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VISCOSITY APPROXIMATION TO COMMON FIXED POINTS OF A FAMILY OF QUASI-NONEXPANSIVE MAPPINGS WITH WEAKLY CONTRACTIVE MAPPINGS

JAMNIAN NANTADILOK

Department of Mathematics
Faculty of Science
Lampang Rajabhat University
Lampang, 52100, Thailand
e-mail: jamnian52@lpru.ac.th

Abstract

Let K be a nonempty closed convex subset of a real reflexive Banach space X that has weakly sequentially continuous duality mapping. In this paper, we consider the viscosity approximation sequence $x_n := \alpha_n f(x_n) + (1-\alpha_n)T_n(x_n)$, $n \geq 1$, where $f : K \to K$ is a weakly contractive mapping, $\{T_n\}$ is a uniformly asymptotically regular sequence of quasi-nonexpansive mappings such that $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and $\{\alpha_n\} \subseteq (0, 1)$. Suppose that $\{x_n\}$ satisfies condition (A). Then it is proved that $\{x_n\}$ converges strongly to a common fixed point p of a family T_i, $i = 1, 2, \ldots$. Our results extend and improve the existing known results in this area.

1. Introduction

Let K be a nonempty closed convex subset of a real Banach space X. Then a

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mapping $T : K \to X$ is called *quasi-nonexpansive* if $\forall x \in K$ and $p \in F(T)$, the following inequality holds:

$$\| Tx - p \| \leq \| x - p \|, \tag{1.1}$$

where $F(T) := \{x \in K : T(x) = x\} \neq \emptyset$, and T is called *nonexpansive* if $\| Tx - Ty \| \leq \| x - y \|$, $\forall x, y \in K$. It is clear that a nonexpansive mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. The converse is not true. For a sequence $\{\alpha_n\} \in (0, 1)$ and an arbitrary $u \in K$, let the sequence $\{x_n\} \in K$ be iteratively defined by $x_0 \in K$,

$$x_{n+1} := \alpha_{n+1} u + (1 - \alpha_{n+1}) T(x_n), \quad n \geq 0, \tag{1.2}$$

where T is a nonexpansive mapping of K into itself.

Hapern [4] was the first to study the convergence of the algorithm (1.2) in the framework of Hilbert spaces. Lions [5] improved the result of Hapern, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T if the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$\text{(i) } \lim_{n\to\infty} \alpha_n = 0; \text{ (ii) } \sum_{n=1}^{\infty} \alpha_n = \infty; \text{ and (iii) } \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0. \tag{1.3}$$

Wittmann [10] proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies the following conditions:

$$\text{(i) } \lim_{n\to\infty} \alpha_n = 0; \text{ (ii) } \sum_{n=1}^{\infty} \alpha_n = \infty; \text{ and (iii) } \sum_{n=1}^{\infty} | \alpha_{n+1} - \alpha_n | < \infty. \tag{1.4}$$

Reich [8] extended the result of Wittmann to the class of Banach spaces which are uniformly smooth and have weakly sequentially continuous duality mappings.

In 2000, Moudafi [6] introduced viscosity approximation method and proved that if X is a real Hilbert space, for given $x_0 \in K$, the sequence $\{x_n\}$ generated by the algorithm:

$$x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n) T(x_n), \quad n \geq 0, \tag{1.5}$$

where $f : K \to K$ is a contraction mapping with constant $\beta \in (0, 1)$ and $\{\alpha_n\} \subseteq (0, 1)$ satisfies certain conditions, converges strongly to a fixed point of T in K

which is the unique solution to the following variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T).$$

Moudafi in [6] generalized Browder's and Hapern's theorems in the direction of viscosity approximations. Viscosity approximations are very important because they are applied to convex optimization, linear programming, monotone inclusions and elliptic differential equations.

In 2004, Xu [11] studied further the viscosity approximation method for nonexpansive mappings in uniformly smooth Banach spaces. This result of Xu [11] extends Theorem 2.2 of Moudafi [6] to Banach space setting. For details on the iterative methods, we refer the reader to [1].

Definition 1.1. Let (X, d) be a complete metric space. Then a mapping $T : X \to X$ is called *weakly contractive* if

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \tag{1.6}$$

where $x, y \in X$ and $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$.

Theorem 1.1 (See, e.g., [7])**.** *Let $T : X \to X$ be weakly contractive mapping, where (X, d) is a complete metric space. Then T has a unique fixed point.*

For $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm

$$x_n := \alpha_n f(x_n) + (1 - \alpha_n)T_n(x_n), \quad n \geq 1, \tag{1.7}$$

where $f : K \to K$ is a weakly contractive mapping and $\{\alpha_n\} \subseteq (0, 1)$ satisfying:

$$\lim_{n\to\infty} \alpha_n = 0.$$

Then $\{x_n\}$ is said to satisfy condition (A) if for any subsequence $x_{n_k} \rightharpoonup x$ and $x_n - T_m(x_n) \to 0$, $\forall m \in \{1, 2, ...\}$, implies $x \in F$.

Razani and Homaeipour [7] proved the convergence of viscosity approximation scheme (1.7) to common fixed points of families of nonexpansive mappings with weakly contractive mappings.

In this paper, inspired by [7] and the above results, we established strong convergence of the sequence (1.7) to a common fixed point a family of quasi-nonexpansive mappings a more general class of mappings in Banach spaces.

2. Preliminaries

Let X be a real Banach space with dual X^*. A guage function is a continuous strictly increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. The duality mapping $J_\varphi : X \to X^*$ associated with a guage function φ is defined by $J_\varphi := \{u^* : \langle x, u^*\rangle = \| x \| \cdot \| u^* \|, \| u^* \| = \varphi(\| x \|)\}$, $x \in X$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the particular case, $\varphi(t) = t$, the duality map $J = J_\varphi$ is called the *normalized duality map*. We denote that $J_\varphi = \frac{\varphi(\| x \|)}{\| x \|} J(x)$. It is known that if X is smooth, then J_φ is single valued and norm to w^* continuous (see, e.g., [3]).

Following Browder [2], we say that a Banach space X has *weakly continuous duality mapping* if there exists a guage function φ for which the duality map J_φ is single valued and weak to weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weak* to $J_\varphi(x)$).

It is known that l^p $(1 < p < \infty)$ spaces have a weakly continuous duality mapping J_φ with a guage $\varphi(t) = t^{p-1}$.

A Banach space X is called *strictly convex* if

$$\| x \| = \| y \| = 1, \quad x \neq y \text{ implies } \frac{\| x + y \|}{2} < 1. \tag{2.1}$$

A Banach space X is called *uniformly convex*, if for all $\varepsilon \in [0, 2]$, there exists $\delta_\varepsilon > 0$ such that

$$\| x \| = \| y \| = 1 \text{ with } \| x - y \| \geq \varepsilon \text{ implies } \frac{\| x + y \|}{2} < 1 - \delta_\varepsilon. \tag{2.2}$$

It is well known that a uniformly convex Banach space X is reflexive and strictly convex.

Proposition 2.1 (See, e.g., [12])**.** (1) $J = I$ *if and only if X is a Hilbert space*.

(2) *J is surjective if and only if X is reflexive*.

(3) $J_\varphi(\lambda x) = sign\,\lambda(\varphi(|\lambda| \cdot \| x \|)/\| x \|)J(x)$, *for all* $x \in X \setminus \{0\}$, $\lambda \in R$; *in particular* $J(-x) = -J(x)$, *for all* $x \in X$.

Definition 2.1 (See [9])**.** Let K be a nonempty closed convex subset of a Banach space X and $T_n : K \to K$, where $n \in \{1, 2, ...\}$. Then the mappings sequence $\{T_n\}$ is called *uniformly asymptotically regular* on K, if for all $m \in \{1, 2, ...\}$ and any bounded subset C of K, we have

$$\lim_{n\to\infty} \sup_{x\in C} \| T_m(T_n x) - T_n x \| = 0. \tag{2.3}$$

3. Main Results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real reflexive Banach space X that has weakly sequentially continuous duality mapping* J_φ *for some guage* φ. *Suppose that* $T_i : K \to K$, $i = 1, 2, ...,$ *is a uniformly asymptotically regular sequence of quasi-nonexpansive mappings such that* $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let sequence* $\{x_n\}$ *be generated by the algorithm*

$$x_n := \alpha_n f(x_n) + (1 - \alpha_n)T_n(x_n), \quad n \geq 1, \tag{3.1}$$

where $f : K \to K$ *is a weakly contractive mapping. Suppose that* $\{\alpha_n\} \subset (0, 1)$ *satisfying* $\lim_{n\to\infty} \alpha_n = 0$. *Suppose that* $\{x_n\}$ *satisfies condition* (A). *Then* $\{x_n\}$ *converges strongly to a common fixed point p of a family* T_i, $i = 1, 2, ...,$ *as* $n \to \infty$. *Moreover, p is the unique solution, in F, to the variational inequality*

$$\langle f(p) - p,\ J_\varphi(y - p)\rangle \leq 0, \quad \forall y \in F. \tag{3.2}$$

Proof. Let $q \in \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. We first show that $\{x_n\}$ is bounded. Then from (1.2) and (3.1), we obtain that

$$\begin{aligned}
\| x_n - q \|^2 &= \langle x_n - q,\ J_\varphi(x_n - q)\rangle \\
&= \langle \alpha_n f(x_n) + (1-\alpha_n)T_n x_n - q,\ J_\varphi(x_n - q)\rangle \\
&= \langle \alpha_n(f(x_n) - q) + (1-\alpha_n)(T_n x_n - q),\ J_\varphi(x_n - q)\rangle \\
&= \alpha_n\langle (f(x_n) - f(q)) + (f(q) - q),\ J_\varphi(x_n - q)\rangle \\
&\quad + (1-\alpha_n)\langle T_n x_n - q,\ J_\varphi(x_n - q)\rangle \\
&\le \alpha_n \| f(x_n) - f(q)\| \| J_\varphi(x_n - q)\| + \alpha_n\langle (f(q) - q),\ J_\varphi(x_n - q)\rangle \\
&\quad + (1-\alpha_n)\| T_n x_n - q\| \| J_\varphi(x_n - q)\| \\
&\le \alpha_n[(\|x_n - q\| - \psi(\| x_n - q\|))\| x_n - q\| + \langle (f(q) - q),\ J_\varphi(x_n - q)\rangle] \\
&\quad + (1-\alpha_n)\| T_n x_n - q\| \| J_\varphi(x_n - q)\| \\
&\le \alpha_n[\|x_n - q\|^2 - \psi(\| x_n - q\|)\| x_n - q\| + \langle (f(q) - q),\ J_\varphi(x_n - q)\rangle] \\
&\quad + (1-\alpha_n)\| x_n - q\|^2 \\
&= \| x_n - q\|^2 - \alpha_n\| x_n - q\|\psi(\| x_n - q\|) + \alpha_n\| f(q) - q\| \| x_n - q\|
\end{aligned}$$

which implies that

$$\psi(\| x_n - q\|) \le \| f(q) - q\|. \tag{3.3}$$

Therefore, $\{x_n\}$ is bounded.

Next, we prove that $\lim_{n\to\infty}\| x_n - T_m x_n\| = 0,\quad \forall m \in \{1, 2, ...\}$. Since $\{x_n\}$ is bounded, so $\{f(x_n)\}$ and $\{T_n x_n\}$ are bounded.

We have that

$$\lim_{n\to\infty} \alpha_n\| T_n x_n - f(x_n)\| = 0,$$

this implies

$$\lim_{n\to\infty} \| x_n - T_n(x_n)\| = 0.$$

Let C be bounded subset of K which contains $\{x_n\}$. Since $\{T_n\}$ is uniformly asymptotically regular on K, we obtain

$$\lim_{n\to\infty} \| T_m(T_n x_n) - T_n x_n \| \leq \lim_{n\to\infty} \sup_{x\in C} \| T_m(T_n x) - T_n x \| = 0.$$

Let $n \to \infty$. Then

$$\begin{aligned} \| x_n - T_m x_n \| &\leq \| x_n - T_n x_n \| + \| T_n x_n - T_m(T_n x_n) \| + \| T_m(T_n x_n) - T_m x_n \| \\ &\leq \| x_n - T_n x_n \| + 2\| T_n x_n - T_m(T_n x_n) \| \to 0. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \| x_n - T_m x_n \| = 0, \tag{3.4}$$

for all $m \in \{1, 2, ...\}$.

Next, we show that sequence $\{x_n\}$ is sequentially compact. Since X is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q \in K$ as $k \to \infty$. Since $\lim_{n\to\infty} \| x_n - T_m x_n \| = 0$, for all $m \in \{1, 2, ...\}$, by condition (A), we conclude that $q \in F$.

We know from above that

$$\begin{aligned} \| x_{n_k} - q \|^2 \leq \alpha_{n_k} [(\| x_{n_k} - q \|) - \psi(\| x_{n_k} - q \|)\| x_{n_k} - q \| + \langle f(q) - q, J_\varphi(x_{n_k} - q)\rangle] \\ + (1 + \alpha_{n_k})\| x_{n_k} - q \|^2. \end{aligned}$$

Hence

$$\| x_{n_k} - q \| \psi(\| x_{n_k} - q \|) \leq \langle f(q) - q, J_\varphi(x_{n_k} - q)\rangle.$$

Since J_φ is single valued and weakly sequentially continuous from X to X^*, we have

$$\limsup_{k\to\infty} \| x_{n_k} - q \| \psi(\| x_{n_k} - q \|) \leq \lim_{k\to\infty} \langle f(q) - q, j_\varphi(x_{n_k} - q)\rangle = 0.$$

Therefore, $\lim_{k\to\infty} x_{n_k} = q$. Hence $\{x_n\}$ is sequentially compact.

We prove that $q \in F$ is a solution to the variational inequality (3.2). Suppose that $y \in F$. Then

$$\begin{aligned}
&\| x_n - y \|^2 \\
&= \alpha_n \langle (f(x_n) - x_n) + (x_n - y),\ J_\varphi(x_n - y) \rangle + (1 - \alpha_n) \langle T_n x_n - y,\ J_\varphi(x_n - y) \rangle \\
&= \alpha_n \langle (f(x_n) - x_n),\ J_\varphi(x_n - y) \rangle + \alpha_n \langle (x_n - y),\ J_\varphi(x_n - y) \rangle \\
&\quad + (1 - \alpha_n) \langle T_n x_n - y,\ J_\varphi(x_n - y) \rangle \\
&\le \alpha_n \langle (f(x_n) - x_n),\ J_\varphi(x_n - y) \rangle + \alpha_n \| (x_n - y) \|^2 + (1 - \alpha_n) \| x_n - y \|^2.
\end{aligned}$$

Hence

$$\langle (f(x_n) - x_n),\ J_\varphi(y - x_n) \rangle \le 0, \tag{3.5}$$

for each $n \in \{1, 2, \ldots\}$.

Since $x_{n_k} \to q$ as $k \to \infty$, we have

$$\| (x_{n_k} - f(x_{n_k})) - (q - f(q)) \| \to 0$$

and therefore

$$\begin{aligned}
&| \langle (x_{n_k} - f(x_{n_k})),\ J_\varphi(x_{n_k} - y) \rangle - \langle (q - f(q)),\ J_\varphi(q - y) \rangle | \\
&= | \langle (x_{n_k} - f(x_{n_k})) - (q - f(q)),\ J_\varphi(x_{n_k} - y) \rangle \\
&\quad + \langle (q - f(q)),\ J_\varphi(x_{n_k} - y) - J_\varphi(q - y) \rangle | \\
&\le \| (x_{n_k} - f(x_{n_k})) - (q - f(q)) \| \| x_{n_k} - y \| \\
&\quad + | \langle (q - f(q)),\ J_\varphi(x_{n_k} - y) - J_\varphi(q - y) \rangle | \to 0
\end{aligned}$$

as $k \to \infty$.

Hence

$$\langle f(q) - q,\ J_\varphi(y - q) \rangle = \lim_{k \to \infty} \langle f(x_{n_k}) - x_{n_k},\ J_\varphi(y - x_{n_k}) \rangle \le 0.$$

This means $q \in F$ is a solution to the variational inequality (3.2). For uniqueness of the solution to the variational inequality (3.2), suppose that $p, q \in F$ are distinct solutions to (3.2). Then adding

$$\langle f(p) - p, J_\varphi(q - p)\rangle \leq 0$$

and

$$\langle f(q) - q, J_\varphi(p - q)\rangle \leq 0$$

we get that $\psi(\| p - q \|)\| p - q \| \leq 0.$ This gives $p = q.$ We denote by p the unique solution, in F, to inequality (3.2). By uniqueness, $q = p,$ and since the sequence $\{x_n\}$ is sequentially compact and each cluster point of it is equal to p, $x_n \to p$ as $n \to \infty.$ This completes our proof. □

Suppose that K is a nonempty closed convex subset of X which is a real reflexive Banach space and $\{T_n\},\ n \in 1, 2, \ldots$ is a uniformly asymptotically regular sequence of nonexpansive mappings from K into itself. Then we obtain the following corollary:

Corollary 3.2. *Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J_φ from X to X^*. Suppose that K is a nonempty closed convex subset of X and $\{T_n\},\ n \in 1, 2, \ldots,$ is a uniformly asymptotically regular sequence of nonexpansive mappings from K into itself such that $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$ defined by* (3.1), *where $f : K \to K$ is a weakly contractive mapping. Suppose that $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$. Suppose that $\{x_n\}$ satisfies condition* (A). *Then $\{x_n\}$ converges strongly to a common fixed point p of a family $T_i,\ i = 1, 2, \ldots,$ as $n \to \infty$. Moreover, p is the unique solution, in F, to the variational inequality*

$$\langle f(p) - p, J_\varphi(y - p)\rangle \leq 0, \quad \forall y \in F.$$

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