Internat. J. Functional Analysis, Operator Theory and Applications
Volume 2, Number 2, 2010, Pages 125-132
Published Online: February 12, 2011
This paper is available online at http://pphmj.com/journals/ijaota.htm

COMPOSITION OPERATORS OF (α, β)-NORMAL OPERATORS

D. SENTHILKUMAR and S. M. SHERINJOY

Post Graduate and Research Department of Mathematics
Govt. Arts College
Coimbatore-641 018
Tamilnadu, India
e-mail: sherinjoy@yahoo.com

Abstract

In this paper, composition operators and weighted composition operators of (α, β)-normal operators on L^2 space and composition operators of (α, β)-normal operators on general weighted Hardy Spaces are characterised.

1. Preliminaries

Let (X, Σ, λ) be a sigma-finite measure space and T be a non-singular measurable transformation from X onto itself. Composition transformation C on $L^2(\lambda)$ induced T is given by $Cf = f \circ T$ for every f in $L^2(\lambda)$. If C is bounded, we call C to be a *composition operator* on $L^2(\lambda)$. It is known that T induces a bounded composition operator C on $L^2(\lambda)$ if and only if the measure λT^{-1} is absolutely

2010 Mathematics Subject Classification: 47B33, 47B20.

Keywords and phrases: hyponormal composition operators, *m*-hyponormal composition operators, (α, β)-normal operators.

Communicated by B. P. Duggal, Editor of Far East Journal of Mathematical Sciences (FJMS)
Published by Pushpa Publishing House.

Received July 13, 2010

continuous with respect to the measure λ and f_0 is essentially bounded, where f_0 is the Radon-Nikodym derivative of the measure λT^{-1} with respect to λ. The Radon-Nikodym derivative of the measure $\lambda(T^k)^{-1}$ with respect to λ is denoted by f_0^k, where T^k is obtained by composing *T*, *k* times [8].

Every essentially bounded complex-valued measurable function f_0 induces the bounded operator M_{f_0} on $L^2(\lambda)$, which is defined by $M_{f_0} f = f_0 f$, for every $f \in L^2(\lambda)$ and it is well known that $C^* C = M_{f_0}$.

A weighted composition operator W (w.c.o) induced by T is a linear transformation acting on the set of complex valued Σ measurable functions *f* of the form $Wf = wf \circ T$, where w is a complex valued Σ measurable function. When $u = 1$, we say that W is a composition operator. Let w_k denote $w(w \circ T)(w \circ T^2) \cdots (w \circ T^{k-1})$ so that $W^k f = w_k (f \circ T)^k$ [11].

For better examination, Lambert [9] associated with each transformation *T*, a condition expectation operator $E(\bullet / T^{-1}\Sigma) = E(\bullet)$ studied in [3], [5], [6].

$E(f)$ is defined for each non-negative measurable function *f*, or for each $f \in L^p$ $(1 \leq p)$ and is uniquely determined by the conditions

(i) $E(f)$ is $T^{-1}\Sigma$ is measurable.

(ii) If B is any $T^{-1}\Sigma$ measurable set for which $\int_B f d\lambda$ converges, then $\int_B E(f) d\lambda$ also converges.

2. (α, β)-Normal Composition Operators on the L^2 Space

Let $B(H)$ be the Banach algebra of all the bounded linear operators on a Hilbert space *H*. Then an operator $T \in B(H)$ is said to be *hyponormal* if $TT^* \leq T^* T$, and *m*-hyponormal if there exists an $m \geq 0$ such that $TT^* \leq m^2 T^* T$. Correspondingly the composition operator *C* is hyponormal if and only if $(f_0 \circ T)P$

$\leq f_0$ a.e., and *m*-hyponormal if there exists an $m \geq 0$ such that $(f_0 \circ T)P \leq m^2 f_0$ a.e., [11].

Definition 2.1 [9]**.** An operator $T \in B(H)$ is said to be an (α, β)-*normal operator* [10], $(0 \leq \alpha \leq 1 \leq \beta)$ if $\alpha^2 T^* T \leq TT^* \leq \beta^2 T^* T$.

When $\alpha = \beta = 1$, (α, β)-normal operator is normal. We need the following Lemma [8] for the characterization of (α, β)-normal composition operators on the L^2 space.

Lemma 2.2. *Let P denote the projection of* L^2 *onto* $\overline{R(C)}$. *Then*

(a) $C^* Cf = f_0 f$ *and* $CC^* f = (f_0 \circ T)Pf$, *for all* $f \in L^2$.

(b) $\overline{R(C)} = \{f \in L^2 : T^{-1}(\Sigma) \textit{ is measurable}\}$.

Theorem 2.3. *Let* $C \in B(L^2(\lambda))$. *Then C is* (α, β)-*normal if and only if* $\alpha^2 f_0 \leq (f_0 \circ T)P \leq \beta^2 f_0$ *a.e.*

Proof. By Definition 2.1, *C* is (α, β)-normal if and only if $\alpha^2 C^* C \leq CC^* \leq \beta^2 C^* C$ for $(0 \leq \alpha \leq 1 \leq \beta)$. Since $C^* C = M_{f_0}$ and $CC^* f = M_{f_0 \circ T} P$, it follows that *C* is (α, β)-normal if and only if $\alpha^2 M_{f_0} \leq M_{f_0 \circ T} P \leq \beta^2 M_{f_0}$ a.e.

Thus $\alpha^2 f_0 \leq (f_0 \circ T)P \leq \beta^2 f_0$, a.e., for $0 \leq \alpha \leq 1 \leq \beta$.

Corollary 2.4. *Let* $C \in B(L^2(\lambda))$. *If C has a dense range, then C is* (α, β)-*normal if and only if* $\alpha^2 f_0 \leq (f_0 \circ T) \leq \beta^2 f_0$, *a.e.*

Corollary 2.5. *Let* $C \in B(L^2(\lambda))$. *If C has a dense range, then* C^* *is* (α, β)-*normal if and only if* $\alpha^2 (f_0 \circ T) \leq f_0 \leq \beta^2 (f_0 \circ T)$ *a.e.*

Example 2.6. Let $X = N$ be the set of all natural numbers and λ be the counting measure on it. Define $T : N \to N$ by $T(1) = 1$, $T(2) = 1$, $T(5n + m - 2) = n + 1$ for $m = 0, 1, 2, \ldots$ and $n \in N$. Then *C* is (α, β)-normal for $0 \leq \alpha \leq 1$, and $\beta \geq \sqrt{5/3}$.

Now, we give an example of an m-hyponormal composition operator on $L^2(\lambda)$ which is neither (α, β)-normal nor hyponormal.

Example 2.7. Let $X = N$ be the set of all natural numbers and λ be the counting measure on it. Define $T : N \to N$ by $T(1) = 2$, $T(2) = 1$, $T(3) = 2$, $T(3n + m) = n + 2$ for $m = 1, 2, \ldots$ and $n \in N$. Then C is m-hyponormal but neither (α, β)- normal nor hyponormal for $n = 1$.

Now, we use the following proposition due to Campbell and Jamison [3] for the characterization of weighted (α, β)-normal composition operators on the L^2 space.

Proposition 2.8. *For* $w \geq 0$,

(a) $W^*Wf = f_0[E(w^2)] \circ T^{-1}f$.

(b) $WW^*f = w(f_0 \circ T)E(wf)$.

Here, we characterize weighted (α, β)-normal composition operators.

Theorem 2.9. *If* $T^{-1}\Sigma = \Sigma$, *then W is* (α, β)*-normal if and only if* $\alpha^2 f_0(w^2) \circ T^{-1} \leq w^2(f_0 \circ T) \leq \beta^2 f_0(w^2) \circ T^{-1}$ *a.e.*

Proof. Since $W^k f = w_k(f \circ T^k)$ and $(W^{*k})f = f_0^{(k)}E(w_k f) \circ T^{-k}$, we have $W^{*k}W^k = f_0^{(k)}E(w_k^2) \circ T^{-k} f$ and $|W^*| f = vE(vf)$, where $v = \dfrac{w\sqrt{f_0 \circ T}}{[E(w\sqrt{f_0 \circ T})^2]^{\frac{1}{4}}}$.

If $T^{-1}\Sigma = \Sigma$, then E becomes the identity operator and hence $WW^*f = v^4 f = w^2(f_0 \circ T)f$; $f \in L^2$. If W is (α, β)-normal, then $\alpha^2 W^*W \leq WW^* \leq \beta^2 W^*W$ and hence $\alpha^2 f_0(w^2) \circ T^{-1} \leq w^2(f_0 \circ T) \leq \beta^2 f_0(w^2) \circ T^{-1}$ a.e.

The Aluthge transformation [1] of T is the operator $\widetilde{T}$ is given by $\widetilde{T} = |T|^{1/2} U|T|^{1/2}$. More generally, we may form the family of operators $A_r : 0 < r \leq 1$, where $A_r = |A|^r U|A|^{1-r}$. For a composition operator C, the polar decomposition is given by $C = U|C|$, where $|C|f = \sqrt{f_0 f}$ and $UF = \frac{1}{\sqrt{f_0 \circ T}} f \circ T$. Lambert et al. [5] has given general Aluthge transformation for composition operators as $C_r = |C|^r U|C|^{1-r}$ and $C_r f = \left(\frac{f_0}{f_0 \circ T}\right)^{\frac{1}{2}} f \circ T$. That is, C_r is weighted composition operator with weight $\pi = \left(\frac{f_0}{f_0 \circ T}\right)^{\frac{1}{2}}$, where $0 < r < 1$. Since C_r is a weighted composition operator it is easy to show that $|C_r|f = \sqrt{f_0}[E(\pi)^2 \circ T^{-1}]f$ and $|C_r^*|f = vE[vf]$, where $v = \frac{\pi\sqrt{f_0 \circ T}}{[E(\pi\sqrt{f_0 \circ T})^2]^{\frac{1}{4}}}$. Also, we have

$$C_r^k f = \pi_k(f \circ T^k).$$

$$C_r^{*k} f = f_0^{(k)} E(\pi_k f) \circ T^{-k}.$$

$$C_r^{*k} C_r^k f = f_0^{(k)} E(\pi_k^2) \circ T^{-k} f.$$

Corollary 2.10. *If* $T^{-1}\Sigma = \Sigma$, *then* C_r *is* (α, β)*-normal if and only if* $\alpha^2 f_0(\pi^2) \circ T^{-1} \leq \pi^2(f_0 \circ T) \leq \beta^2 f_0(\pi^2) \circ T^{-1}$ *a.e.*

The second Aluthge transformation of T described by Duggal [4] is given by $\widetilde{\widetilde{T}} = |\widetilde{T}|^{\frac{1}{2}} V |\widetilde{T}|^{\frac{1}{2}}$, where $\widetilde{T} = V|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$.

Senthilkumar and Prasad [14] studied that the operator $\widetilde{\widetilde{C}} = |C_r|^{\frac{1}{2}} V |C_r|^{\frac{1}{2}}$, where $C_r = V|C_r|$ is the polar decomposition of the generalised Aluthge transformation $C_r : 0 < r < 1$, is weighted composition operator with weight $w' = J^{\frac{1}{4}} \pi \left(\frac{\chi \sup J}{J^{1/4}} \circ T\right)$, where $J = f_0 E(\pi^2) \circ T^{-1}$.

Corollary 2.11. *If* $T^{-1}\Sigma = \Sigma$, *then* W *is* (α, β)*-normal if and only if* $\alpha^2 f_0(w'^2) \circ T^{-1} \leq w'^2(f_0 \circ T) \leq \beta^2 f_0(w'^2) \circ T^{-1}$ *a.e.*

3. (α, β)-normal Composition Operators on Weighted Hardy Spaces

The set $H^2(\gamma)$ of formal complex power series $f(z) = \sum_{n=0}^{\infty} a_n Z^n$ such that $\| f \|_{\gamma}^2 = \sum_{n=0}^{\infty} | a_n |^2 \gamma_n^2 < \infty$ is the general Hardy space of functions analytic in the unit disc with the inner product

$$\langle f, g \rangle_{\gamma} = \sum_{n=0}^{\infty} a_n \overline{b_n} \gamma_n^2$$

for *f* as above and $g(z) = \sum_{n=0}^{\infty} b_n Z^n$ and $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ is a sequence of positive numbers with $\gamma_0 = 1$ and $\frac{\gamma_{n+1}}{\gamma_n} \to 1$ as $n \to \infty$.

If ϕ is an analytic function mapping the unit disc D into itself, we define the composition operator C_{ϕ} on the spaces $H^2(\gamma)$ by

$$C_{\phi} f = f_0 \phi.$$

Though the operator C_{ϕ} is defined everywhere on the classical Hardy space H^2 (the case when $\gamma_n = 1$, for all *n*), they are not necessarily defined on all of $H^2(\beta)$. The composition operator C_{ϕ} is defined on $H^2(\gamma)$ only when the function ϕ is analytic on some open set containing the closed unit disc having supremum norm strictly smaller than one [16].

The properties of composition operator on the general Hardy spaces $H^2(\gamma)$ are studied in [7], [12], [15].

In this Section, we investigate the properties of (α, β)-normal composition operators on general Hardy spaces $H^2(\gamma)$.

For a sequence γ as above and a point *w* in *D*, let

$$k_w \gamma_{(z)} = \sum_{n=0}^{\infty} \frac{1}{\gamma_2^n} (\overline{w_z})^n.$$

Then the function $k_w \gamma$ is a point evaluation for $H^2(\gamma)$, i.e., for *f* in $H^2(\gamma)$,

$$(f, k_w \gamma)_{\gamma} = f(w).$$

Then $k_0\gamma = 1$ and $C_\phi^* k_w\gamma = k_{\phi(w)}\gamma$.

Theorem 3.1. *If* C_ϕ *is* (α, β)*-normal on* $H^2(\gamma)$, *then* $\alpha \leq \frac{1}{\| k_{\phi(0)}^{\gamma} \|} \leq \beta$.

Proof. Let C_ϕ be (α, β)-normal on $H^2(\gamma)$. By the definition of (α, β) normality,

$$\alpha \| C_\phi^* f \|_\gamma \leq \| C_\phi f \|_\gamma \leq \beta \| C_\phi^* f \| \quad \forall f \in H^2(\gamma)$$

and if $f = k_0\gamma$, we have

$$\alpha \| C_\phi^* k_0 \gamma \|_\gamma \leq \| C_\phi k_0 \gamma \|_\gamma \leq \beta \| C^* k_0 \gamma \|_\gamma,$$

$$\alpha \| k_{\phi(0)}^{\gamma} \|_\gamma \leq \| k_0 \gamma \|_\gamma \leq \beta \| k_{\phi(0)}^{\gamma} \|_\gamma,$$

$$\alpha \leq \frac{1}{\| k_{\phi(0)}^{\gamma} \|} \leq \beta.$$

Theorem 3.2. *If* C_ϕ *is* (α, β)*-normal, then* ϕ *is univalent in the unit disk. Moreover, there is a subset E of the unit circle with measure zero, such that off E, the radial limits of* ϕ *exists and are distinct at distinct points.*

Proof. For non-constant ϕ, $\ker(C_\phi) = (0)$. If C_ϕ is (α, β)-normal, $\ker(C_\phi) = \ker(C_\phi^*)$, so $\ker(C_\phi^*) = (0)$.

This implies ran (C_ϕ) is dense. In particular, there is a sequence of polynomials p_n so that $C_\phi p_n = p_n \circ \phi$ converges to *z* in $H^2(\gamma)$.

Since $p_n \circ \phi(\gamma)$ converges to γ for each γ in the disk, ϕ is univalent in the disk.

By possibly passing to a subsequence, we may assume that the boundary functions of the $p_n \circ \phi$ are defined and converge pointwise to *z* off a set *E* of measure zero. If $e^{i\theta_1}$ and $e^{i\theta_2}$ are distinct and not in *E*, then the convergence of the redial limit functions implies that infinitely many of the $p_n \circ \phi$ have distinct values at $e^{i\theta_1}$ and $e^{i\theta_2}$ which implies that $\phi(e^{i\theta_1})$ and $\phi(e^{i\theta_2})$ are distinct.

References

[1] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Integr. Equat. Oper. Th. 13 (1990), 307-315.

[2] C. Burnap, I. L. Bong Jung and A. Lambert, Separating partial normality class with composition operators, J. Operator Theory 53(2) (2005), 381-397.

[3] J. Campbell and J. Jamison, On some classes of weighted composition operators, Glasgow Math J. 32 (1990), 82-94.

[4] B. P. Duggal, Quasi similar p-hyponormal operators, Integr. Equat. Oper. Th. 26 (1996), 338-345.

[5] M. Embry Wardrop and A. Lambert, Measurable transformations and centered composition operators, Proc. Royal Irish Acad. 90A (1990), 165-172.

[6] S. R. Foguel, Selected topics in the study of markov operators, Carolina Lectures Series No. 9, Dept. Math., UNC-CH, Chapel Hill, NC. 27514, 1980.

[7] G. Gunatillake, Weighted composition operators, Doctoral Dissertation, Purdue University, 2005.

[8] D. J. Harrington and R. Whitley, Seminormal composition operators, J. Operator Theory 11 (1984), 125-135.

[9] Lambert, Hyponormal composition operators, Bull. London Math. Soc. 18 (1986), 395-400.

[10] M. S. Moslehian, On (α, β)-normal operators in Hilbert spaces, IMAGE 39 (2007), Problem 39-4.

[11] S. Panayappan, Non-hyponormal weighted composition operators, Indian J. Pure Appl. Math. 27(10) (1996), 979-983.

[12] S. Panayappan, D. Senthilkumar and R. Mohanraj, M-quasihyponormal composition operators on weighted hardy spaces, Int. J. Math. Anal. 2 (2008), 1163-1170.

[13] R. Pushpa Suri and N. Singh, M-quasi-hyponormal composition operators, Internat. J. Math. Math. Sci. 10(3) (1987), 621-623.

[14] D. Senthilkumar and T. Prasad, Composition operators of class Q^*, Int. J. Math. Anal. 4(21) (2010), 1035-1040.

[15] T. Veluchamy and T. Thulasimani, Posinormal composition operators on weighted Hardy space, Int. Math. Forum 5(24) (2010), 1195-1205.

[16] N. Zorboska, Composition operators induced by functions with supremum strictly smaller than one, Proc. Amer. Math. Soc. 106 (1989), 679-684.