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CHARACTERIZATION OF A NEW BIMODAL NORMAL DISTRIBUTION

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Abstract

In applied statistics there are many situations in which the data are not satisfied with unimodal distributions. The distribution of uncontrolled data in quality control or outlier observations in linear models and time series may require to be considered as bimodal version. Such situations occur when the recorded data have the probability proportional to absolute value of deviations. In this paper, a new distribution called absolute-double normal distribution is introduced and characterized. This distribution has two symmetric modes about the mean. Point and interval estimations of its parameters are derived. Three pivotal values are introduced. Confidence intervals for the parameters by numerical methods are given.

1. Introduction

Frequently, in applied statistics, the common distributions are considered to be unimodal. However in practice, there are many situations in which the data are not satisfied with unimodal distributions. Univariate elliptical family of distributions and Kotz type distributions as a generalization of normal include some bimodal types (Nadarajah [9] and Fang et al. [5]). For example, the distribution of uncontrolled data in quality control or outlier observations in linear models and time series may

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require to be considered as bimodal. Such situations occur when the recorded data have the probability proportional to absolute value of deviations, so that the recorded data are considered as values of a weighted distribution (Misra et al. [8], Kim [6], Alavi [1], Alavi and Chinipardaz [2, 3], Patil [10], Barmi and Simonoff [4] and, Patil and Rao [11]). In this paper, a new distribution is introduced. This distribution is called *absolute-double normal distribution* and denoted by $ADN(\mu, \sigma)$ in which μ and σ are location and scale parameters, respectively. This distribution has two symmetric modes about μ. Probability density function (pdf) of $ADN(\mu, \sigma)$ is given by

$$f(x; \mu, \sigma) = \frac{|x-\mu|}{2\sigma^2} \exp\left\{\frac{1}{2\sigma^2}(x-\mu)^2\right\}, \quad -\infty < x < \infty, \tag{1}$$

where $-\infty < \mu < \infty$ and $\sigma > 0$.

In fact, ADN is a weighted normal distribution with weight function $w(x) = |x - \mu|$ which belongs to univariate case of symmetric Kotz type distributions. This distribution can be used in ballistic data in which the observations far from the target are matter of interest for researchers. In Section 2, properties and characterizations of $ADN(\mu, \sigma)$ are studied. Sections 3 and 4 are devoted to point estimations and confidence intervals. Some simulations are done to give the confidence intervals.

2. Properties and Characterizations of Absolute-double Normal

In this section, some properties and characterizations of absolute-double normal distribution are studied. When $\mu = 0$ and $\sigma = 1$, we call this distribution as *standard absolute-double normal* and denote it by *SADN*. pdf's of *SADN* and standard normal distribution are shown in Figure 1. As it is seen, this distribution is symmetric and bimodal.

Table 1 gives cumulative distribution function, cdf, of the standard absolute-double normal variable Z for a given positive value z. cdf for negative value of z are obtained by symmetry.

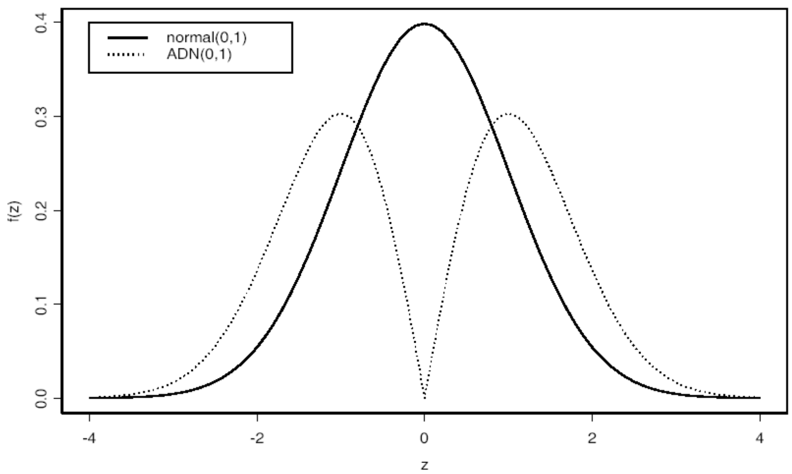

Figure 1. pdf's of standard normal and standard absolute-double normal.

Table 1. Cumulative standard absolute-double normal probabilities $P\,(Z \leq z)$ for positive value z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5000	0.5001	0.5002	0.5004	0.5006	0.5009	0.5012	0.5016	0.5020
0.1	0.5025	0.5030	0.5036	0.5042	0.5049	0.5056	0.5064	0.5072	0.5080	0.5089
0.2	0.5099	0.5109	0.5120	0.5131	0.5142	0.5154	0.5166	0.5179	0.5192	0.5206
0.3	0.5220	0.5235	0.5250	0.5265	0.5281	0.5297	0.5314	0.5331	0.5348	0.5366
0.4	0.5384	0.5403	0.5422	0.5442	0.5461	0.5481	0.5502	0.5523	0.5544	0.5566
0.5	0.5588	0.5610	0.5632	0.5655	0.5678	0.5702	0.5726	0.5750	0.5774	0.5799
0.6	0.5824	0.5849	0.5874	0.5900	0.5926	0.5952	0.5979	0.6005	0.6032	0.6059
0.7	0.6086	0.6114	0.6142	0.6170	0.6198	0.6226	0.6254	0.6283	0.6311	0.6340
0.8	0.6369	0.6398	0.6428	0.6457	0.6486	0.6516	0.6546	0.6575	0.6605	0.6635
0.9	0.6665	0.6695	0.6725	0.6755	0.6786	0.6816	0.6846	0.6876	0.6907	0.6937
1.0	0.6967	0.6998	0.7028	0.7058	0.7089	0.7119	0.7149	0.7179	0.7209	0.7240
1.1	0.7270	0.7300	0.7330	0.7359	0.7389	0.7419	0.7449	0.7478	0.7508	0.7537
1.2	0.7566	0.7595	0.7624	0.7653	0.7682	0.7711	0.7739	0.7768	0.7796	0.7824
1.3	0.7852	0.7880	0.7908	0.7935	0.7963	0.7990	0.8017	0.8044	0.8071	0.8097
1.4	0.8123	0.8150	0.8176	0.8201	0.8227	0.8252	0.8278	0.8303	0.8328	0.8352
1.5	0.8377	0.8401	0.8425	0.8449	0.8472	0.8496	0.8519	0.8542	0.8565	0.8587
1.6	0.8610	0.8632	0.8654	0.8676	0.8697	0.8718	0.8739	0.8760	0.8781	0.8801
1.7	0.8821	0.8841	0.8861	0.8880	0.8900	0.8919	0.8937	0.8956	0.8974	0.8993
1.8	0.9011	0.9028	0.9046	0.9063	0.9080	0.9097	0.9113	0.9130	0.9146	0.9162
1.9	0.9178	0.9193	0.9208	0.9224	0.9238	0.9253	0.9268	0.9282	0.9296	0.9310
2.0	0.9323	0.9337	0.9350	0.9363	0.9376	0.9388	0.9401	0.9413	0.9425	0.9437
2.1	0.9449	0.9460	0.9472	0.9483	0.9494	0.9504	0.9515	0.9525	0.9535	0.9546
2.2	0.9555	0.9565	0.9575	0.9584	0.9593	0.9602	0.9611	0.9620	0.9628	0.9637
2.3	0.9645	0.9653	0.9661	0.9669	0.9676	0.9684	0.9691	0.9699	0.9706	0.9713

2.4	0.9719	0.9726	0.9733	0.9739	0.9745	0.9751	0.9757	0.9763	0.9769	0.9775
2.5	0.9780	0.9786	0.9791	0.9796	0.9801	0.9806	0.9811	0.9816	0.9821	0.9825
2.6	0.9830	0.9834	0.9838	0.9843	0.9847	0.9851	0.9855	0.9858	0.9862	0.9866
2.7	0.9869	0.9873	0.9876	0.9880	0.9883	0.9886	0.9889	0.9892	0.9895	0.9898
2.8	0.9901	0.9904	0.9906	0.9909	0.9911	0.9914	0.9916	0.9919	0.9921	0.9923
2.9	0.9925	0.9928	0.9930	0.9932	0.9934	0.9936	0.9937	0.9939	0.9941	0.9943
3.0	0.9944	0.9946	0.9948	0.9949	0.9951	0.9952	0.9954	0.9955	0.9956	0.9958
3.1	0.9959	0.9960	0.9962	0.9963	0.9964	0.9965	0.9966	0.9967	0.9968	0.9969
3.2	0.9970	0.9971	0.9972	0.9973	0.9974	0.9975	0.9975	0.9976	0.9977	0.9978
3.3	0.9978	0.9979	0.9980	0.9980	0.9981	0.9982	0.9982	0.9983	0.9983	0.9984
3.4	0.9985	0.9985	0.9986	0.9986	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989
3.5	0.9989	0.9989	0.9990	0.9990	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992
3.6	0.9992	0.9993	0.9993	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994
3.7	0.9995	0.9995	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996
3.8	0.9996	0.9996	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997
3.9	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998

The mean, variance and cdf of *ADN* are obtained from the following theorem.

Theorem 2.1. *Suppose that* $X \sim ADN(\mu, \sigma)$. *Then*

(a) *The mean and variance of X are* μ *and* $2\sigma^2$, *respectively*.

(b) *Cumulative distribution function of ADN is*

$$F_X(x) = \begin{cases} \frac{1}{2} - \frac{1}{2} F_T\left(\frac{(x-\mu)^2}{\sigma^2}\right), & x < \mu, \\ \frac{1}{2} + \frac{1}{2} F_T\left(\frac{(x-\mu)^2}{\sigma^2}\right), & x \geq \mu, \end{cases}$$

where $F_T(\cdot)$ *is cdf of the random variable T distributed as exponential distribution with rate* 0.5.

Proof. (a) It is sufficient to show that $E(X - \mu) = 0$. We have

$$E(X - \mu) = \int_{-\infty}^{\infty} (x - \mu) \frac{|x - \mu|}{2\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}(x - \mu)^2\right\} dx$$

$$= \sigma \int_{-\infty}^{\infty} z \frac{|z|}{2} \exp\left\{-\frac{1}{2} z^2\right\} dz = 0,$$

where $z = \frac{x - \mu}{\sigma}$. Because the intergrand is an odd function related to z, for variance, we have

$$E(X-\mu)^2 = \int_{-\infty}^{\infty}(x-\mu)^2\,\frac{|x-\mu|}{2\sigma^2}\exp\left\{\frac{1}{2\sigma^2}(x-\mu)^2\right\}dx$$

$$= \sigma^2\int_{-\infty}^{\infty}\frac{|z|^3}{2}\exp\left\{\frac{1}{2}z^2\right\}dz$$

$$= 2\sigma^2.$$

(b)

$$F_X(u) = \int_{-\infty}^{u}\frac{|x-\mu|}{2\sigma^2}\exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}dx$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\frac{u-\mu}{\sigma}}\frac{|z|}{2}\exp\left\{-\frac{1}{2}z^2\right\}dz, & u < \mu, \\[2ex] \displaystyle\frac{1}{2}+\int_{0}^{\frac{u-\mu}{\sigma}}\frac{|z|}{2}\exp\left\{-\frac{1}{2}z^2\right\}dz, & u \geq \mu \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{2}\int_{\left(\frac{u-\mu}{\sigma}\right)^2}^{\infty}\frac{1}{2}\exp\left\{-\frac{1}{2}y\right\}dy, & u < \mu, \\[2ex] \displaystyle\frac{1}{2}+\frac{1}{2}\int_{0}^{\left(\frac{u-\mu}{\sigma}\right)^2}\frac{1}{2}\exp\left\{-\frac{1}{2}y\right\}dy, & u \geq \mu \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{2}\exp\left\{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2\right\}, & u < \mu, \\[2ex] \displaystyle 1-\frac{1}{2}\exp\left\{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2\right\}, & u \geq \mu. \end{cases}$$

This completes the proof.

The following theorem explains that any linear combination of absolute-double normal distribution is also absolute-double normal.

Theorem 2.2. *Suppose that* $X \sim ADN(\mu, \sigma)$. *Then for* $a \neq 0$, $aX + b \sim ADN(a\mu + b, |a|\sigma)$.

Proof. Straightforward.

Theorem 2.3. *Suppose that* $X \sim ADN(\mu, \sigma)$. *Let* $Z = \frac{X - \mu}{\sigma}$. *Then*

(a) $Z \sim SADN$.

(b) $Z^2 \sim Exp(0.5)$, *where* $Exp(q)$ *denotes exponential distribution with rate* q.

Proof. Proof of (a) is resulted from Theorem 2.2. For (b) use transformation $Y = Z^2$. Then the cdf of Y is

$$F_Y(y) = 2F_z(\sqrt{y}),$$

where $F_Z(\cdot)$ is cdf of *SADN*. Thus pdf of Y is given by

$$f_Y(y) = \frac{f_Z(\sqrt{y})}{\sqrt{y}} = \frac{1}{2}\exp\left\{-\frac{1}{2}y\right\}, \quad y > 0.$$

Moment generating function (mgf) of *SADN* is brought in the following theorem.

Theorem 2.4. *Suppose that* $Z \sim SADN$. *Then*

(a) *mgf of* Z, $M_Z(t)$, *is*

$$M_Z(t) = 1 + \frac{\sqrt{2\pi}}{2} t \exp\left\{\frac{1}{2}t^2\right\}(1 - 2\Phi(-t)).$$

(b) *The qth quantile of Z is*

$$F_Z^{-1}(q) = \begin{cases} -\sqrt{-2\ln 2q}, & 0 < q < 0.5, \\ \sqrt{-2\ln(1-q)}, & 0.5 \le q < 1, \end{cases}$$

where $\Phi(\cdot)$ *is cdf of standard normal distribution*.

Proof. We have

$$M_Z(t) = \int_{-\infty}^{\infty} \exp\{tz\}\frac{1}{2}|z|\exp\left\{-\frac{1}{2}z^2\right\}dz$$

$$= \frac{\sqrt{2\pi}}{2}\exp\left\{\frac{1}{2}t^2\right\}\int_{-\infty}^{\infty}\frac{|z|}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(z-t)^2\right\}dz$$

$$= 1 + \frac{\sqrt{2\pi}}{2} t \exp\left\{\frac{1}{2}t^2\right\}(1 - 2\Phi(-t)).$$

Proof of (b) is resulted from Theorem 2.1. Some programs are provided by the author to generate a sample of size n form $ADN(\mu, \sigma)$ in SPLUS/2000 software. We use these programs in Section 5 for our numerical study.

3. Point Estimation of Parameters

Suppose that $X_1, X_2, ..., X_n$ is a random sample from $ADN(\mu, \sigma)$. The moment estimator of μ and σ^2 are $\overline{X} = \frac{\sum_{i=1}^n X_i}{n}$ and $\frac{\sum_{i=1}^n (X_i - \overline{X})^2}{2n}$, respectively.

The estimates of μ and σ^2 in maximum likelihood (ML) method are not that easy as in the moment method. If μ is known, then the ML estimator (MLE) of σ is given by

$$\hat{\sigma} = \sqrt{\frac{\sum (x_i - \mu)^2}{2n}}. \tag{2}$$

Now, suppose that σ^2 is known, then the MLE of μ can be obtained from

$$\frac{\partial \log L}{\partial \mu} = -\sum_{i=1}^{n} \frac{(x_i - \mu)}{|x_i - \mu|} + \frac{\sum_{i=1}^n (x_i - \mu)}{\sigma^2} = 0, \quad x_i \neq \mu \tag{3}$$

because of

$$\frac{\partial^2 \log L}{\partial \mu^2} = -\frac{n}{\sigma^2} < 0, \tag{4}$$

where $\log L$ is the logarithm of likelihood function. To solve (3), note that for $n = 1$, (3) leads to

$$|x_i - \mu| = \sigma^2$$

and therefore $\hat{\mu}_1 = x_1 - \sigma^2$ and $\hat{\mu}_1 = x_1 + \sigma^2$ both are MLEs of μ. For $n \geq 2$, (3) has at most $2n$ roots, say k roots $\hat{\mu}_1, \hat{\mu}_2, ..., \hat{\mu}_k$. Notice that $\hat{\mu}_j$ $(j = 1, 2, ..., k)$ is locally maximum value. Therefore, the global MLE is

$$\hat{\hat{\mu}} = \arg\max_j \left\{ \prod_{i=1}^{n} \frac{|x_i - \hat{\mu}_j|}{2\sigma^2} \exp\left\{ -\frac{1}{2\sigma^2}(x_i - \hat{\mu}_j)^2 \right\} \right\}.$$

Now, suppose μ and σ^2 are unknown and to be estimated from data $x_1, x_2, \ldots, x_n$. Using $L = \prod_{i=1}^{n} \frac{|x_i - \mu|}{2\sigma^2} \exp\left\{ -\frac{1}{2\sigma^2}(x_i - \mu)^2 \right\}$, $\frac{\partial \log L}{\partial \mu} = 0$ and $\frac{\partial \log L}{\partial \sigma} = 0$, we have

$$\hat{\sigma}^2 = \frac{\sum (x_i - \hat{\mu})^2}{2n} \quad \text{and} \quad \sum \frac{x_i - \hat{\mu}}{|x_i - \hat{\mu}|} = \frac{\sum (x_i - \hat{\mu})}{\hat{\sigma}^2}$$

which are highly correlated. The roots of these equations have to be obtained numerically. Such equations are usually solved by Newton-Raphson iterative process

$$\hat{\boldsymbol{\theta}}^{(k+1)} = \hat{\boldsymbol{\theta}}^{(k)} - \left[\frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j} \right]^{-1}_{\hat{\boldsymbol{\theta}}^{(k)}} \left[\frac{\partial \log L}{\partial \boldsymbol{\theta}} \right]_{\hat{\boldsymbol{\theta}}^{(k)}},$$

where $\hat{\boldsymbol{\theta}}^{(j)} = (\hat{\mu}^{(j)}, \hat{\sigma}^{(j)})'$ is the jth step of estimation of $(\mu, \sigma)'$. This procedure continues to $|\hat{\theta}_i^{(k+1)} - \hat{\theta}_i^{(k)}| < \varepsilon_i$, $i = 1, 2$, where ε_i is an arbitrary small value specified by the researcher.

For our job we have

$$\left[\frac{\partial \log L}{\partial \boldsymbol{\theta}} \right]_{\hat{\boldsymbol{\theta}}^{(k)}} = \left(-\sum \frac{x_i - \hat{\mu}^{(k)}}{|x_i - \hat{\mu}^{(k)}|} + \frac{\sum (x_i - \hat{\mu}^{(k)})}{\hat{\sigma}^{2(k)}}, -\frac{2n}{\hat{\sigma}^{(k)}} + \frac{\sum (x_i - \hat{\mu}^{(k)})^2}{\hat{\sigma}^{3(k)}} \right)'$$

and

$$\left[\frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j} \right]_{\hat{\boldsymbol{\theta}}^{(k)}} = \begin{pmatrix} -\frac{n}{\hat{\sigma}^{2(k)}} & -2\sum_{i=1}^{n} \frac{(x_i - \hat{\mu}^{(k)})}{\hat{\sigma}^{3(k)}} \\ -2\sum_{i=1}^{n} \frac{(x_i - \hat{\mu}^{(k)})}{\hat{\sigma}^{3(k)}} & \frac{2n}{\hat{\sigma}^{2(k)}} - 3\sum_{i=1}^{n} \frac{(x_i - \hat{\mu}^{(k)})^2}{\hat{\sigma}^{4(k)}} \end{pmatrix}.$$

Another method, converges slower but will often give better results, is the method of scoring (see Knight [7] and Stuart et al. [12] for more details). Based on this method

$\frac{\partial^2 \log L}{\partial\theta_i \partial\theta_j}$ is replaced with its expectation, $E\left(\frac{\partial^2 \log L}{\partial\theta_i \partial\theta_j}\right) = (\mathrm{Var}(\hat{\boldsymbol{\theta}}))^{-1}$ and obtain

$$\hat{\boldsymbol{\theta}}^{(k+1)} = \hat{\boldsymbol{\theta}}^{(k)} - [\mathrm{Var}(\hat{\boldsymbol{\theta}})]_{\hat{\boldsymbol{\theta}}^{(k)}} \left[\frac{\partial \log L}{\partial \boldsymbol{\theta}}\right]_{\hat{\boldsymbol{\theta}}^{(k)}}.$$

In our case

$$\left[\frac{\partial \log L}{\partial \theta}\right]_{\hat{\theta}^{(k)}} = \left(-\sum \frac{x_i - \hat{\mu}^{(k)}}{|x_i - \hat{\mu}^{(k)}|} + \frac{\sum (x_i - \hat{\mu}^{(k)})}{\hat{\sigma}^{2(k)}}, -\frac{2n}{\hat{\sigma}^{(k)}} + \frac{\sum (x_i - \hat{\mu}^{(k)})^2}{\hat{\sigma}^{3(k)}}\right)'$$

and

$$[\mathrm{Var}(\hat{\boldsymbol{\theta}})]_{\hat{\boldsymbol{\theta}}^{(k)}} = \begin{pmatrix} \frac{\hat{\sigma}^{2(k)}}{2n} & 0 \\ 0 & \frac{\hat{\sigma}^{2(k)}}{4n} \end{pmatrix}.$$

These methods need the initial value, $\hat{\boldsymbol{\theta}}^{(0)} = (\hat{\mu}^{(0)}, \hat{\sigma}^{(0)})'$. One appropriate suggestion can be estimators μ and σ^2 based on moment method, i.e., $\hat{\boldsymbol{\theta}}^{(0)} = \left(\bar{x}, \sqrt{\frac{\sum (x_i - \bar{x})^2}{2n}}\right)'$. One important problem in MLE's is their asymptotic properties. For large n, we have

$$\begin{pmatrix} \hat{\mu} \\ \hat{\sigma} \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu \\ \sigma \end{pmatrix}, \begin{pmatrix} \frac{\sigma^2}{2n} & 0 \\ 0 & \frac{\sigma^2}{4n} \end{pmatrix}\right). \tag{5}$$

4. Confidence Interval of Parameters

In this section, confidence interval (CI) for μ and σ are studied. When n is large, approximately CI's at level $(1 - \alpha)$ for μ and σ are obtained from (5) as

$$\left(\hat{\mu} - z_{1-\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{2n}}, \hat{\mu} + z_{1-\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{2n}}\right) \text{ and } \left(\frac{\sqrt{4n}\hat{\sigma}}{\sqrt{4n} + z_{1-\frac{\alpha}{2}}}, \frac{\sqrt{4n}\hat{\sigma}}{\sqrt{4n} - z_{1-\frac{\alpha}{2}}}\right),$$

respectively, where z_q is the qth quantile of standard normal distribution. Frequently for small n, distribution of the estimators are complicated.

At first we assume that μ is known. In this case an explicit CI can be drawn for σ using the following theorem.

Theorem 4.1. *Suppose* $X_1, X_2, \ldots, X_n$ *is a random sample from* $ADN(\mu, \sigma)$. *Then*

$$\frac{\sum (X_i - \mu)^2}{\sigma^2} \sim \chi^2_{(2n)}. \tag{6}$$

Proof. Proof can be done from $\frac{X_i - \mu}{\sigma} \sim SADN$ and independence of X_i's.

The pivotal in (6) leads to the following CI for σ

$$\left(\frac{\sqrt{2n}\hat{\sigma}}{\sqrt{\chi^2_{(2n, 1-\frac{\alpha}{2})}}}, \; \frac{\sqrt{2n}\hat{\sigma}}{\sqrt{\chi^2_{(2n, \frac{\alpha}{2})}}} \right),$$

where $\hat{\sigma}$ is MLE of σ resulted in (2) and $\chi^2_{(k, q)}$ is the qth quantile of χ^2_k. In the case of known σ, we can use the pivotal value

$$\begin{aligned} \Lambda &= \frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \\ &= \sqrt{n}\,\frac{\sum \left(\frac{X_i - \mu}{\sigma} \right)}{n} = \sqrt{n}\overline{Z}, \end{aligned} \tag{7}$$

where $\overline{Z} = \frac{\sum_{i=1}^{n} Z_i}{n}$, $Z_i = \frac{X_i - \mu}{\sigma} \sim SADN$. It seems that distribution of Λ is symmetric about zero, approximately (see Table 2). One CI for μ based on Λ is given by $\left(\overline{X} - \Lambda_{(n, 1-\frac{\alpha}{2})} \frac{\sigma}{\sqrt{n}}, \overline{X} + \Lambda_{(n, 1-\frac{\alpha}{2})} \frac{\sigma}{\sqrt{n}} \right)$, where $\Lambda_{(n, q)}$ is the qth quantile of Λ based on random sample of size $n > 1$. This quantile can be obtained by numerical study (see Table 2).

Table 2. The simulated quantiles of distribution of Λ

n	0.10	0.05	0.025	0.01	0.005	0.995	0.99	0.975	0.95	0.90
2	–1.92	–2.33	–2.68	–3.02	–3.31	3.27	3.01	2.64	2.30	1.88
3	–1.86	–2.37	–2.75	–3.16	–3.41	3.36	3.11	2.73	2.35	1.87
4	–1.82	–2.30	–2.74	–3.18	–3.52	3.45	3.11	2.71	2.33	1.83
5	–1.84	–2.35	–2.76	–3.25	–3.55	3.54	3.25	2.75	2.37	1.87
6	–1.84	–2.34	–2.75	–3.27	–3.51	3.44	3.14	2.73	2.30	1.80
7	–1.86	–2.36	–2.77	–3.29	–3.63	3.60	3.28	2.78	2.38	1.80
8	–1.87	–2.38	–2.75	–3.19	–3.54	3.57	3.21	2.75	2.32	1.41
9	–1.83	–2.35	–2.75	–3.24	–3.59	3.55	3.21	2.71	2.26	1.76
10	–1.80	–2.33	–2.75	–3.27	–3.66	3.57	3.26	2.74	2.28	1.78
20	–1.82	–2.31	–2.72	–3.16	–3.46	3.59	3.15	2.72	2.31	1.82
30	–1.82	–2.30	–2.75	–3.22	–3.48	3.58	3.24	2.72	2.28	1.81
40	–1.82	–2.30	–2.76	–3.28	–3.58	3.63	3.25	2.76	2.33	1.80
50	–1.84	–2.35	–2.77	–3.34	–3.75	3.68	3.26	2.75	2.30	1.82

For the case of unknown σ, we introduce the new pivotal value based on sample mean and sample standard deviation, $S = \sqrt{\frac{\sum (X_i - \bar{X})^2}{n-1}}$, as

$$\Lambda' = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\frac{1}{n}\sum (X_i - \mu)}{\sqrt{\frac{\sum (X_i - \bar{X})^2}{n(n-1)}}}$$

$$= \frac{\frac{1}{n}\sum\left(\frac{x_i - \mu}{\sigma}\right)}{\sqrt{\frac{\sum\left(\left(\frac{X_i - \mu}{\sigma}\right) - \left(\frac{\bar{X} - \mu}{\sigma}\right)\right)^2}{n(n-1)}}} = \frac{\bar{Z}}{\sqrt{\frac{\sum (Z_i - \bar{Z})^2}{n(n-1)}}}. \tag{8}$$

Note that $\frac{S^2}{2}$ is an unbiased estimator of σ^2. The distribution of this pivotal value is also complicated but independent of μ and σ parameters. The qth quantile of Λ', $\Lambda'_{(n,q)}$, based on random sample of size $n > 1$ is obtained by numerical study (see Table 3). It seems that the distribution of Λ' will also be symmetric about zero from Table 3. However, in this case one CI for μ is given by

$$\left(\bar{X} - \Lambda'_{(n, 1-\frac{\alpha}{2})}\frac{S}{\sqrt{n}},\ \bar{X} + \Lambda'_{(n, 1-\frac{\alpha}{2})}\frac{S}{\sqrt{n}}\right).$$

Table 3. The simulated quantiles of distribution of Λ'

n	0.10	0.05	0.025	0.01	0.005	0.995	0.99	0.975	0.95	0.90
2	−4.68	−9.58	−20.00	−54.65	−140.11	113.40	54.18	22.00	11.08	5.11
3	−2.47	−4.26	−6.24	−10.11	−14.52	14.73	10.62	6.40	4.29	2.69
4	−1.53	−3.04	−4.55	−6.59	−9.42	9.44	7.14	4.61	3.07	1.56
5	−1.55	−2.21	−3.80	−5.59	−6.86	6.90	5.42	3.63	2.18	1.52
6	−1.52	−2.15	−2.76	−4.24	−5.42	5.76	4.46	2.74	2.09	1.45
7	−1.41	−1.94	−2.55	−3.51	−4.85	4.79	3.70	2.68	2.06	1.48
8	−1.40	−1.90	−2.40	−3.18	−3.81	3.91	3.30	2.51	1.95	1.43
9	−1.38	−1.90	−2.42	−3.18	−3.87	3.69	3.14	2.43	1.90	1.36
10	−1.36	−1.81	−2.24	−2.93	−3.53	3.58	2.94	2.34	1.84	1.35
20	−1.32	−1.74	−2.11	−2.65	−2.99	2.81	2.53	2.10	1.71	1.33
30	−1.33	−1.69	−2.02	−2.42	−2.79	2.82	2.45	2.03	1.68	1.29
40	−1.30	−1.65	−2.04	−2.43	−2.73	2.63	2.36	2.01	1.66	1.25
50	−1.30	−1.71	−2.05	−2.50	−2.74	2.61	2.36	2.00	1.67	1.30

We can also introduce the following pivotal value to obtain CI for σ

$$Q = \frac{(n-1)S^2}{\sigma^2} = \sum\left(\left(\frac{X_i - \mu}{\sigma}\right) - \left(\frac{\overline{X} - \mu}{\sigma}\right)\right)^2 = \sum (Z_i - \overline{Z})^2.$$

The distribution of Q is also complicated but independent of μ and σ parameters. Based on Q the following CI for σ is introduced

$$\left(\frac{\sqrt{n-1}S}{\sqrt{Q_{(n, 1-\frac{\alpha}{2})}}},\ \frac{\sqrt{n-1}S}{\sqrt{Q_{(n, \frac{\alpha}{2})}}}\right),$$

where $Q_{(n,q)}$ denotes the qth quantile of Q which is obtained by numerical study (see Table 4).

Table 4. The simulated quantiles of distribution of Q

n	0.10	0.05	0.025	0.01	0.005	0.995	0.99	0.975	0.95	0.90
2	0.0263	0.0070	0.0019	0.0004	0.0001	11.67	10.54	8.728	7.175	5.523
3	0.4142	0.1868	0.0924	0.0394	0.0206	15.889	14.185	11.977	10.178	8.481
4	1.397	0.7034	0.4059	0.2066	0.1177	19.65	17.73	14.83	12.93	10.95
5	2.933	1.854	1.129	0.643	0.380	22.30	20.96	18.27	16.26	14.05
6	4.446	3.139	2.192	1.295	0.915	25.76	23.80	21.12	19.07	16.52
7	5.878	4.544	3.517	2.147	1.424	28.61	26.59	23.70	21.34	18.87
8	7.428	6.015	4.755	3.427	2.542	31.57	29.51	26.83	24.17	21.53
9	8.936	7.393	6.155	4.650	3.676	34.56	32.59	29.32	26.92	23.97
10	10.50	8.781	7.479	6.040	4.983	37.16	35.23	31.73	29.07	26.35
20	27.00	24.49	22.25	19.40	18.04	64.69	62.46	57.71	53.93	50.01
30	44.06	40.85	37.90	34.97	32.67	90.24	86.42	80.85	77.20	72.33
40	62.25	58.38	55.02	51.23	48.84	114.1	110.1	104.4	99.73	94.58
50	125.17	119.57	115.12	108.81	106.17	194.66	190.43	182.89	176.61	170.05

5. Quantiles of Λ, Λ' and Q

A simulation study has been carried out to find the quantiles of the distribution of Λ, Λ' and Q using SPLUS/2000 software. For every n, the following algorithm was used:

Step 1. One sample of size n is generated from $Z \sim SADN$.

Step 2. Λ, Λ' and Q were calculated. Both steps were repeated K times. Then αth quantiles are obtained. Different sample sizes, n, have been considered. The results are given in Tables 2, 3 and 4 for Λ, Λ' and Q, respectively $(K = 10000)$. In conclusion we evaluate the results by random sample of size 50 from $ADN(40, 5)$ using the provided programs in SPLUS (see Section 4).

34.50, 44.10, 47.61, 47.82, 52.42, 33.11, 31.75, 44.42, 44.58, 32.89
33.94, 47.65, 39.19, 46.64, 45.47, 30.78, 34.90, 45.78, 43.59, 47.61
48.49, 21.95, 34.58, 33.40, 27.86, 44.34, 37.98, 37.73, 22.50, 30.34
41.00, 44.83, 44.85, 33.46, 28.43, 32.56, 30.07, 36.32, 47.79, 25.16
46.65, 45.11, 43.52, 31.32, 35.54, 35.79, 46.15, 49.68, 47.65, 44.16

Summary of data is $\bar{x} = 39.16$ and $S^2 = 61.19$ $(S = 7.82)$.

Thus the estimates of μ and σ in the moment method are 39.16 and 5.53. MLE's of μ and σ are 39.34 and 5.13 with 3 iterations and start point, moment estimations. One CI for μ at level 95% is

$$\left(39.16 - \frac{7.82}{\sqrt{50}} \times 2.00,\ 39.16 + \frac{7.82}{\sqrt{50}} \times 2.00\right) = (36.95,\ 41.37).$$

Note that from Table 3, we have $\Lambda'_{(50,\, 0.975)} = 2.00$. At level 95% one CI for σ based on Q is given by (4.05, 5.10).

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