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A NOTE ON THE SECOND KIND (h, q)-EULER POLYNOMIALS

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Abstract

In this paper, we construct the second kind (h, q)-Euler numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$. From these numbers and polynomials, we establish some interesting identities and relations.

1. Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, ...\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one

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normally assume that $|q|<1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral is defined as

$$I_{-1}(g) = \lim_{q \to -1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{0 \leq x < p^N} g(x)(-1)^x. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0), \text{ (see [1-3])}. \tag{1.2}$$

First, we introduce the second kind Euler numbers E_n and polynomials $E_n(x)$. The second kind Euler numbers E_n are defined by the generating function:

$$F(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.3}$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$F(x, t) = \frac{2e^t}{e^{2t}+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{1.4}$$

2. The Second Kind (h, q)-Euler Numbers and Polynomials

In this section, we introduce the second kind (h, q)-Euler numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$ and investigate their properties. In (1.2), if we take $g(x) = q^{hx}e^{(2x+1)t}$, then we easily see that

$$I_{-1}(q^{hx}e^{(2x+1)t}) = \int_{\mathbb{Z}_p} q^{hx}e^{(2x+1)t} d\mu_{-1}(x) = \frac{2e^t}{q^h e^{2t}+1}.$$

Let us define the second kind (h, q)-Euler numbers $E_{n,q}^{(h)}$ and polynomials

$E_{n,q}^{(h)}(x)$ as follows:

$$I_{-1}(q^{hy}e^{(2y+1)t}) = \int_{\mathbb{Z}_p} q^{hy}e^{(2y+1)t}d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!}, \tag{2.1}$$

$$I_{-1}(q^{hy}e^{(2y+1+x)t}) = \int_{\mathbb{Z}_p} q^{hy}e^{(x+2y+1)t}d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{2.2}$$

By (2.1) and (2.2), we obtain the following Witt's formula.

Theorem 1. *For* $h \in \mathbb{Z}$ *and* $q \in \mathbb{C}_p$ *with* $|q-1|_p < p^{-\frac{1}{p-1}}$, we have

$$\int_{\mathbb{Z}_p} q^{hx}(2x+1)^n d\mu_{-1}(x) = E_{n,q}^{(h)},$$

$$\int_{\mathbb{Z}_p} q^{hy}(x+2y+1)^n d\mu_{-1}(y) = E_{n,q}^{(h)}(x).$$

Let q be a complex number with $|q| < 1$ and $h \in \mathbb{Z}$. By the meaning of (1.3) and (1.4), let us define the second kind (h, q)-Euler numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$ as follows:

$$F_q^{(h)}(t) = \frac{2e^t}{q^h e^{2t} + 1} = \sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!}, \tag{2.3}$$

$$F_q^{(h)}(x, t) = \frac{2e^t}{q^h e^{2t} + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{2.4}$$

We have the following remark.

Remark. Note that

(1) $E_{n,q}^{(h)}(0) = E_{n,q}^{(h)}$,

(2) If $q \to 1$, then $E_{n,q}^{(h)}(x) = E_n(x)$, $E_{n,q}^{(h)} = E_n$,

(3) If $q \to 1$, then $F_q^{(h)}(x, t) = F(x, t)$, $F_q^{(h)}(t) = F(t)$.

By the above definition, we obtain

$$\sum_{l=0}^{\infty} E_{l,q}^{(h)}(x)\frac{t^l}{l!} = \frac{2e^t}{q^h e^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}\frac{t^n}{n!}\sum_{m=0}^{\infty} x^m \frac{t^m}{m!}$$

$$= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} E_{n,q}^{(h)}\frac{t^n}{n!}x^{l-n}\frac{t^{l-n}}{(l-n)!}\right)$$

$$= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n} E_{n,q}^{(h)} x^{l-n}\right)\frac{t^l}{l!}.$$

By using comparing coefficients of $\frac{t^l}{l!}$, we have the following theorem.

Theorem 2. *For any positive integer n, we have*

$$E_{n,q}^{(h)}(x) = \sum_{k=0}^{n}\binom{n}{k} E_{k,q}^{(h)} x^{n-k}.$$

Because

$$\frac{\partial}{\partial x}F_q(x, t) = tF_q(x, t) = \sum_{n=0}^{\infty}\frac{d}{dx}E_{n,q}^{(h)}(x)\frac{t^n}{n!},$$

it follows the important relation

$$\frac{d}{dx}E_{n,q}^{(h)}(x) = nE_{n-1,q}^{(h)}(x).$$

We also obtain the following integral formula

$$\int_a^b E_{n-1,q}^{(h)}(x)dx = \frac{1}{n}(E_{n,q}^{(h)}(b) - E_{n,q}^{(h)}(a)).$$

Theorem 3. *The second kind (h, q)-Euler numbers $E_{n,q}^{(h)}$ are defined inductively by*

$$q^h(E_q^{(h)}+1)^n + (E_q^{(h)}-1)^n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

with the usual convention about replacing $(E_q^{(h)})^n$ by $E_{n,q}^{(h)}$ in the binomial expansion.

Proof. From (2.3), we have

$$\frac{2}{q^h e^t + e^{-t}} = \sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} (E_q^{(h)})^n \frac{t^n}{n!} = e^{E_q^{(h)} t}$$

which yields

$$2 = (q^h e^t + e^{-t}) e^{E_q^{(h)} t} = q^h e^{(E_q^{(h)}+1)t} + e^{(E_q^{(h)}-1)t}.$$

Using Taylor expansion of exponential function, we obtain

$$2 = \sum_{n=0}^{\infty} \{q^h (E_q^{(h)} + 1)^n + (E_q^{(h)} - 1)^n\} \frac{t^n}{n!}$$

$$= q^h (E_q^{(h)} + 1)^0 + (E_q^{(h)} - 1)^0 + \sum_{n=1}^{\infty} \{q^h (E_q^{(h)} + 1)^n + (E_q^{(h)} - 1)^n\} \frac{t^n}{n!}.$$

The result follows by comparing the coefficients.

Since

$$\sum_{l=0}^{\infty} E_{l,q}^{(h)}(x+y) \frac{t^l}{l!} = \frac{2e^t}{q^h e^{2t} + 1} e^{(x+y)t}$$

$$= \sum_{l=0}^{\infty} \left(\sum_{n=0}^{l} E_{n,q}^{(h)}(x) \frac{t^n}{n!} y^{l-n} \frac{t^{l-n}}{(l-n)!} \right)$$

$$= \sum_{l=0}^{\infty} \left(\sum_{n=0}^{l} \binom{l}{n} E_{n,q}^{(h)}(x) y^{l-n} \right) \frac{t^l}{l!},$$

we have the following addition theorem.

Theorem 4. *The second kind* (h, q)*-Euler polynomials* $E_{n,q}^{(h)}(x)$ *satisfy the following relation*:

$$E_{k,q}^{(h)}(x+y) = \sum_{n=0}^{k} \binom{k}{n} E_{n,q}^{(h)}(x) y^{k-n}.$$

It is easy to see that

$$\sum_{n=0}^{\infty} E_{n,q}^{(h)}(x)\frac{t^n}{n!} = \frac{2e^t}{q^h e^{2t}+1}e^{xt}$$

$$= \sum_{a=0}^{m-1}(-1)^a q^{ha}\sum_{n=0}^{\infty} E_{n,q^m}^{(h)}\left(\frac{2a+x+1-m}{m}\right)\frac{(mt)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(m^n\sum_{a=0}^{m-1}(-1)^a q^{ha} E_{n,q^m}^{(h)}\left(\frac{2a+x+1-m}{m}\right)\right)\frac{t^n}{n!}.$$

Hence we have the below theorem.

Theorem 5. *For any positive integer* $m(=odd)$, *we have*

$$E_{n,q}^{(h)}(x) = m^n\sum_{a=0}^{m-1}(-1)^a q^{ha} E_{n,q^m}^{(h)}\left(\frac{2a+x+1-m}{m}\right), \quad \textit{for } n \geq 0.$$

3. Directions for Further Research

In [4], we observed the behavior of complex roots of the second kind Euler polynomials $E_n(x)$ by using numerical investigation. Since

$$\sum_{n=0}^{\infty} E_n(-x)\frac{(-t)^n}{n!} = F(-x, -t)\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

we have

$$E_n(x) = (-1)^n E_n(-x), \quad \text{for } n \in \mathbb{N}.$$

Prove that $E_n(x)$, $x \in \mathbb{C}$, has $\mathrm{Re}(x) = 0$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions. The obvious corollary is that the zeros of $E_n(x)$ will also inherit these symmetries.

$$\text{If } E_n(x_0) = 0, \text{ then } E_n(-x_0) = 0 = E_n(x_0^*) = E_n(-x_0^*), \tag{3.1}$$

* denotes complex conjugation (see [4]). The question is: what happens with the reflection symmetry (3.1), when one considers the second kind (h, q)-Euler polynomials $E_{n,q}^{(h)}(x)$? In general, how many roots does $E_{n,q}^{(h)}(x)$ have? This is open

problem. Prove or disprove: $E_{n,q}^{(h)}(x) = 0$ have n distinct solutions. Find the numbers of complex zeros $C_{E_{n,q}^{(h)}(x)}$ of $E_{n,q}^{(h)}(x)$, $Im(x) \neq 0$. Since n is the degree of the polynomial $E_{n,q}^{(h)}(x)$, the number of real zeros $R_{E_{n,q}^{(h)}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{E_{n,q}^{(h)}(x)} = n - C_{E_{n,q}^{(h)}(x)}$, where $C_{E_{n,q}^{(h)}(x)}$ denotes complex zeros. Observe that the structure of the zeros of the second Euler polynomials $E_n(x)$ resembles the structure of the zeros of the second kind (h, q)-Euler polynomials $E_{n,q}^{(h)}(x)$ as $q \to 1$. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the second kind (h, q)-Euler polynomials $E_{n,q}^{(h)}(x)$ to appear in mathematics and physics. The reader may refer to [4-6] for the details.

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