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GENERALIZATION OF OSTROWSKI-TYPE INEQUALITIES FOR DIFFERENTIABLE REAL (s, m)-CONVEX MAPPINGS

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Abstract

In this article, the author obtained new generalizations of Ostrowski-type inequalities for differentiable s-convex, m-convex and (s, m)-convex mappings.

1. Introduction

In recent years many authors have established error estimations for the Simpson's inequality and Ostrowski inequality; for refinements, counterparts, generalizations and new Ostrowski-type inequalities, see [4, 7, 10, 16, 17].

For $s \in (0, 1]$ and $m \in [0, 1]$, $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ is said to be a (s, m)-*convex mapping* on $\mathbb{I}$ if

$$f(tx + m(1-t)y) \le t^s f(x) + m(1-t)^s f(y), \tag{1}$$

for all $x, y \in \mathbb{I}$ and $t \in [0, 1]$. [1, 2, 5, 8, 12, 13, 14].

In (1), if we let $m = s = 1$, $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ is said to be a *convex mapping* on $\mathbb{I}$, [1, 2, 6, 7, 14].

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In (1), if we let $m = 1$, $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ is said to be a *s-convex mapping in the second sense* on $\mathbb{I}$, and if we let $s = 1$, $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ is said to be a *m-convex mapping* on $\mathbb{I}$.

In (1), if we let $m = 0$, $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ is said to be a *s-starshaped mapping* on $\mathbb{I}$, [1, 2, 6, 7, 14].

For $\alpha \in [0, 1]$, $f : \mathbb{I} \to [0, \infty)$ is said to be a (α, m)*-convex mapping* on $\mathbb{I}$ if

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1 - t^{\alpha}) f(y), \tag{2}$$

for all $x, y \in \mathbb{I}$ and $t \in [0, 1]$, [10, 11].

Denote the sets of convex, s-convex in the second sense, m-convex, (α, m)-convex, (s, m)-convex and s-starshaped mappings on $[a, b]$ by $K[a, b]$, $K_s^2[a, b]$, $K_m[a, b]$, $K_m^{\alpha}[a, b]$, $K_{s,m}^2[a, b]$ and $S_s^*[a, b]$, respectively, [15].

Dragomir in [3] pointed out some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

In [5, 6, 12, 14], Dragomir et al. proved a variant of Hermit-Hadamard's inequality for s-convex functions in second sense.

Theorem 1.1. *For* $a, b \in \mathbb{I}^0$ *with* $a < b$, *if* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *is in* $K_s^2[a, b]$ *for some fixed* $s \in (0, 1]$ *and* $f \in L^1([a, b])$, *then the following inequality holds*:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}. \tag{3}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (3).

In [1, 2], Alomari et al. proved the following inequality of Ostrowski type for mappings whose derivative in absolute value are s-convex in the second sense.

In [1, 2, 6] some inequalities of Hermite-Hadamard type for differentiable convex mappings are presented as follows:

Theorem 1.2. *Let* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^0$, *where* $a, b \in \mathbb{I}$ *with* $a < b$, *and let* $p > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $| f' |^q \in K[a, b]$, *then the following inequality holds*:

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|$$

$$\leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{\frac{1}{p}} [(| f'(a) |^q + 3| f'(b) |^q)^{\frac{1}{q}} + (3| f'(a) |^q + | f'(b) |^q)^{\frac{1}{q}}]. \tag{4}$$

In [7, 9, 12, 13, 15], Kirmaci establish a more general result related to this theorem.

In [10, 11], Pečarić et al. proved a variant of Hermit-Hadamard's inequality for s-convex and (α, m)-convex functions.

Theorem 1.3. *Suppose that* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *is in* $K_m[a, b]$ *with* $m \in (0, 1]$, *where* $a, b \in \mathbb{I}$ *with* $a < b$. *If* $f \in L^1([a, b])$, *then the following inequality holds*:

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\}. \tag{5}$$

For recent results and generalizations concerning Hermit-Hadamard's inequality, see [1, 8, 9] and [12]. The aim of this article is to establish Ostrowski type inequalities based on (s, m)-convexity.

2. Ostrowski Type Inequalities for (s, m)-convex Mappings

To begin with, we begin the following lemma:

Lemma 1. *Let* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^0$ *such that* $f' \in L^1([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$. *Then the following equality holds*:

$$f(x) - \frac{1}{b-a} \int_a^b f(u) du = (b-a) \int_0^1 p(t) f'(tb + (1-t)a) dt, \tag{6}$$

where $p(t) = \begin{cases} t, & t \in \left[0, \frac{x-a}{b-a}\right] \\ t-1, & t \in \left(\frac{x-a}{b-a}, 1\right] \end{cases}.$

Theorem 2.1. *Let* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^0$ *such that* $f' \in L^1([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$. *If* $f' \in K_{s,m}^2[a, b]$ *for some fixed* $s, m \in (0, 1]$, *then one has the inequality*:

$$\frac{1}{b-a}\int_a^b f(u)du \le \min\left\{\frac{f(b) + mf\left(\frac{a}{m}\right)}{s+1}, \frac{mf\left(\frac{b}{m}\right) + f(a)}{s+1}\right\}. \tag{7}$$

Proof. By the properties of (s, m)-convex mappings, for any $t \in [0, 1]$ we obtain the following inequalities: for $x, y \in \mathbb{I}$,

$$f(tx + (1-t)y) \le t^s f(x) + m(1-t)^s f\left(\frac{y}{m}\right), \tag{8}$$

for all $t \in [0, 1]$. By integrating (8) on $[0, 1]$ this is proved.

Theorem 2.2. *For* $a, b \in \mathbb{I}$ *with* $0 \le a < b$ *if a mapping* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *is in* $K_{s,m}^2[a, b]$ *for some fixed* $s, m \in (0, 1]$, *then one has the inequality*:

$$\frac{1}{b-a}\int_a^b \left[f(x) + mf\left(\frac{x}{m}\right)\right]dx$$

$$\le \frac{1}{s+1}\left[\left(\frac{f(a) + f(b)}{2}\right) + 2m\left(\frac{f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)}{2}\right) + m^2\left(\frac{f\left(\frac{a}{m^2}\right) + f\left(\frac{b}{m^2}\right)}{2}\right)\right].$$

Proof. By the properties of (s, m)-convex mappings, for any $t \in [0, 1]$ we obtain the following inequalities:

$$f\left(\frac{x+y}{2}\right) \le \left(\frac{1}{2}\right)^s f(x) + m\left(\frac{1}{2}\right)^s f\left(\frac{y}{m}\right) = \frac{f(x) + mf\left(\frac{y}{m}\right)}{2^s}$$

or

$$f\left(\frac{x+y}{2}\right) \le m\left(\frac{1}{2}\right)^s f\left(\frac{x}{m}\right) + \left(\frac{1}{2}\right)^s f(y) = \frac{mf\left(\frac{x}{m}\right) + f(y)}{2^s}.$$

By choosing $x = ta + (1-t)b$ and $y = (1-t)a + tb$, and integrating the result over $t \in [0, 1]$, we get:

$$\int_0^1 f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2^s} dx,$$

which implies that:

$$\frac{1}{b-a}\int_a^b \left(f(x) + mf\left(\frac{x}{m}\right)\right)dx \le \frac{f(a) + m\left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right) + m^2 f\left(\frac{b}{m^2}\right)}{s+1}. \quad (9)$$

By Theorem 2.1, analogously we also have:

$$\frac{1}{b-a}\int_a^b \left(mf\left(\frac{x}{m}\right) + f(x)\right)dx \le \frac{f(b) + m\left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right) + m^2 f\left(\frac{a}{m^2}\right)}{s+1}. \quad (10)$$

By (9) and (10), we have:

$$\frac{1}{b-a}\int_a^b \left(f(x) + mf\left(\frac{x}{m}\right)\right)dx$$

$$\le \frac{f(a) + f(b) + 2m\left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right) + m^2\left(f\left(\frac{a}{m^2}\right) + f\left(\frac{b}{m^2}\right)\right)}{2(s+1)}.$$

Corollary 1. *In Theorem* 2.2, *if we choose* $m = 1$, *then*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a) + f(b)}{s+1},$$

which implies the second inequality of Theorem 1.1.

Theorem 2.3. *Suppose that* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *is in* $\in L_1([am, b])$ *for* $m \in (0, 1]$ *and* a, b *with* $0 \le a < b$. *If* $f \in K_{s,m}^2[a, b]$ *for some fixed* $s \in (0, 1]$, *then we have the inequality*:

$$\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{ma}^{b} f(x)dx \leq \frac{(m+1)[f(a)+f(b)]}{s+1}. \qquad (11)$$

Proof. By the (s, m)-convexity of *f* we can write: for all $t \in [0, 1]$ and a, b as above,

$$f(ta + m(1-t)b) \leq t^s f(a) + m(1-t)^s f(b),$$

$$f((1-t)a + mtb) \leq (1-t)^s f(a) + mt^s f(b). \qquad (12)$$

By adding the above inequalities (12) side by side and integrating over $t \in [0, 1]$, (11) is proved.

Remark 1. In Theorem 2.3, if we choose $m = 1$, then we obtain the result of the right side of Theorem 1.1.

Theorem 2.4. *Let* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^0$ *such that* $f' \in L^1([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$. *If* $|f'| \in K_{s,m}^2[a, b]$ *for some fixed* $s, m \in (0, 1)$, *then the following inequalities hold*:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq (b-a)\min\left\{\mu_1 |f'(b)| + \nu_1 m \left| f'\left(\frac{a}{m}\right)\right|, \mu_1 m \left| f'\left(\frac{b}{m}\right)\right| + \nu_1 |f'(a)|\right\},$$

where

$$\mu_1 = \frac{1}{s+1}\left\{1 - \left(\frac{x-a}{b-a}\right)^{s+1}\right\} - \frac{1}{s+2}\left\{1 - 2\left(\frac{x-a}{b-a}\right)^{s+2}\right\},$$

$$\nu_1 = \frac{1}{s+1}\left\{1 - \left(\frac{b-x}{b-a}\right)^{s+1}\right\} - \frac{1}{s+2}\left\{1 - 2\left(\frac{b-x}{b-a}\right)^{s+2}\right\}.$$

Proof. By Lemma 1 and the properties of (s, m)-convex mappings, for any $t \in [0, 1]$ we obtain the following inequalities:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right|$$

$$\leq (b-a) \left[| f'(b) | \int_0^{\frac{x-a}{b-a}} t^{s+1} dt + m \left| f'\left(\frac{a}{m}\right) \right| \int_0^{\frac{x-a}{b-a}} t(1-t)^s dt \right.$$

$$\left. + | f'(b) | \int_{\frac{x-a}{b-a}}^1 (1-t) t^s dt + m \left| f'\left(\frac{a}{m}\right) \right| \int_{\frac{x-a}{b-a}}^1 (1-t)^{s+1} dt \right]$$

$$\leq (b-a) \left[\left\{ \frac{1}{s+1} \left(1 - \left(\frac{x-a}{b-a}\right)^{s+1} \right) - \frac{1}{s+2} \left(1 - 2 \left(\frac{x-a}{b-a}\right)^{s+2} \right) \right\} | f'(b) | \right.$$

$$\left. + \left\{ \frac{1}{s+1} \left(1 - \left(\frac{b-x}{b-a}\right)^{s+1} \right) - \frac{1}{s+2} \left(1 - 2 \left(\frac{b-x}{b-a}\right)^{s+2} \right) \right\} m \left| f'\left(\frac{a}{m}\right) \right| \right].$$

Analogously, we have:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right|$$

$$\leq (b-a) \left[\left\{ \frac{1}{s+1} \left(1 - \left(\frac{x-a}{b-a}\right)^{s+1} \right) - \frac{1}{s+2} \left(1 - 2 \left(\frac{x-a}{b-a}\right)^{s+2} \right) \right\} m \left| f'\left(\frac{b}{m}\right) \right| \right.$$

$$\left. + \left\{ \frac{1}{s+1} \left(1 - \left(\frac{b-x}{b-a}\right)^{s+1} \right) - \frac{1}{s+2} \left(1 - 2 \left(\frac{b-x}{b-a}\right)^{s+2} \right) \right\} | f'(a) | \right].$$

By Theorem 1.4, this is proved.

Corollary 2. *In Theorem* 2.4, *if we choose* $x = \frac{a+b}{2}$ *and* $m = 1$, *then we have*:

$$\mu_1 = \frac{1}{(s+1)(s+2)} \left(1 - \left(\frac{1}{2}\right)^{s+1} \right) = \nu_1.$$

Hence

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq (b-a)\frac{\left(1-\left(\frac{1}{2}\right)^{s+1}\right)}{(s+1)(s+2)}(|f'(a)|+|f'(b)|).$$

Especially, additionally if $s = 1$, then we obtain:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|+|f'(b)|}{4}\right].$$

Theorem 2.5. *Let* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^0$ *such that* $f' \in L^1([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$. *If* $|f'|^q \in K_{s,m}^2[a, b]$ *for some fixed* $s, m \in (0, 1]$ *and* $p > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequalities hold*:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\min\{\mu_2, \nu_2\},$$

where

$$\mu_2 = \left(\frac{x-a}{b-a}\right)^{\frac{p+1}{p}}\left\{\left(\frac{x-a}{b-a}\right)^{s+1}|f'(b)|^q + m\left(1-\left(\frac{b-x}{b-a}\right)^{s+1}\right)\left|f'\left(\frac{a}{m}\right)\right|^q\right\}^{\frac{1}{q}}$$

$$+\left(\frac{b-x}{b-a}\right)^{\frac{p+1}{p}}\left\{\left(1-\left(\frac{x-a}{b-a}\right)^{s+1}\right)|f'(b)|^q + m\left(\left(\frac{b-x}{b-a}\right)^{s+1}\right)\left|f'\left(\frac{a}{m}\right)\right|^q\right\}^{\frac{1}{q}},$$

$$\nu_2 = \left(\frac{x-a}{b-a}\right)^{\frac{p+1}{p}}\left\{\left(\frac{x-a}{b-a}\right)^{s+1}m\left|f'\left(\frac{b}{m}\right)\right|^q + \left(1-\left(\frac{b-x}{b-a}\right)^{s+1}\right)|f'(a)|^q\right\}^{\frac{1}{q}}$$

$$+\left(\frac{b-x}{b-a}\right)^{\frac{p+1}{p}}\left\{\left(1-\left(\frac{x-a}{b-a}\right)^{s+1}\right)m\left|f'\left(\frac{b}{m}\right)\right|^q + \left(\left(\frac{b-x}{b-a}\right)^{s+1}\right)|f'(a)|^q\right\}^{\frac{1}{q}}.$$

Proof. By Lemma 1 and the properties of (s, m)-convex mappings, for any $t \in [0, 1]$ we obtain the following inequalities:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq (b-a)\left[\left(\int_0^{\frac{x-a}{b-a}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{x-a}{b-a}} | f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{x-a}{b-a}}^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{x-a}{b-a}}^1 | f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}}\right],$$

$$\leq \frac{b-a}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{p+1}{p}}\left\{\left(\frac{x-a}{b-a}\right)^{s+1} | f'(b)|^q\right.\right.$$

$$\left.+ m\left(1-\left(\frac{b-x}{b-a}\right)^{s+1}\right)\left| f'\left(\frac{a}{m}\right)\right|^q\right\}^{\frac{1}{q}}$$

$$\left.+\left(\frac{b-x}{b-a}\right)^{\frac{p+1}{p}}\left\{\left(1-\left(\frac{x-a}{b-a}\right)^{s+1}\right)| f'(b)|^q + m\left(\frac{b-x}{b-a}\right)^{s+1}\left| f'\left(\frac{a}{m}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

Analogously, also we have:

$$\left| f(x)-\frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq \frac{b-a}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{p+1}{p}}\left\{\left(\frac{x-a}{b-a}\right)^{s+1} m\left| f'\left(\frac{b}{m}\right)\right|^q\right.\right.$$

$$\left.+\left(1-\left(\frac{b-x}{b-a}\right)^{s+1}\right)| f'(a)|^q\right\}^{\frac{1}{q}}$$

$$\left.+\left(\frac{b-x}{b-a}\right)^{\frac{p+1}{p}}\left\{\left(1-\left(\frac{x-a}{b-a}\right)^{s+1}\right)m\left| f'\left(\frac{b}{m}\right)\right|^q + \left(\left(\frac{b-x}{b-a}\right)^{s+1}\right)| f'(a)|^q\right\}^{\frac{1}{q}}\right],$$

which completes the proof by Theorem 2.1.

Remark 2. In Theorem 2.5, if we choose $x = \frac{a+b}{2}$ and $s = m = 1$, then we get Theorem 1.3:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq \left(\frac{b-a}{16}\right)\left(\frac{4}{p+1}\right)^{\frac{1}{p}}[\{|f'(a)|^q + 3|f'(b)|^q\}^{\frac{1}{q}} + \{3|f'(a)|^q + |f'(b)|^q\}^{\frac{1}{q}}].$$

Theorem 2.6. *Let* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on an interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *such that* $f' \in L^1([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$. *If* $|f'|^q \in K^2_{s,m}[a, b]$ *for some fixed* $s, m \in (0, 1]$ *and* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then for each* $x \in [a, b]$ *the following inequality holds*:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq \frac{b-a}{(p+1)^{\frac{1}{p}}}\left\{\left(\frac{x-a}{b-a}\right)^{\frac{p+1}{p}} + \left(\frac{b-x}{b-a}\right)^{p+1}\right\}^{\frac{1}{p}}$$

$$\times \min\left\{\left(\frac{|f'(b)|^q + m\left|f'\left(\frac{a}{m}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}, \left(\frac{m\left|f'\left(\frac{b}{m}\right)\right|^q + |f'(a)|^q}{s+1}\right)^{\frac{1}{q}}\right\}.$$

Proof. By Lemma 1 and the properties of *m*-convexity, we have:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq (b-a)\left(\int_0^1 |p(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tb + (1-t)a)|^q dt\right)^{\frac{1}{q}}. \tag{13}$$

Since $|f'|^q$ is (s, m)-convex, we have:

$$\int_0^1 |f'(tb + (1-t)a)|^q dt \leq \frac{1}{s+1}\left[|f'(b)|^q + \left|f'\left(\frac{a}{m}\right)\right|^q\right]. \tag{14}$$

By using (13) and (14), we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq \frac{b-a}{(p+1)^{\frac{1}{p}}}\left\{\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right\}^{\frac{1}{p}}\left\{\frac{|f'(b)|^q + \left|f'\left(\frac{a}{m}\right)\right|^q}{s+1}\right\}^{\frac{1}{q}}. \tag{15}$$

Analogously, we also have:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq \frac{b-a}{(p+1)^{\frac{1}{p}}}\left\{\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right\}^{\frac{1}{p}}\left\{\frac{m\left|f'\left(\frac{b}{m}\right)\right|^q + |f'(a)|^q}{s+1}\right\}^{\frac{1}{q}}, \tag{16}$$

which completes the proof by (13)-(16) and Theorem 2.1.

Corollary 3. *Under assumptions in Theorem* 2.6 *with* $p = q = 2$ *and* $m = 1$, *we have*:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq \frac{b-a}{3^{\frac{1}{2}}}\left\{\frac{1}{4} + 3\left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right\}^{\frac{1}{2}}\left[\frac{|f'(a)|^2 + |f'(b)|^2}{s+1}\right]^{\frac{1}{2}}.$$

In addition, *if we choose* $x = \frac{a+b}{2}$ *and* $s = 1$, *then we obtain the following midpoint inequality*:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{2}\left(\frac{|f'(a)|^2 + |f'(b)|^2}{6}\right)^{\frac{1}{2}}.$$

Theorem 2.7. *Let* $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ *be a differentiable mapping on an interior* $\mathbb{I}^0$ *of* $\mathbb{I}$ *such that* $f' \in L^1([a, b])$, *where* $a, b \in \mathbb{I}$ *with* $a < b$. *If* $|f'|^q \in K_{s,m}^2[a, b]$ *for some fixed* $s, m \in (0, 1]$ *and* $q \geq 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then for each* $x \in [a, b]$ *the following inequality holds*:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}} \min\left\{\left(\frac{x-a}{b-a}\right)^{\frac{2}{p}}\left[\mu_3 |f'(b)|^q + \nu_3 m\left|f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}\right.$$

$$+\left(\frac{b-x}{b-a}\right)^{\frac{2}{p}}\left[\nu_3 |f'(b)|^q + \mu_3 m\left|f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}, \left(\frac{x-a}{b-a}\right)^{\frac{2}{p}}\left[\mu_3 m\left|f'\left(\frac{b}{m}\right)\right|^q\right.$$

$$\left.\left. + \nu_3 m |f'(a)|^q\right]^{\frac{1}{q}} + \left(\frac{b-x}{b-a}\right)^{\frac{2}{p}}\left[\nu_3 m\left|f'\left(\frac{b}{m}\right)\right|^q + \mu_3 m |f'(a)|^q\right]^{\frac{1}{q}}\right\},$$

where

$$\mu_3 = \frac{1}{s+2}\left(\left(\frac{x-a}{b-a}\right)^{s+2} - 1\right),$$

$$\nu_3 = \frac{1}{s+2}\left(\left(\frac{x-a}{b-a}\right)^{s+2} - 1\right) + \frac{1}{s+1}\left(1 - \left(\frac{x-a}{b-a}\right)^{s+1}\right).$$

Proof. By Lemma 1 and using the power mean inequality, we have:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq (b-a)\left[\left(\int_0^{\frac{x-a}{b-a}} tdt\right)^{\frac{1}{p}}\left(\int_0^{\frac{x-a}{b-a}} t|f'(tb + (1-t)a)|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_{\frac{x-a}{b-a}}^1 (1-t)dt\right)^{\frac{1}{p}}\left(\int_{\frac{x-a}{b-a}}^1 (1-t)|f'(tb + (1-t)a)|^q dt\right)^{\frac{1}{q}}\right]. \quad (17)$$

By (17) and the (s, m)-convexity $|f'|^q$, we also have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{x-a}{b-a}\right)\frac{2}{p}\left[\left\{\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\right\}|f'(b)|^q\right.$$

$$\left. + \left\{\frac{1}{s+1}\left(1-\left(\frac{b-x}{b-a}\right)^{s+1}\right) + \frac{1}{s+2}\left(\left(\frac{x-a}{b-a}\right)^{s+2} - 1\right)\right\}\left|f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}$$

$$+ (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{b-x}{b-a}\right)\frac{2}{p}\left[\left\{\frac{1}{s+1}\left(1-\left(\frac{x-a}{b-a}\right)^{s+1}\right) + \frac{1}{s+2}\right.\right.$$

$$\left.\left.\times\left(\left(\frac{x-a}{b-a}\right)^{s+2} - 1\right)\right\}|f'(b)|^q + \left\{\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\right\}|f'(b)|^q\right]^{\frac{1}{q}}. \tag{18}$$

Analogously, also we have:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{x-a}{b-a}\right)\frac{2}{p}\left[\left\{\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\right\}m\left|f'\left(\frac{b}{m}\right)\right|^q\right.$$

$$\left. + \left\{\frac{1}{s+1}\left(1-\left(\frac{b-x}{b-a}\right)^{s+1}\right) + \frac{1}{s+2}\left(\left(\frac{x-a}{b-a}\right)^{s+2} - 1\right)\right\}|f'(a)|^q\right]^{\frac{1}{q}}$$

$$+ (b-a)\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{b-x}{b-a}\right)\frac{2}{p}\left[\left\{\frac{1}{s+1}\left(1-\left(\frac{x-a}{b-a}\right)^{s+1}\right) + \frac{1}{s+2}\right.\right.$$

$$\left.\left.\times\left(\left(\frac{x-a}{b-a}\right)^{s+2} - 1\right)\right\}m\left|f'\left(\frac{b}{m}\right)\right|^q + \left\{\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\right\}|f'(b)|^q\right]^{\frac{1}{q}}, \tag{19}$$

which completes the proof by (18), (19) and Theorem 2.3.

Corollary 4. *In Theorem* 2.7, *if we let* $x = \frac{a+b}{2}$, *then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right|$$

$$\leq \frac{b-a}{8} 2^{\frac{1-s}{q}} \left(\frac{1}{(s+1)(s+2)}\right) \min\left\{\left[(s+1)|f'(b)|^q + (2^{s+3} - s - 3) \times m\left|f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}\right.$$

$$+ \left[(2^{s+3} - s - 3)m|f'(b)|^q + (s+1)\left|f'\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}},$$

$$(s+1)m\left|f'\left(\frac{b}{m}\right)\right|^q + (2^{s+3} - s - 3)|f'(a)|^q\Bigg]^{\frac{1}{q}}$$

$$\left. + \left[(2^{s+3} - s - 3)m\left|f'\left(\frac{b}{m}\right)\right|^q + (s+1)|f'(a)|^q\right]^{\frac{1}{q}}\right\}.$$

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