAx$. □

References

[1] Y. J. Cho, Fixed points in fuzzy metric spaces, J. Fuzzy Math. 5(4) (1997), 949-962.

[2] M. Grabiec, Fixed point in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988), 385-389.

[3] G. Jungck, P. P. Murthy and Y. J. Cho, Compatible mappings of type(A) and common fixed points, Math. Japon. 38(2) (1993), 381-390.

[4] S. N. Mishra, Common fixed points of compatible mappings in *PM*-spaces, Math. Japon. 36 (1991), 283-289.

[5] J. H. Park, J. S. Park and Y. C. Kwun, A common fixed point theorem in the intuitionistic fuzzy metric space, Advances in Natural Comput. Data Mining, Proc. 2nd ICNC and 3rd FSKD, 2006, pp. 293-300.

[6] J. S. Park, J. H. Park and Y. C. Kwun, On some results for five mappings using compatibility of type (α) in intuitionistic fuzzy metric space, Internat. J. Fuzzy Logic Intell. Syst. 8(4) (2008), 299-305.

[7] H. K. Pathak, S. M. Kang and J. H. Baek, Weak compatible mappings of type (A) and common fixed points in Menger spaces, Comm. Korean Math. Soc. 10(1) (1995), 67-83.

[8] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 314-334.