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GEOMETRICAL LOGARITHMIC AND TRIGONOMETRIC HYPERCOMPLEX FUNCTIONS OF QUATERNIONIC TYPE

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Abstract

Quaternionic theory has greatly been developed in recent years [1-12]. Thus, in our view, the study of trigonometric and logarithmic type quaternionic functions is important for the determination and realization of a hypercomplex theory. In this paper, we intend to give a geometrical foundation for both logarithmic and trigonometric hypercomplex functions based on the exponential function of quaternionic type recently introduced by Borges, Marão and Machado in their paper entitled Geometrical octonions II: Hyper regularity and hyper periodicity of the exponential function appearing in Int. J. Pure Appl. Math. 48 (2008), 495-500.

1. Introduction and Motivation: The Trigonometric Function

The trigonometric functions of quaternionic type, are determined using the

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exponential quaternionic type, analogous to the octonionic case [11]. So let q be a quaternion $q = q_1 + q_2 i + q_3 j + q_4 k,$ where it can be written as $q = q(1) + vec(q).$ The exponential function is given by

$$e^{q} = e^{q_1}\left\{\cos|\vec{q}\,| + \vec{q}\left(\frac{\sin|\vec{q}\,|}{|\vec{q}\,|}\right)\right\}. \tag{1}$$

Moreover,

$$e^{-q} = e^{q_1}\left\{\cos|\vec{q}\,| - \vec{q}\left(\frac{\sin|\vec{q}\,|}{|\vec{q}\,|}\right)\right\}. \tag{2}$$

Thus, the sine and cosine functions of quaternionic type are given by

$$\sin q = \frac{e^{q} - e^{-q}}{2\dfrac{\vec{q}}{|\vec{q}\,|}}, \tag{3}$$

$$\cos q = \frac{e^{q} + e^{-q}}{2}. \tag{4}$$

The functions outlined above suggest the determination of other trigonometric functions of quaternionic type, namely,

$$tgq = \frac{\sin q}{\cos q} = -\frac{\vec{q}}{|\vec{q}\,|}\frac{e^{q} - e^{-q}}{e^{q} + e^{-q}}, \tag{5}$$

$$\cot gq = \frac{\cos q}{\sin q} = \frac{\vec{q}}{|\vec{q}\,|}\frac{e^{q} + e^{-q}}{e^{q} - e^{-q}}, \tag{6}$$

$$\sec q = \frac{1}{\cos q} = \frac{2}{e^{q} + e^{-q}}, \tag{7}$$

$$\operatorname{cosec} q = \frac{1}{\sin q} = \frac{2}{e^{q} - e^{-q}}. \tag{8}$$

It should be emphasized that these functions have the domain as a subset of H, i.e., the domain is formed by all points of the space except the quaternion $q = (0, 0, 0, 0)$. It is important to note that

$$\left(\frac{\vec{q}}{|\vec{q}\,|}\right)^2 = -1,$$

because $(\vec{q})^2 = -q_2^2 - q_3^2 - q_4^2 = |\vec{q}|^2$. Some other facts are important in developing this work, and are provided below:

Theorem 1. *If* sin q *and* cos q *are trigonometric functions of quaternionic type, then the following equality is applied*:

$$\sin^2 q + \cos^2 q = 1.$$

Demonstration. Using the definitions of quaternionic sin and cos, it follows that

$$\sin^2 q + \cos^2 q = \left(\frac{e^q - e^{-q}}{2\frac{\vec{q}}{|\vec{q}|}}\right)^2 + \left(\frac{e^q + e^{-q}}{2}\right) = \frac{e^{2q} - 2 + e^{-2q}}{(-2)^2} + \frac{e^q + 2 + e^{-2q}}{(2)^2},$$

$$\sin^2 q + \cos^2 q = 1.$$

The above result is similar to results found for the case of a function of one complex variable [13].

Theorem 2. *If* e^q *is the exponential function of quaternionic type, then the following equality is valid*

$$|e^q| = e^{q_1},$$

where $q = q_1 + q_1 i + q_3 j + q_4 k$.

Demonstration. Using the definition of quaternionic exponential, it follows that

$$|e^q| = |e^{q_1}| \cdot \left|\cos|\vec{q}| + \vec{q}\left(\frac{\sin|\vec{q}|}{|\vec{q}|}\right)\right|,$$

$$|e^q| = |e^{q_1}| \cdot \left|\cos|\vec{q}| + \sum_{n=1}^{4} \vec{q}_n \left(\frac{\sin|\vec{q}|}{|\vec{q}|}\right)\right|,$$

$$|e^q| = |e^{q_1}| \cdot \left|\cos|\vec{q}| + (q_2^2 + q_3^2 + q_4^2)\left(\frac{\sin|\vec{q}|}{|\vec{q}|}\right)\right|,$$

$$|e^q| = |e^{q_1}| \cdot \left(\cos^2 q + |\vec{q}|^2 \cdot \frac{\sin^2 q}{|\vec{q}|^2}\right),$$

$$|e^q| = e^{q_1}.$$

Theorem 3. *If* e^q *is the exponential function of quaternionic type, then* $|e^q| = 1.$

Definition. Let p be a quaternion. Then a function of one quaternionic variable is said to be *hyperperiodic* if

$$f(q + p) = f(q), \quad \forall q \in H.$$

Definition. Let u_j be a quaternion whose coordinates are zero except the jth position, and let f be a quaternionic periodic function, i.e.,

$$f(q + u_j) = f(q), \quad \forall q \in H.$$

Then the *hyperperiod* of f is given by the value of the jth coordinate of u_j.

Theorem 4. *Let* $q = q_1 + q_2 i + q_3 j + q_4 k$ *and* u_i, $i = 1, 2, 3, 4$ *and the vectors below*:

$$\vec{u}_1 = (0, 2\pi, 0, 0),$$

$$\vec{u}_2 = (0, 0, 2\pi, 0),$$

$$\vec{u}_3 = (0, 0, 0, 2\pi).$$

If e^q *is the exponential function of quaternionic type, then this is hyperperiodic with hyperperiod of* 2π.

Demonstration. The exponential function of quaternionic type is given by

$$e^q = e^{q_1}\left\{\cos|\vec{q}| + \vec{q}\left(\frac{\sin|\vec{q}|}{|\vec{q}|}\right)\right\},$$

where $q = q_1 + \vec{q}$. So,

$$e^{u_1} = e^0\left\{\cos\sqrt{0^2 + (2\pi)^2 + 0^2 + 0^2} + (0, 2\pi, 0, 0)\left(\frac{\sin\sqrt{0^2 + (2\pi)^2 + 0^2 + 0^2}}{\sqrt{0^2 + (2\pi)^2 + 0^2 + 0^2}}\right)\right\},$$

$$e^{u_1} = 1 \cdot \left\{\cos(2\pi) + (0, 2\pi, 0, 0)\frac{\sin(2\pi)}{2\pi}\right\} = 1,$$

$$e^{u_2} = e^0\left\{\cos\sqrt{0^2+0^2(2\pi)^2+0^2}+(0,\ 0,\ 2\pi,\ 0)\left(\frac{\sin\sqrt{0^2+0^2+(2\pi)^2+0^2}}{\sqrt{0^2+0^2+(2\pi)^2+0^2}}\right)\right\},$$

$$e^{u_2} = 1\cdot\left\{\cos(2\pi)+(0,\ 0,\ 2\pi,\ 0)\frac{\sin(2\pi)}{2\pi}\right\} = 1,$$

$$e^{u_3} = e^0\left\{\cos\sqrt{0^2+0^2 0^2+(2\pi)^2}+(0,\ 0,\ 0,\ 2\pi)\left(\frac{\sin\sqrt{0^2+0^2+0^2+(2\pi)^2}}{\sqrt{0^2+0^2+0^2+(2\pi)^2}}\right)\right\},$$

$$e^{u_3} = 1\cdot\left\{\cos(2\pi)+(0,\ 0,\ 0,\ 2\pi)\frac{\sin(2\pi)}{2\pi}\right\} = 1.$$

Theorem 5. *Let both* $\sin q$ *and* $\cos q$ *be the functions sine and cosine of quaternionic type*; $q = q_1 + q_2 i + q_3 j + q_4 k$. *Then, these functions are hyperperiodic with hyperperiod of* 2π.

Demonstration. Let

$$\vec{u}_1 = (0,\ 2\pi,\ 0,\ 0),$$

$$\vec{u}_2 = (0,\ 0,\ 2\pi,\ 0),$$

$$\vec{u}_3 = (0,\ 0,\ 0,\ 2\pi).$$

Then

$$\sin(q+u_i) = \frac{e^{q+u_i}-e^{-q-u_i}}{2\frac{\vec{q}}{|\vec{q}|}} = \frac{e^q e^{u_i}-e^{-q}e^{-u_i}}{2\frac{\vec{q}}{|\vec{q}|}},$$

$$\sin(q+u_i) = \frac{e^q - e^{-q}}{2\frac{\vec{q}}{|\vec{q}|}} = \sin q,$$

where $i = 2,\ 3,\ 4.$ Similarly, it is shown that

$$\cos(q+u_i) = \cos q,$$

where $i = 2,\ 3,\ 4.$

Theorem 6. *Let q be a quaternion q and* u_i, $i = 2, 3, 4$ *be the vectors*:

$$\vec{u}_1 = (0, \pi, 0, 0),$$

$$\vec{u}_2 = (0, 0, \pi, 0),$$

$$\vec{u}_3 = (0, 0, 0, \pi).$$

If tgq and $\cot gq$ *are quaternionic functions named tangent and cotangent, then these are hyperperiodic with hyperperiod of* π.

Demonstration. In fact, let

$$tg(q + u_i) = -\frac{e^{(q+u_i)} - e^{(-q-u_i)}}{\dfrac{e^{(q+u_i)} - e^{q-u_i}}{\dfrac{\vec{u}}{|\vec{u}|}}} = tgq,$$

$$\cot g(q + u_i) = \frac{\vec{u}}{|\vec{u}|}\frac{e^{(q+u_i)} + e^{(-q-u_i)}}{e^{(q+u_i)} - e^{(-q-u_i)}} = \cot gq.$$

2. Logarithmic Function of Quaternionic Type

In order to determine the logarithmic function of quaternionic type, it is necessary to implement the generalized spherical coordinates. Therefore, we have

$$u_1 = r\cos\theta_1\cos\theta_2\cos\theta_3, \quad 0 < r < \infty,$$

$$u_2 = r\cos\theta_1\cos\theta_2\sin\theta_3, \quad 0 < \theta_3 < 2\pi,$$

$$u_3 = r\cos\theta_1\sin\theta_2, \quad 0 < \theta_2 < \frac{\pi}{2},$$

$$u_4 = r\sin\theta_1, \quad 0 < \theta_1 < \frac{\pi}{2}.$$

Identifying, in the expression

$$e^q = e^{q_1}\left\{\cos|\vec{q}| + \vec{q}\left(\frac{\sin|\vec{q}|}{|\vec{q}|}\right)\right\},$$

the value of $e((\vec{q}))$, it follows that

$$\phi = \left(\cos|\vec{q}|,\ q_2\frac{\sin|\vec{q}|}{|\vec{q}|},\ q_3\frac{\sin|\vec{q}|}{|\vec{q}|},\ q_4\frac{\sin|\vec{q}|}{|\vec{q}|}\right),$$

where

$$e^{q} = e^{u_1}\phi.$$

Identifying the last expression in spherical coordinates, we have

$$e^{q} = e^{u_1}(\cos\theta_1 \cos\theta_2 \cos\theta_3 + \cos\theta_1 \cos\theta_2 \sin\theta_3 i + \cos\theta_1 \sin\theta_2 j + \sin\theta_1 k).$$

The logarithm of q will be represented by lnq and will be the inverse of the exponential function of quaternionic type. Thus $w = lnq,$ satisfies the relation

$$e^{w} = q,$$

where $q \neq 0.$ We take $w = w_1 + w_2 i + w_3 j + w_4 k$ and use the generalized spherical coordinates:

$$u'_1 = r\cos\theta_1 \cos\theta_2 \cos\theta_3, \quad 0 < r < \infty,$$

$$u'_2 = r\cos\theta_1 \cos\theta_2 \sin\theta_3, \quad 0 < \theta_3 < 2\pi,$$

$$u'_3 = r\cos\theta_1 \sin\theta_2, \quad 0 < \theta_2 < \frac{\pi}{2},$$

$$u'_4 = r\sin\theta_1, \quad 0 < \theta_1 < \frac{\pi}{2}.$$

Now let w be given by

$$w = u'_1 + u'_2 i + u'_3 j + u'_4 k,$$

$$w = r\left(\frac{u'_1}{r} + \frac{u'_2}{r} i + \frac{u'_3}{r} j + \frac{u'_4}{r} k\right),$$

$$w = r(\cos\theta_1 \cos\theta_2 \cos\theta_3 + \cos\theta_1 \cos\theta_2 \sin\theta_3 i + \cos\theta_1 \sin\theta_2 j + \sin\theta_1 k),$$

where $r > 0.$ Thus,

$$e^{w} = e^{u'_1}\left\{\cos|\vec{u}'| + \vec{u}'\left(\frac{\sin|\vec{u}'|}{|\vec{u}'|}\right)\right\},$$

where $w = u'_1 + \vec{u}'.$ Now it follows that $e^{u'_1} e\vec{u}' = q$.

Taking $e^{\vec{u}'} = r,\ u_1' = ln|\, r\,|$ and

$$\vec{u}' = ln(\cos\theta_1 \cos\theta_2 \cos\theta_3 + \cos\theta_1 \cos\theta_2 \sin\theta_3 i + \cos\theta_1 \sin\theta_2 j + \sin\theta_1 k)$$

and putting $w = u_1 + \vec{u}'$, we have

$$w = lnq = ln|\, r\,| + ln(\cos\theta_1 \cos\theta_2 \cos\theta_3 + \cos\theta_1 \cos\theta_2 \sin\theta_3 i$$
$$+ \cos\theta_1 \sin\theta_2 j + \sin\theta_1 k).$$

3. Conclusion

The functions represented here, will serve for the determination and solution of classical equations in mathematical physics, as the wave equation which has its solutions in sines and cosines. Moreover, it follows that the Riemann hypersurface can be drawn through the logarithmic function of quaternionic type.

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