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ON THE NICHOLS ALGEBRA ASSOCIATED TO $(q_{ij}) = \begin{pmatrix} -\omega & -\omega & -2\omega \\ \omega & -1 & \omega \\ -\frac{1}{2\omega} & \omega & \omega \end{pmatrix}$, OF RANK 3

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Abstract

We examine the defining relations of the Nichols algebra associated to $(q_{ij}) = \begin{pmatrix} -\omega & -\omega & -2\omega \\ \omega & -1 & \omega \\ -\frac{1}{2\omega} & \omega & \omega \end{pmatrix}$, of rank 3 by using the results by Angiono [2] and the method by Nichols [1].

1. Introduction

Nichols algebras are graded braided Hopf algebras with the base field in degree 0 and which are coradically graded and generated by its primitive elements ([3-7]). Let V be a vector space and $c : V \otimes V \to V \otimes V$ be a linear isomorphism. Then (V, c) is called a *braided vector space*, if c is a solution of the braid equation, that

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is, $(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c)$. The pair (V, c) determines the Nichols algebras up to isomorphism. Let G be a group. A Yetter-Drinfeld module V over $\mathbb{K}G$ is a G-graded vector space $V = \bigoplus_{g \in G} V_g$, which is a G-module such that $g \cdot V_h \subset V_{ghg^{-1}}$ for all $g, h \in G$. The category ${}_G^G YD$ of $\mathbb{K}G$-Yetter-Drinfeld module is braided. For $V, W \in {}_G^G YD$, the braiding $c : V \otimes W \to W \otimes V$ is defined by $c(v \otimes w) = (g \cdot w) \otimes v$, $v \in V_g$, $w \in W$. Let V be a Yetter-Drinfeld module over G and let $T(V) = \bigoplus_{n \geq 0} T(V)(n)$ denote the tensor algebra of the vector space V. Let S be the set of all ideals and coideals I of $T(V)$ which are generated as ideals by $\mathbb{N}$-homogeneous elements of degree ≥ 2, and which are Yetter-Drinfeld submodules of $T(V)$. Let $I(V) = \sum_{I \in S} I$. Then $B(V) := T(V)/I(V)$ is called the *Nichols algebra* of $V \in {}_G^G YD$. In this article, we examine the defining relations of the Nichols algebra $B(V)$ associate to

$$(q_{ij}) = \begin{pmatrix} -\omega & -\omega & -2\omega \\ \omega & -1 & \omega \\ -\frac{1}{2\omega} & \omega & \omega \end{pmatrix}, \text{ of rank 3.}$$

2. Nichols Algebras of Cartan Type

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Let G be an abelian group and V be a finite dimensional Yetter-Drinfeld module. Then the braiding is given by a non-zero scalar $q_{ij} \in \mathbb{K}$, $1 \leq i, \ j \leq \theta$, in the form $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, where $x_1, \ldots, x_\theta$, is a basis of V. If there is a basis such that $g \cdot x_i = \chi_i(g) x_i$ and $x_i \in V_{g_i}$, then V is called *diagonal type*. For the braiding, we have $c(x_i \otimes x_j) = \chi_j(g_i) x_j \otimes x_i$ for $1 \leq i, \ j \leq \theta$. Hence we have $(q_{ij})_{1 \leq i, j \leq \theta} = (\chi_j(g_i))_{1 \leq i, j \leq \theta}$. Let $B(V)$ be the Nichols algebra of V. We can construct the Nichols algebra by $B(V) \cong T(V)/I$, where I denote the sum of all ideals of $T(V)$ that are generated by homogeneous elements of degree ≥ 2 and that are coideals. If $B(V)$ is finite-dimensional, then the matrix (a_{ij}) defined by for all $1 \leq i \neq j \leq \theta$ by $a_{ii} := 2$ and $a_{ij} := -\min\{r \in \mathbb{N} \mid q_{ij} q_{ji} q_{ii}^r = 1 \text{ or } (r+1)_{q_{ii}} = 0\}$ is a generalized Cartan matrix fulfilling $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$ or $ord\ q_{ii} = 1 - a_{ij}$. (a_{ij}) is called *Cartan matrix* associated to $B(V)$. To examine the defining relations of $B(V)$, we use the results [3] and [2].

Proposition 2.1 ([3])**.** (1) *For all* $1 \leq i \leq \theta$, *there exists a uniquely determined* (id, σ)*-derivation* $D_i : B(V) \to B(V)$ *with* $D_i(x_j) = \delta_{ij}$ (*Kronecker* δ) *for all* j.

(2) $\bigcap_{i=1}^{\theta} ker(D_i) = \mathbb{K}1.$

Proposition 2.2 ([2])**.** *Let* (V, c) *be a braided vector space such that* $dim\ V = 3$, *and the corresponding generalized Dynkin diagram is* $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ}\overset{r^{-1}}{\text{———}}\overset{r}{\circ}$. *Then* $B(V)$ *is presented by generators* x_1, x_2, x_3, *and relations*

(2.2.1) $x_1^M = x_2^2 = x_3^N = x_{\alpha_1+2\alpha_2+\alpha_3}^P = 0,$

(2.2.2) $(ad_c x_1)^2 x_2 = (ad_c x_3)^2 x_2 = (ad_c x_1)x_3 = 0,$

(2.2.3) $[x_{\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2+\alpha_3}]_c = [x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_2+\alpha_3}]_c = 0.$

Using this, we obtain the following:

Proposition 2.3. *Let* $(q_{ij}) = \begin{pmatrix} -\omega & -\omega & -2\omega \\ \omega & -1 & \omega \\ -\frac{1}{2\omega} & \omega & \omega \end{pmatrix}$, $\left(\overset{-\omega}{\circ}\overset{-\omega^2}{\text{———}}\overset{-1}{\circ}\overset{\omega^2}{\text{———}}\overset{\omega}{\circ}\right)$

(*where* ω *is a primitive cube root of unity*). *Then the Nichols algebra* $B(V)$ *is described as follows*:

Generators: x_1, x_2, x_3.

Relations: $x_1^6 = 0,\ x_2^2 = 0,\ x_3^3 = 0,$

$x_1^2x_2 + (\omega - \omega^2)x_1x_2x_1 - x_2x_1^2 = 0, \quad x_3^2x_2 + x_3x_2x_3 + x_2x_3^2 = 0,$

$(x_2x_1)^2 = -(x_1x_2)^2, \quad (x_3x_2)^2 = (x_2x_3)^2, \quad x_1x_3 = x_3x_1,$

$x_2x_1x_2x_3 + (1 - \omega^2)x_2x_1x_3x_2 + 2\omega x_2x_3x_2x_1 + \omega^2 x_1x_2x_3x_2 + 2x_3x_2x_1x_2 = 0.$

Its basis is given as follows:

$\{1,\ x_1,\ x_2,\ x_3,\ x_1^2,\ x_1x_2,\ x_1x_3,\ x_2x_1,\ x_2x_3,\ x_3x_2,\ x_3^2,\ x_1x_2x_1,\ x_1x_2x_3,\ x_2x_1x_2,\ x_2x_1x_3,$

$x_2x_3x_2,\ x_3x_1x_2,\ x_3x_2x_1,\ x_3x_2x_3,\ x_2x_1^2,\ x_3x_1^2,\ x_2x_3^2,\ x_1x_3^2,\ x_1^3,\ (x_1x_2)^2,\ x_1x_2x_1x_3,$

$x_1x_2x_3x_2,\ x_2x_1x_2x_3,\ x_2x_1x_3x_2,\ x_2x_3x_2x_1,\ (x_2x_3)^2,\ x_3x_1x_2x_3,\ x_3x_1x_2x_1,$

$x_3x_2x_3x_1,\ x_1x_2x_1^2,\ x_3x_2x_1^2,\ x_2x_3x_1^2,\ x_1x_2x_3^2,\ x_3x_2x_3^2,\ x_2x_1x_3^2,\ x_2x_1^3,\ x_3x_1^3,\ x_1^4,$

$x_1^2x_3^2,\ (x_1x_2)^2x_3,\ x_1x_2x_3x_2x_1,\ x_1(x_2x_3)^2,\ x_2x_1x_2x_3x_2,\ (x_2x_3)^2x_1,\ x_3x_1x_2x_1x_3,$

$x_3(x_1x_2)^2,\ x_3x_2x_1x_2x_3,\ x_2x_1x_2x_1^2,\ x_3x_1x_2x_1^2,\ x_2x_3x_2x_1^2,\ x_1x_2x_3x_1^2,\ x_3x_2x_3x_1^2,$

$x_2x_1x_2x_3^2,\ x_3x_1x_2x_3^2,\ x_2x_3x_2x_3^2,\ x_1x_2x_1x_3^2,\ x_3x_2x_1x_3^2,\ x_1x_2x_1^3,\ x_3x_2x_1^3,\ x_2x_3x_1^3,$

$x_2x_1^4,\ x_3x_1^4,\ x_1^5,\ x_2x_1^2x_3^2,\ x_3^2x_1^3,\ x_1(x_2x_3)^2x_1,\ x_2x_1x_2x_3x_2x_1,\ x_3(x_1x_2)^2x_3,$

$x_3x_2x_1x_2x_3x_2,\ x_3x_2x_1x_2x_1^2,\ x_1x_2x_3x_2x_1^2,\ (x_2x_3)^2x_1^2,\ x_2x_1x_2x_3x_1^2,\ x_1x_3x_2x_3x_1^2,$

$x_3x_2x_1x_2x_3^2,\ x_1x_2x_3x_2x_3^2,\ (x_1x_2)^2x_3^2,\ x_3x_1x_2x_1x_3^2,\ x_2x_3x_2x_1x_3^2,\ x_2x_1x_2x_1^3,$

$x_3x_1x_2x_1^3,\ x_2x_3x_2x_1^3,\ x_1x_2x_3x_1^3,\ x_3x_2x_3x_1^3,\ x_1x_2x_1^4,\ x_3x_2x_1^4,\ x_2x_3x_1^4,\ x_2x_1^5,\ x_3x_1^5,$

$x_1x_2x_1^2x_3^2,\ x_3x_2x_1^2x_3^2,\ x_2x_3^2x_1^3,\ x_3^2x_1^4,\ x_3x_2x_1x_2x_3x_2x_1,\ x_2x_1x_2x_3x_2x_1^2,$

$x_1(x_2x_3)^2x_1^2,\ x_3x_2x_1x_2x_3x_1^2,\ x_3(x_1x_2)^2x_3^2,\ x_2x_1x_2x_3x_2x_3^2,\ x_1x_2x_3x_2x_1x_3^2,$

$x_3x_2x_1x_2x_1^3,\ x_1x_2x_3x_2x_1^3,\ (x_2x_3)^2x_1^3,\ x_2x_1x_2x_3x_1^3,\ x_1x_3x_2x_3x_1^3,\ x_2x_1x_2x_1^4,$

$x_3x_1x_2x_1^4,\ x_2x_3x_2x_1^4,\ x_1x_2x_3x_1^4,\ x_3x_2x_3x_1^4,\ x_1x_2x_1^5,\ x_3x_2x_1^5,\ x_2x_3x_1^5,\ x_2x_1x_2x_1^2x_3^2,$

$x_3x_1x_2x_1^2x_3^2,\ x_2x_3x_2x_1^2x_3^2,\ x_1x_3x_2x_1^2x_3^2,\ x_1x_2x_3^2x_1^3,\ x_3x_2x_3^2x_1^3,\ x_2x_3^2x_1^4,$

$x_3^2x_1^5,\ x_3x_2x_1(x_2x_3)^2x_1,\ x_3x_2x_1x_2x_3x_2x_1^2,\ x_2x_1(x_2x_3)^2x_1^2,\ (x_1x_2)^2x_3x_2x_3^2,$

$x_3x_2x_1x_2x_3x_2x_3^2,\ x_2x_1x_2x_3x_2x_1x_3^2,\ x_2x_1x_2x_3x_2x_1^3,\ x_1(x_2x_3)^2x_1^3,\ x_3x_2x_1x_2x_3x_1^3,$

$x_1x_2x_3x_2x_1^4,\ x_3x_2x_1x_2x_1^4,\ x_2x_3x_1x_2x_1^4,\ (x_2x_3)^2x_1^4,\ x_1x_3x_2x_3x_1^4,\ x_2x_1x_2x_3x_1^4,$

$x_2x_1x_2x_1^5,\ x_3x_1x_2x_1^5,\ x_2x_3x_2x_1^5,\ x_1x_2x_3x_1^5,\ x_3x_2x_3x_1^5,\ x_3x_2x_1x_2x_1^2x_3^2,$

$x_1x_2x_3x_2x_1^2x_3^2,\ x_2x_1x_2x_3^2x_1^3,\ x_3x_1x_2x_3^2x_1^3,\ x_2x_3x_2x_3^2x_1^3,\ x_1x_2x_3^2x_1^4,\ x_3x_2x_3^2x_1^4,$

$x_2x_3^2x_1^5,\ x_3x_2x_1(x_2x_3)^2x_1x_2,\ x_3(x_1x_2)^2x_3x_2x_1^2,\ x_3x_2x_1(x_2x_3)^2x_1^2,$

$(x_1x_2)^2x_3x_2x_1x_3^2,\ x_3x_2x_1x_2x_3x_2x_1x_3^2,\ x_1x_2x_1x_2x_3x_2x_1^3,\ x_3x_2x_1x_2x_3x_2x_1^3,$

$x_2x_1(x_2x_3)^2x_1^3,\ x_2x_1x_2x_3x_2x_1^4,\ x_1(x_2x_3)^2x_1^4,\ x_3x_2x_3x_1x_2x_1^4,\ x_3(x_1x_2)^2x_3x_2x_3^2,$

$x_3x_2x_1x_2x_3x_1^4,\ x_3x_2x_1x_2x_1^5,\ x_1x_2x_3x_2x_1^5,\ (x_2x_3)^2x_1^5,\ x_2x_1x_2x_3x_1^5,\ x_1x_3x_2x_3x_1^5,$

$x_2x_1x_2x_3x_2x_1^2x_3^2,\ x_3x_2x_1x_2x_3^2x_1^3,\ x_1x_2x_3x_2x_3^2x_1^3,\ x_2x_1x_2x_3^2x_1^4,\ x_3x_1x_2x_3^2x_1^4,$

$x_2x_3x_2x_3^2x_1^4,\ x_1x_2x_3^2x_1^5,\ x_3x_2x_3^2x_1^5,\ x_3x_2x_1(x_2x_3)^2x_1x_2x_1,\ x_3x_2x_1(x_2x_3)^2x_1x_2x_3,$

$x_3x_1x_2x_1(x_2x_3)^2x_1^2,\ x_3(x_1x_2)^2x_3x_2x_1^3,\ x_3x_2x_1(x_2x_3)^2x_1^3,\ x_1x_2x_1(x_2x_3)^2x_1^3,$

$x_3x_2x_1x_2x_3x_2x_1^4,\ x_2x_1(x_2x_3)^2x_1^4,\ x_2x_1x_2x_3x_2x_1^5,\ x_1(x_2x_3)^2x_1^5,\ x_3x_2x_1x_2x_3x_1^5,$

$(x_1x_2)^2x_3x_2x_1^2x_3^2,\ x_3x_2x_1x_2x_3x_2x_1^2x_3^2,\ x_2x_1x_2x_3x_2x_3^2x_1^3,\ x_3x_2x_1x_2x_3^2x_1^4,$

$x_2x_3x_1x_2x_3^2x_1^4,\ x_1x_2x_3x_2x_3^2x_1^4,\ x_2x_1x_2x_3^2x_1^5,\ x_3x_1x_2x_3^2x_1^5,\ x_2x_3x_2x_3^2x_1^5,$

$x_3x_2x_1(x_2x_3)^2x_1x_2x_1x_3,\ x_2x_3x_1x_2x_1(x_2x_3)^2x_1^2,\ x_3x_1x_2x_1(x_2x_3)^2x_1^3,$

$x_1x_2x_1(x_2x_3)^2x_1^4,\ x_3x_2x_1(x_2x_3)^2x_1^4,\ (x_1x_2)^2x_3x_2x_1^5,\ x_3x_2x_1x_2x_3x_2x_1^5,$

$x_2x_1(x_2x_3)^2x_1^5,\ x_3x_2x_1x_2x_3x_2x_3^2x_1^3,\ x_2x_1x_2x_3x_2x_3^2x_1^4,\ x_3x_2x_1x_2x_3^2x_1^5,$

$x_2x_3x_1x_2x_3^2x_1^5,\ x_1x_2x_3x_2x_3^2x_1^5,\ x_3x_2x_1(x_2x_3)^2x_1x_2x_1x_3x_2,\ (x_3x_2x_1)^2(x_2x_3)^2x_1^2,$

$x_2x_3x_1x_2x_1(x_2x_3)^2x_1^3,\ x_1x_2x_1(x_2x_3)^2x_1^5,\ x_3x_2x_1(x_2x_3)^2x_1^5, x_3x_2x_1x_2x_3x_2x_3^2x_1^4,$

$x_2x_1x_2x_3x_2x_3^2x_1^5,\ x_3x_2x_1(x_2x_3)^2x_1x_2x_1x_3x_2x_1,\ x_3x_2x_1(x_2x_3)^2(x_1x_2x_3)^2,$

$x_3x_2x_1x_2x_3x_2x_3^2x_1^5,\ x_3(x_1x_2)^2x_3x_2x_3^2x_1^5\}.$

Hence the Hilbert polynomial of $B(V)$ *is given as follows*:

$$P(t) = 1 + 3t + 7t^2 + 13t^3 + 20t^4 + 26t^5 + 28t^6 + 28t^7 + 28t^8 + 26t^9 + 20t^{10} + 13t^{11} + 7t^{12} + 3t^{13} + t^{14}.$$

Proof. They are directly calculated. □

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