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BOUNDS FOR THE PERRON ROOTS OF NONNEGATIVE MATRICES

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Abstract

An estimate for the Perron roots of nonnegative matrices is obtained. The results generalize or improve the bounds found in the related literature.

1. Introduction

This paper is concerned with the estimation for the Perron root of a nonnegative matrix. Let $A = (a_{ij})$ be a nonnegative matrix of order n with the Perron root r, and denote by $r_i(A)$ and $c_i(A)$ $(i = 1, 2, ..., n)$ the ith row sum and ith column sum of the matrix A, respectively. Let I be the identity matrix.

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For the bounds for the Perron root of a nonnegative matrix, it is well known that the Perron root satisfies the Frobenius' inequality [2]

$$\min_i r_i(A) \le r \le \max_i r_i(A). \tag{1}$$

An analogous result holds for column sums $c_1, c_2, \ldots, c_n$. For nonnegative matrices without zero row sums, eq. (1) was improved by Minc [5]. For positive matrices, we have the results of Ledermann [3], Ostrowski [6] and Brauer [1]. These results improved bound in (1). Recently, Liu [4] and Yin [7] extended these results further.

In this paper, we investigate estimation for the Perron root of a nonnegative matrix further, the results generalize or improve the bounds obtained in the related literature.

2. Estimation for Perron Root

Lemma 1 [5]**.** *Let* α *be a characteristic root of a matrix* A *of order* n, *and let* $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *be characteristic vectors corresponding to* α *of* A^T *and* A, *respectively. Then*

$$\alpha \sum_{i=1}^{n} x_i = \sum_{t=1}^{n} x_t r_t(A), \tag{2}$$

$$\alpha \sum_{j=1}^{n} y_j = \sum_{t=1}^{n} y_t c_t(A). \tag{3}$$

Lemma 2 [5]**.** *If* $q_1, q_2, \ldots, q_n$ *are positive numbers for any real numbers* $p_1, p_2, \ldots, p_n$, *then*

$$\min_i \frac{p_i}{q_i} \le \frac{p_1 + p_2 + \cdots + p_n}{q_1 + q_2 + \cdots + q_n} \le \max_i \frac{p_i}{q_i}. \tag{4}$$

Lemma 3 [4]**.** *Let* A *be an arbitrary matrix of order* n, *and denote by*

$r_i(A^k)$ *and* $c_i(A^k)$ $(i = 1, 2, \ldots, n)$ *the ith row sum and ith column sum of the matrix* A^k, *respectively. Then*

$$r_i(A^{k+1}) = \sum_{t=1}^{n} a_{it} r_t(A^k), \tag{5}$$

$$c_i(A^{k+1}) = \sum_{t=1}^{n} a_{ti} c_t(A^k). \tag{6}$$

Lemma 4. *Let* A *be a nonnegative matrix of order* n, *and define* $B = A + \alpha I$, $\alpha > 0$. *And let m and k be positive integers. Then*

$$\min_i \frac{r_i(AB^k)}{r_i(B^k)} \leq \min_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)}\right)^{\frac{1}{m}} \leq \max_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)}\right)^{\frac{1}{m}} \leq \max_i \frac{r_i(AB^k)}{r_i(B^k)}. \tag{7}$$

Proof. We prove the right inequality of (7) by induction. The left inequality can be proved similarly. The middle inequality is obvious.

When $m = 1$, the right inequality of (7) becomes an equality.

When $m = 2$, by (5), we have

$$\frac{r_i(A^2B^k)}{r_i(B^k)} = \frac{1}{r_i(B^k)} \sum_{t=1}^{n} \left[a_{it} r_t(B^k) \frac{r_t(AB^k)}{r_t(B^k)}\right] \leq \max_i \left(\frac{r_i(AB^k)}{r_i(B^k)}\right) \frac{r_i(AB^k)}{r_i(B^k)}.$$

So,

$$\max_i \frac{r_i(A^2B^k)}{r_i(B^k)} \leq \max_i \left(\frac{r_i(AB^k)}{r_i(B^k)}\right)^2.$$

Hence the right inequality of (7) is valid when $m = 2$. Suppose inductively that (7) holds for $m = h$. Then, by (5), we have

$$\frac{r_i(A^{h+1}B^k)}{r_i(B^k)} = \frac{1}{r_i(B^k)}\sum_{t=1}^{n}\left[a_{it}r_t(B^k)\frac{r_t(A^hB^k)}{r_t(B^k)}\right] \le \max_i\left(\frac{r_i(A^hB^k)}{r_i(B^k)}\right)\frac{r_i(AB^k)}{r_i(B^k)}.$$

Further, by the inductive hypothesis, we have

$$\max_i \frac{r_i(A^{h+1}B^k)}{r_i(B^k)} \le \max_i\left(\frac{r_i(AB^k)}{r_i(B^k)}\right)^{h+1}.$$

Hence the right inequality of (7) holds for $m = h + 1$.

Remark 1. A result similar to (7) holds for column sums $c_1, c_2, \ldots, c_n$.

Theorem 1. *Let A be a nonnegative matrix of order n, and define $B = A + \alpha I$, $\alpha > 0$. Then for arbitrary positive integers m, k and r,*

$$\min_i\left(\frac{r_i(A^mB^k)}{r_i(B^k)}\right)^{\frac{1}{m}} \le r \le \max_i\left(\frac{r_i(A^mB^k)}{r_i(B^k)}\right)^{\frac{1}{m}}, \tag{8}$$

$$\min_i\left(\frac{c_i(A^mB^k)}{c_i(B^k)}\right)^{\frac{1}{m}} \le r \le \max_i\left(\frac{c_i(A^mB^k)}{c_i(B^k)}\right)^{\frac{1}{m}}. \tag{9}$$

Proof. Assume that, without loss of generality, $(x_1, x_2, \ldots, x_n)$ is a nonnegative characteristic vector of A^T corresponding to r such that $\sum_{i=1}^{n} x_i = 1$. Hence, for any positive integers m and k, $(x_1, x_2, \ldots, x_n)$ is a characteristic vector of $(A^mB^k)^T$ and $(B^k)^T$ corresponding to $r^m(r+\alpha)^k$ and $(r+\alpha)^k$, respectively. By the above statements, applying (2) to $(A^mB^k)^T$ and $(B^k)^T$, we have

$$r^m = \frac{r^m(r+\alpha)^k}{(r+\alpha)^k} = \frac{\sum_{i=1}^{n} x_i r_i(A^mB^k)}{\sum_{i=1}^{n} x_i r_i(B^k)}.$$

Further, by (4), we get

$$\min_i \frac{r_i(A^m B^k)}{r_i(B^k)} \le \min_{x_i \ne 0} \frac{x_i r_i(A^m B^k)}{x_i r_i(B^k)} \le r^m \le \max_{x_i \ne 0} \frac{x_i r_i(A^m B^k)}{x_i r_i(B^k)} \le \max_i \frac{r_i(A^m B^k)}{r_i(B^k)}.$$

Hence the inequality of (8) holds. The column sums case can be proved similarly.

Corollary 2. *Let A be a nonnegative matrix of order n, and define $B = A + I$. Then for any positive integers m, k and r,*

$$\min_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)} \right)^{\frac{1}{m}} \le r \le \max_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)} \right)^{\frac{1}{m}}, \tag{10}$$

$$\min_i \left(\frac{c_i(A^m B^k)}{c_i(B^k)} \right)^{\frac{1}{m}} \le r \le \max_i \left(\frac{c_i(A^m B^k)}{c_i(B^k)} \right)^{\frac{1}{m}}. \tag{11}$$

Proof. By taking $\alpha = 1$ in Theorem 1, we obtain (10) and (11).

Remark 2. By (7) in Lemma 4 and (10) in Corollary 2, we have

$$\min_i \frac{r_i(AB^k)}{r_i(B^k)} \le \min_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)} \right)^{\frac{1}{m}} \le r \le \max_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)} \right)^{\frac{1}{m}} \le \max_i \frac{r_i(AB^k)}{r_i(B^k)}.$$

Hence the estimation in Corollary 2 improves the bound given by Yin in [7].

Remark 3. If we take $m = 1$ in Corollary 2, then we get the bound given by Yin.

Corollary 3. *Let A be a nonnegative matrix of order n, and define $B = A + I$. Then for any positive integers m and r,*

$$\min_i \left(\frac{r_i(A^m B)}{r_i(B)} \right)^{\frac{1}{m}} \le r \le \max_i \left(\frac{r_i(A^m B)}{r_i(B)} \right)^{\frac{1}{m}}, \tag{12}$$

$$\min_i \left(\frac{c_i(A^m B)}{c_i(B)} \right)^{\frac{1}{m}} \le r \le \max_i \left(\frac{c_i(A^m B)}{c_i(B)} \right)^{\frac{1}{m}}. \tag{13}$$

Proof. By taking $k = 1$ in Corollary 2, we obtain (12) and (13).

Theorem 4. *Let A be a nonnegative matrix of order n, and define $B = A + \alpha I$, $\alpha > 0$. And let m and k be any positive integers. Then the limits* $\lim_{k\to\infty} \min_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)}\right)^{\frac{1}{m}}$ *and* $\lim_{k\to\infty} \max_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)}\right)^{\frac{1}{m}}$ *exist for any fixed numbers α, m and r, and*

$$\lim_{k\to\infty} \min_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)}\right)^{\frac{1}{m}} \le r \le \lim_{k\to\infty} \max_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)}\right)^{\frac{1}{m}}. \tag{14}$$

Proof. Let $B = (b_{ij})$. By (4) in Lemma 2 and (5) in Lemma 3, for any positive integer k, the following inequality holds:

$$\frac{r_i(A^m B^{k+1})}{r_i(B^{k+1})} = \frac{\sum_{t=1}^{n} b_{it} r_t(A^m B^k)}{\sum_{t=1}^{n} b_{it} r_t(B^k)} \le \max_{b_{it}\neq 0} \frac{b_{it} r_t(A^m B^k)}{b_{it} r_t(B^k)} \le \max_t \frac{r_t(A^m B^k)}{r_t(B^k)}.$$

For any $i \, (1 \le i \le n)$, we have

$$\max_i \left(\frac{r_i(A^m B^{k+1})}{r_i(B^{k+1})}\right)^{\frac{1}{m}} \le \max_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)}\right)^{\frac{1}{m}}.$$

Thus the sequence $\left\{\max_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)}\right)^{\frac{1}{m}}\right\}_{k=1}^{\infty}$ decreases monotonically and has lower bound r. Similarly, we can deduce that the sequence $\left\{\min_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)}\right)^{\frac{1}{m}}\right\}_{k=1}^{\infty}$ increases monotonically and has upper bound r. Then, we finish the proof of Theorem 4.

Remark 4. For any nonnegative matrix A of order n, if we take $m = 1$ and $k = 0$ in Theorem 1 and assume $A^0 = I$, then Theorem 1 yields the Frobenius' bounds, i.e., (1). And by the proof of Theorem 4, we have

$$r \le \max_i \left(\frac{r_i(A^m B^k)}{r_i(B^k)} \right)^{\frac{1}{m}} \le \max_i \left(\frac{r_i(A^m B^{k-1})}{r_i(B^{k-1})} \right)^{\frac{1}{m}}$$

$$\le \cdots \le \max_i \left(\frac{r_i(A^m B)}{r_i(B)} \right)^{\frac{1}{m}} \le \max_i r_i(A).$$

Thus the estimation in Theorem 1 improves the bound given by Frobenius [2].

Remark 5. Assume that A has nonzero row sums, and if we take $\alpha = 0$ in Theorem 1, then we get the bound given by Liu [4]. Moreover, by Lemma 4 and Theorem 4, the above bound improves the Minc's bounds [5].

Example. Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 1 & 1 \end{bmatrix}.$$

The accurate value $r = 5.74165738\ldots$. If we take $\alpha = 1$, $m = 3$, $k = 4$, then by Theorem 1, bounds for the Perron root of A are shown in the following table.

	Row	Column
Frobenius	$4 \le r \le 8$	$5 \le r \le 7$
Minc	$5 \le r \le 6.25$	$5.6 \le r \le 5.8572$
Ledermann	$4.1547 \le r \le 7.8661$	$5.080 \le r \le 6.9259$
Ostrowski	$4.5275 \le r \le 7.6547$	$5.2247 \le r \le 6.8165$
Brauer	$4.8284 \le r \le 7.4642$	$5.3722 \le r \le 6.7016$
Yin [7]	$5.7297 \le r \le 5.7564$	$5.7349 \le r \le 5.7484$
Theorem 1 of this paper	$5.7368 \le r \le 5.74759$	$5.73897 \le r \le 5.74438$

The choice of the optimal α in estimations is difficult and investigation on it is interesting.

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