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ALGEBRAIC PROPERTIES OF THE DISTANCE BETWEEN SOBOLEV SPACES

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Abstract

In this paper, it is shown that the value of the distance between the standard Sobolev spaces $W^{1,2}(\mathbb{R})$ and $W^{2,2}(\mathbb{R})$ is an irrational algebraic number.

1. Introduction

Sobolev space is de Branges space. Since de Branges space is deeply concerned with an operator theory, there is a new approach to Sobolev spaces by operational methods of de Branges space. Indeed, the standard Sobolev space $W^{m,2}(\mathbb{R})$ $(m \geq 1)$ in $L^2(\mathbb{R})$ is isometrically isomorphic to de Branges space $\mathcal{M}(A_m)$ for a positive contraction $A_m = (I + \mathcal{D}_1^2 + \cdots + \mathcal{D}_1^{2m})^{-\frac{1}{2}}$ on $L^2(\mathbb{R})$, where $\mathcal{D}_1$ is a differential operator $\frac{1}{i}\frac{d}{dx}$. Further, the Fourier type Sobolev space $H^{\sigma}(\mathbb{R}^N)$ $(N \geq 1)$ is isometrically isomorphic to de Branges space $\mathcal{M}(\tilde{A}_\sigma)$ for the Bessel potential

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$\tilde{A}_\sigma = (I - \Delta)^{-\frac{\sigma}{2}}$ on $L^2(\mathbb{R}^N)$ with the order $\sigma > 0$. Using these operators A_m and $\tilde{A}_\sigma$, we can define the metric (we call the ρ-*metric*) among Sobolev spaces, and based on them, we further define the metric (we call the *q-metric*) among differential operators whose domains are Sobolev spaces ([2]).

The ρ-metric can be defined not only for Sobolev spaces, but also for more general situations, the set of all semiclosed subspaces in any Hilbert space *H*. Thus, corresponding to their general cases of the ρ-metric, the *q*-metric also can be defined not only for differential operators, but also for the set $\mathcal{S}(H)$ of all semiclosed operators in *H*. The set $\mathcal{S}(H)$ is larger than the set $\mathcal{CD}(H)$ of all closed and densely defined operators in *H*. That is, $\mathcal{S}(H) \supset \mathcal{CD}(H) \supset \mathcal{B}(H)$, where $\mathcal{B}(H)$ is the set of all bounded operators in *H*. The sums, the products of closed operators are not closed in general, but necessarily semiclosed. Therefore, it is significant to consider the relations between $\mathcal{CD}(H)$ and $\mathcal{S}(H)$. In this paper [2], it is shown that $\mathcal{CD}(H)$ is an open set in $\mathcal{S}(H)$ with respect to the *q*-metric. Hence the set $\mathcal{CD}(H)$ is stable to the small perturbation with respect to the *q*-metric. Moreover, it is shown that $\mathcal{B}(H)$ is a connected component in $\mathcal{S}(H)$ with respect to the same metric. The *q*-metric is so natural in a sense. Because, on $\mathcal{B}(H)$, the *q*-metric coincides with the usual metric induced from the operator norm on it. Therefore, it is important to handle the ρ-metric which plays a role of the distance between domains of given two semiclosed operators.

In this paper, we focus on the special cases of the ρ-metric for Sobolev spaces which are domains of differential operators. Especially, we study the algebraic properties of the distance between the standard Sobolev spaces $W^{1,2}(\mathbb{R})$ and $W^{2,2}(\mathbb{R})$. We shall show that the value of the distance between them is the irrational and the algebraic number.

2. Sobolev Space is de Branges Space

Let *H* be an infinite dimensional, complex Hilbert space. We denote by $(\cdot, \cdot)$ the original inner product in *H* and put $\|\cdot\| := (\cdot, \cdot)^{\frac{1}{2}}$. For a linear operator *T* defined in *H*, denote $\mathrm{dom}(T)$ and $\mathrm{ran}(T)$ by the domain and the range of *T*, respectively.

Throughout this paper, operator means linear operator. Let $\mathcal{B}(H)$ be the set of all bounded operators with their domains H. Then $S \in \mathcal{B}(H)$ is said to be *positive*, in short $S \geq 0$, if $(Su, u) \geq 0$ for all $u \in H$. For a subspace M in H, M is said to be *semiclosed* in H if there exists an inner product $(\cdot, \cdot)_M$ on M such that M is a complete inner product space with respect to $(\cdot, \cdot)_M$ and that the inclusion mapping $J : (M, \|\cdot\|_M) \to H$ is continuous with respect to the norm $\|\cdot\|_M$ induced by $(\cdot, \cdot)_M$. That is, $\|u\| \leq c\|u\|_M$, $u \in M$ for some $c > 0$. When the inclusion mapping J is continuous, we simply write $J : (M, \|\cdot\|_M) \hookrightarrow H$. Clearly, any closed subspace is semiclosed. A semiclosed subspace is an operator range of some element in $\mathcal{B}(H)$. For if M is a semiclosed subspace in H, then it is the operator range of the positive bounded operator $(JJ^*)^{\frac{1}{2}}$, that is, $M = (JJ^*)^{\frac{1}{2}} H$ for the inclusion mapping $J : (M, \|\cdot\|_M) \hookrightarrow H$. Conversely, if M is an operator range for some $S \in \mathcal{B}(H)$, i.e., $M = SH$, then the inner product $(\cdot, \cdot)_M$ defined by

$$(Su, Sv)_M := (u, v), \quad u, v \in (\ker S)^{\perp}$$

gives Hilbert space structures for $M = SH$ so that $(M, \|\cdot\|_M) \hookrightarrow H$. Therefore, M is semiclosed.

Definition. For $S \in \mathcal{B}(H)$, we define the inner product $(\cdot, \cdot)_S$ on the operator range SH by $(Su, Sv)_S := (u, v)u$, $v \in (\ker S)^{\perp}$. Then $(SH, (\cdot, \cdot)_S)$ is a complete inner product space, that is, a Hilbert space and $(SH, (\cdot, \cdot)_S) \hookrightarrow H$. We call $(SH, (\cdot, \cdot)_S)$ the *de Branges space* induced by S and denote it by $\mathcal{M}(S)$.

A connection between semiclosed subspaces and de Branges spaces is given by the next proposition.

Proposition 2.1 ([1]). *Let $M \subset H$ be a semiclosed subspace and let $\|\cdot\|_M$ be a Hilbert norm on M such that $J : (M, \|\cdot\|_M) \hookrightarrow H$. Then there uniquely exists a positive bounded operator $A \in \mathcal{B}(H)$ such that*

$$(M, \|\cdot\|_M) = \mathcal{M}(A) \ (\textit{isometrically isomorphic}).$$

In this case, A is given by $(JJ^*)^{\frac{1}{2}} \geq 0$.

As an example, let $H := L^2(\mathbb{R})$ and $M := \mathrm{dom}(\mathcal{D}_1^m)$ which stands for the domain of $\mathcal{D}_1^m$ for integers $m \geq 1$. $\mathcal{D}_1 := \frac{1}{i}\frac{d}{dx} : \mathrm{dom}(\mathcal{D}_1) \to L^2(\mathbb{R})$ is a (selfadjoint) differential operator in a weak derivative sense. Here, a subspace

$$\mathrm{dom}(\mathcal{D}_1^m) := \{f \in L^2(\mathbb{R}) : \mathcal{D}_1 f, \ldots, \mathcal{D}_1^m f \in L^2(\mathbb{R})\}$$

is a Hilbert space with the standard Hilbert structure

$$\| f \|_{W^{m,2}} := (\| f \|^2 + \| \mathcal{D}_1 f \|^2 + \cdots + \| \mathcal{D}_1^m f \|^2)^{\frac{1}{2}}.$$

Clearly,

$$(\mathrm{dom}(\mathcal{D}_1^m), \| \cdot \|_{W^{m,2}}) \hookrightarrow L^2(\mathbb{R}).$$

Therefore, we see that $\mathrm{dom}(\mathcal{D}_1^m)$ is a semiclosed subspace in $L^2(\mathbb{R})$ and we call $W^{m,2}(\mathbb{R}) := (\mathrm{dom}(\mathcal{D}_1^m), \| \cdot \|_{W^{m,2}})$ the *standard Sobolev spaces* with the order m. Let the inclusion mapping

$$J_m : W^{m,2}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \quad (m \geq 1).$$

Then it follows from Proposition 2.1 that there exists a unique positive bounded operator $A_m = (J_m J_m^*)^{\frac{1}{2}} \in \mathcal{B}(H)$ such that $W^{m,2}(\mathbb{R}) = \mathcal{M}(A_m)$ in the sense of isometrically isomorphic. In this case, we define the real valued function by

$$\rho(W^{k,2}(\mathbb{R}), W^{l,2}(\mathbb{R})) := \| A_k - A_l \| \quad (k, l \geq 1),$$

where $\| \cdot \|$ means the operator norm in $\mathcal{B}(L^2(\mathbb{R}))$. It is clear that the function ρ is a metric among the standard Sobolev spaces. But it is difficult to calculate the distance $\| A_k - A_l \|$ for the form $(J_m J_m^*)^{\frac{1}{2}}$ $(= A_m)$.

Now on, we construct another form of A_m for available treatment of calculations. The next lemma is essentially proved by Kaufman [4].

Lemma 2.2. *Let T be a closed and densely defined operator in a Hilbert space H. Then* $(I + T^*T)^{-\frac{1}{2}} \in \mathcal{B}(H)$ *and* $\mathrm{dom}(T) = (I + T^*T)^{-\frac{1}{2}} H$. *Further, the graph norm of T and de Branges norm of* $(I + T^*T)^{-\frac{1}{2}}$ *are equal, that is,*

$$(\operatorname{dom}(T), \|\cdot\|_{graph}) = \mathcal{M}((I + T^*T)^{-\frac{1}{2}}) \ \textit{(isometrically isomorphic)}.$$

Proof. It is shown in [4] that a closed and densely defined operator T in H is represented by a quotient $T = B/(I - B^*B)^{\frac{1}{2}}$ for a unique pure contraction $B(= T(I + T^*T)^{-\frac{1}{2}})$ such that $\operatorname{dom}(T) = (I - B^*B)^{\frac{1}{2}}H$ and $\operatorname{ran}(T) = BH$.

In general, under the kernel condition $\ker E \subseteq \ker F$ for $E, F \in \mathcal{B}(H)$, a mapping $Eu \to Fu$ for $u \in H$ is called a *quotient* of bounded operators E and F. We denote it by F/E. Since $(I + T^*T)^{-\frac{1}{2}} = (I - B^*B)^{\frac{1}{2}}$, we see that $\operatorname{dom}(T) = (I + T^*T)^{-\frac{1}{2}}H$. Hence it is sufficient to show that the norm condition $\|\cdot\|_{\text{graph}} = \|\cdot\|_{(I+T^*T)^{-\frac{1}{2}}}$. For any $f \in \operatorname{dom}(T)$ and $f = (I - B^*B)^{\frac{1}{2}}u$, $u \in H$,

$$\| f \|_{\text{graph}}^2 := \| f \|^2 + \| Tf \|^2 = \| (I - B^*B)^{\frac{1}{2}}u \| + \| Bu \|^2$$

$$= \| u \|^2 = \| (I - B^*B)^{\frac{1}{2}}u \|^2_{(I-B^*B)^{\frac{1}{2}}} = \| f \|^2_{(I+T^*T)^{-\frac{1}{2}}}. \qquad \square$$

To apply Lemma 2.2, we remark that the standard Sobolev norm $\|\cdot\|_{W^{m,2}(\mathbb{R})}$ is a graph norm of some differential operator. Now we explain this. For $f \in W^{m,2}(\mathbb{R})$,

$$\| f \|^2_{W^{m,2}(\mathbb{R})} = \| f \|^2 + \| \mathcal{D}_1 f \|^2 + \cdots + \| \mathcal{D}_1^m f \|^2$$

$$= \| f \|^2 + \| (\mathcal{D}_1^2 + \cdots + \mathcal{D}_1^{2m})^{\frac{1}{2}} f \|^2$$

$$= \| f \|^2_{\text{graph}} \ (\text{the graph norm of } (\mathcal{D}_1^2 + \cdots + \mathcal{D}_1^{2m})^{\frac{1}{2}}).$$

Hence the standard Sobolev norm is equal to the graph norm of $(\mathcal{D}_1^2 + \cdots + \mathcal{D}_1^{2m})^{\frac{1}{2}}$, $\|\cdot\|_{W^{m,2}(\mathbb{R})} = \|\cdot\|_{\text{graph}}$. Let T in Lemma 2.2 be $\left(\sum_{k=1}^m \mathcal{D}_1^{2k}\right)^{\frac{1}{2}}$. Then, since $T =$

$\left(\sum_{k=1}^{m}\mathcal{D}_1^{2k}\right)^{\frac{1}{2}}$ is closed (in fact, selfadjoint) operator with the domain $\mathrm{dom}(\mathcal{D}_1^m)$ which is dense in $L^2(\mathbb{R})$, it follows from Lemma 2.2 that

$$(\mathrm{dom}(T), \|\cdot\|_{\mathrm{graph}}) = W^{m,2}(\mathbb{R}) = \mathcal{M}((I + \mathcal{D}_1^2 + \cdots + \mathcal{D}_1^{2m})^{-\frac{1}{2}})$$

in a sense of isometrically isomorphic. Thus, by the uniqueness condition of positivity in Proposition 2.1, we have

Proposition 2.3 ([2, Lemma 3.2]). *Let* $W^{m,2}(\mathbb{R})\,(m \geq 1)$ *be the standard Sobolev spaces* $(J_m : W^{m,2}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}))$. *Then there exists a unique positive bounded operator* $A_m \in \mathcal{B}(H)$ *such that* $W^{m,2}(\mathbb{R})$ *is isometrically isomorphic to de Branges space* $\mathcal{M}(A_m)$ *for*

$$A_m = (J_m J_m^*)^{\frac{1}{2}} = (I + \mathcal{D}_1^2 + \cdots + \mathcal{D}_1^{2m})^{-\frac{1}{2}} \geq 0. \tag{2.1}$$

3. The Metric Calculations for $W^{1,2}(\mathbb{R})$ and $W^{2,2}(\mathbb{R})$

We shall calculate the metric $\rho(W^{1,2}(\mathbb{R}), W^{2,2}(\mathbb{R}))$ ([2, Example 4.1]). By Proposition 2.3,

$$\begin{aligned}\rho(W^{1,2}(\mathbb{R}), W^{2,2}(\mathbb{R})) = \| A_1 - A_2 \| &= \sup_{g\in L^2, \|g\|\leq 1} \| A_1 g - A_2 g \| \\ &= \sup_{g\in L^2, \|g\|\leq 1} \| (I + \mathcal{D}_1^2)^{-\frac{1}{2}} g - (I + \mathcal{D}_1^2 + \mathcal{D}_1^4)^{-\frac{1}{2}} g \| \\ &= \sup_{\hat{g}\in L^2, \|\hat{g}\|\leq 1} \| (1 + \xi^2)^{-\frac{1}{2}} \hat{g} - (1 + \xi^2 + \xi^4)^{-\frac{1}{2}} \hat{g} \| \\ &= \| (1 + \xi^2)^{-\frac{1}{2}} - (1 + \xi^2 + \xi^4)^{-\frac{1}{2}} \|_\infty. \end{aligned} \tag{3.1}$$

In the above equations, we use the L^2-Fourier transform by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx, \quad f \in L^2(\mathbb{R}).$$

We can see that the final term (3.1) is about $0.229365 \cdots$. We have some questions. Is it rational or irrational? Is it an algebraic number or not? In general, a complex number is said to be *algebraic* if it is a solution of a polynomial with integer coefficients. From now, we consider this problem. Let

$$S(\xi) := \frac{1}{\sqrt{1+\xi^2}} - \frac{1}{\sqrt{1+\xi^2+\xi^4}} \quad (\geq 0) \quad \text{for} \quad \xi \in \mathbb{R}.$$

Since we are interested in the maximum of the function $S(\xi)$, it is sufficient to consider the case for $\xi > 0$. To obtain the maximal value, we consider the derivative function

$$S'(\xi) = \frac{-\xi}{(1+\xi^2)\sqrt{1+\xi^2}} + \frac{\xi(1+2\xi^2)}{(1+\xi^2+\xi^4)\sqrt{1+\xi^2+\xi^4}}$$

$$= \frac{-\xi(1+\xi^2+\xi^4)^{\frac{3}{2}} + \xi(1+2\xi^2)(1+\xi^2)^{\frac{3}{2}}}{(1+\xi^2)(1+\xi^2+\xi^4)\sqrt{1+\xi^2}\sqrt{1+\xi^2+\xi^4}},$$

so that we have $S'(\xi) = 0$ if and only if $(1+2\xi^2)(1+\xi^2)^{\frac{3}{2}} - (1+\xi^2+\xi^4)^{\frac{3}{2}} = 0$. Here let $t := 1+\xi^2 (> 1)$. Then this is equivalent to the equation

$$t^6 - 7t^5 + 10t^4 - 8t^3 + 6t^2 - 3t + 1 = 0$$

or

$$(t-1)(t^5 - 6t^4 + 4t^3 - 4t^2 + 2t - 1) = 0.$$

Hence, we have

$$P(t) := t^5 - 6t^4 + 4t^3 - 4t^2 + 2t - 1 = 0. \tag{3.2}$$

By the way, we can see that the equation $P(t) = 0$ has the only one real solution t_0 $(5 < t_0 < 6)$ with the help of computational methods. Since we now consider on the domain $t > 1$, the solution t_0 $(P(t_0) = 0)$ is important for us. It is not in $\mathbb{Q}$ (equivalently, is not in $\mathbb{Z}$) by the monic property of $P(t)$. Let $\xi_0 > 0$ such that $t_0 = 1 + \xi_0^2$. Then $S(\xi)$ attains the maximal value at ξ_0. Therefore,

$$\rho(W^{1,2}(\mathbb{R}), W^{2,2}(\mathbb{R})) = S(\xi_0) = \frac{1}{\sqrt{1+\xi_0^2}} - \frac{1}{\sqrt{1+\xi_0^2+\xi_0^4}}$$

$$= \frac{1}{\sqrt{t_0}} - \frac{1}{\sqrt{t_0^2 - t_0 + 1}}. \tag{3.3}$$

We shall show that the value (3.3) is the irrational number. Suppose that it is rational, that is, irreducible fraction n/m (m and n are positive integers):

$$\frac{1}{\sqrt{t_0}} - \frac{1}{\sqrt{t_0^2 - t_0 + 1}} = \frac{n}{m}.$$

Then

$$\frac{\sqrt{t_0^2 - t_0 + 1} - \sqrt{t_0}}{\sqrt{t_0(t_0^2 - t_0 + 1)}} = \frac{n}{m} \quad \text{or} \quad \frac{t_0^2 + 1 - 2\sqrt{t_0(t_0^2 - t_0 + 1)}}{t_0(t_0^2 - t_0 + 1)} = \left(\frac{n}{m}\right)^2.$$

Thus,

$$-n^2 t_0^3 + (m^2 + n^2)t_0^2 - n^2 t_0 + m^2 = 2m^2\sqrt{t_0(t_0^2 - t_0 + 1)}.$$

Squaring both sides, we have the following which is in 6-degree with respect to t_0:

$$n^4 t_0^6 - 2n^2(m^2 + n^2)t_0^5 + \{(m^2 + n^2)^2 + 2n^4\}t_0^4 - 2\{(m^2 + n^2)^2 + m^4\}t_0^3$$

$$+ \{(m^2 + n^2)^2 + 5m^4\}t_0^2 - 2m^2(n^2 + 2m^2)t_0 + m^4 = 0. \tag{3.4}$$

We replace t_0 by t in the left side of (3.4), and let be $Q(t)$. We divide $Q(t)$ by $P(t)$, then the remainder $R(t)$ is 4-degree polynomial on $\mathbb{Q}$ and $R(t_0) = 0$. Further, we divide $P(t)$ by $R(t)$, then the remainder $R_1(t)$ is 3-degree polynomial on $\mathbb{Q}$ and $R_1(t_0) = 0$. Furthermore, we divide $R(t)$ by $R_1(t)$, then the remainder $R_2(t)$ is 2-degree polynomial on $\mathbb{Q}$ and $R_2(t_0) = 0$. We continue the process once again and getting the remainder $R_3(t)$ is 1-degree polynomial on $\mathbb{Q}$ and $R_3(t_0) = 0$. This means that t_0 is in $\mathbb{Q}$. This is a contradiction. Therefore, the value of the distance (3.3) is the irrational number.

Moreover, we consider the next problem which is whether the distance is an algebraic number or not. The set of algebraic numbers is a field and is closed operations of square root and inverse. Since t_0 is the algebraic number by (3.2), we can see that the distance $\rho(W^{1,2}(\mathbb{R}), W^{2,2}(\mathbb{R}))$ is the algebraic number by (3.3).

Therefore, we obtain the next theorem.

Theorem 3.1. *The distance* $\rho(W^{1,2}(\mathbb{R}), W^{2,2}(\mathbb{R}))$ *is the irrational number and the algebraic number*.

We do not know that whether the distance between another order cases is the irrational number. In other words, is $\rho(W^{k,2}(\mathbb{R}), W^{l,2}(\mathbb{R}))$ for $k \geq 2,\ l \geq 2\ (k \neq l)$ an irrational and an algebraic number?

Remark 3.1. If we replace the standard Sobolev norm $\|\cdot\|_{W^{j,2}(\mathbb{R})}$ $(j = 1, 2)$ for the Sobolev norm of Fourier type $\|\cdot\|_{H^\sigma}$ $(\sigma = 1, 2)$ defined by

$$\| f \|_{H^\sigma} := \| (1 + |\xi|^2)^{\frac{\sigma}{2}} \hat{f} \|, \quad f \in H^\sigma(\mathbb{R}) \ \ (\sigma > 0),$$

then the distance $\rho(H^1(\mathbb{R}), H^2(\mathbb{R})) = 0.25$ which is a rational number ([2]).

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